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ON RAPID EQUIVALENCE AND TRANSLATIONAL RAPID EQUIVALENCE

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Abstract. In this paper we will prove some properties of the rapid equivalence and consider some selection principles and games related to rapidly varying sequences.

1. Introduction

Let $\mathbb{S}$ be the set of sequences of positive real numbers, and $\mathbb{S}_1$ be the set of nondecreasing sequences from $\mathbb{S}$ [5]. Let $c = (c_n)_{n\in\mathbb{N}} \in \mathbb{S}$. A sequence c is said to be *rapidly varying* in the sense of de Haan, if

$$\lim_{n\to+\infty} \frac{c_{[\lambda n]}}{c_n} = +\infty \tag{1.1}$$

holds for each $\lambda > 1$. The set of all these sequences is denoted by $R_{\infty,s}$. These sequences are objects in rapid variation theory in the sense of de Haan, which is very important in asymptotic analysis and applications (see, e.g., [1–3, 8, 10, 15]). The theory of rapid variation is an important modification of Karamata's theory of regular variation [13], and its relation can be seen on example of slow and rapid variation within generalized inverse (see, e.g., [7]). Elements of the class $R_{\infty,s}$ are important objects in dynamic systems theory [10, 11, 15], infinite topological games theory and selection principles theory [3–6].

A sequence c is *translationally slowly varying* (in the sense of Karamata) if

$$\lim_{n\to+\infty} \frac{c_{[\lambda+n]}}{c_n} = 1 \tag{1.2}$$

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holds for each $\lambda \geqslant 1$. Translationally slowly varying sequences form the class $Tr(SV_s)$ (see, e.g., [4–6]), and it holds $R_{\infty,s} \cap Tr(SV_s) \neq \varnothing$, $R_{\infty,s} \setminus Tr(SV_s) \neq \varnothing$ and $Tr(SV_s) \setminus R_{\infty,s} \neq \varnothing$.

A sequence c is *translationally rapidly varying* (in the sense of de Haan) if

$$\lim_{n\to+\infty} \frac{c_{[\lambda+n]}}{c_n} = +\infty \tag{1.3}$$

holds for each $\lambda \geqslant 1$.

The class of translationally rapidly varying sequences is denoted by $Tr(R_{\infty,s})$. It holds $Tr(R_{\infty,s}) \subsetneq R_{\infty,s}$ for each $\lambda \geqslant 1$ (see, e.g., [5]).

The classes of sequences mentioned above have nice and deep connections with selection principles theory and infinitely long two-person game theory (see, for example, [2, 3, 5, 6]).

Motivated by the study of some equivalence relations on classes of functions and sequences given in [7, 8, 14], in this paper we define a relation on the class of translationally rapidly varying sequences and investigate some properties of this relation. In particular, we study relationships of this relation with selection principles and game theory complementing the research in [2, 3, 5, 6]. We also obtain some additional information on the classes of rapidly varying and translationally rapidly varying sequences.

Definition 1.1. Sequences c and d of positive real numbers are *mutually translationally rapidly equivalent*, denoted by

$$c \overset{tr}{\sim} d \quad \text{as} \quad n \to +\infty,$$

if

$$\lim_{n\to+\infty} \frac{c_{[\lambda+n]}}{d_n} = +\infty \tag{1.4}$$

and

$$\lim_{n\to+\infty} \frac{d_{[\lambda+n]}}{c_n} = +\infty \tag{1.5}$$

hold for each $\lambda \geqslant 1$.

The previous relation is a modification of the rapid equivalence relation between sequences c and d given by

$$\lim_{n\to+\infty} \frac{c_{[\lambda n]}}{d_n} = +\infty \tag{1.6}$$

and

$$\lim_{n\to+\infty} \frac{d_{[\lambda n]}}{c_n} = +\infty, \tag{1.7}$$

for each $\lambda > 1$. We denote it by $c \overset{r}{\sim} d$ as $n \to +\infty$ (see, e.g., [8, 14]).

Let c be a nondecreasing sequence from a subset $\mathcal{V}$ of $\mathbb{S}$. The capacity of c with respect to $\mathcal{V}$ is the subfamily of $\mathbb{S}$ given by $\mathcal{M}_c^{\mathcal{V}} = \{x = (x_n) \in \mathbb{S} \mid c_n \leqslant x_n \leqslant c_{n+1} \text{ for each } n \in \mathbb{N}\}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subfamilies of $\mathbb{S}$. Let us adduce two selection principles which we need in this paper:

(a) (Rotberger, see, e.g.,[12]) $S_1(\mathcal{A},\mathcal{B})$: for each sequence $(A^n)_{n\in\mathbb{N}}$ of elements from $\mathcal{A}$ there is an element $b\in\mathcal{B}$, so that $b_n\in\ A^n$ for each $n\in\mathbb{N}$;

(b) (Kočinac, see, e.g., [9]) $\alpha_2(\mathcal{A},\mathcal{B})$: for each sequence $(A^n)_{n\in\mathbb{N}}$ of elements from $\mathcal{A}$, there is an element $b\in\mathcal{B}$, so that $b\cap A^n$ is infinite for each $n\in\mathbb{N}$.

Games associated to the previous two selection principles are the following.

$G_1(\mathcal{A},\mathcal{B})$. Two players, I and II, play a round for each positive integer. In m^{th} round, $m\in\mathbb{N}$, the player I plays a sequence $A^m\in\mathcal{A}$, and the player II plays an element $b_m\in A^m$. II wins the play $A^1,b_1;A^2,b_2;\ldots$ if and only if $b=(b_n)\in\mathcal{B}$.

The symbol $G_{\alpha_2}(\mathcal{A},\mathcal{B})$ denotes the following infinitely long game for two players, I and II, who play a round for each natural number n. In the first round the player I plays an arbitrary element $A^1\in\mathcal{A}$, and the player II chooses a subsequence $A^{r_1(j)}, j\in\mathbb{N}$, of the sequence A^1. At the k^{th} round, $k\geqslant 2$, the player I plays an arbitrary element $A^k\in\mathcal{A}$ and the player II chooses a subsequence $A^{r_k(j)}$ of the sequence A^k, such that $A^{r_k(j)}\cap A^{r_p(j)}=\emptyset$ is satisfied, for each $p\leqslant k-1$. The player II wins the play $A^1,A^{r_1(j)};\ldots;A^k,A^{r_k(j)};\ldots$ if and only if all elements from $Y=\cup_{k\in\mathbb{N}}\cup_{j\in\mathbb{N}}A^{r_k(j)}$ form a subsequence $y\in\mathcal{B}$.

Note that if II has a winning strategy (even if I does not have a winning strategy) in a game defined above, then the corresponding selection principle holds.

Note that in the paper [5] it is proven that the player II does not have a winning strategy in the game $G_1(Tr(SV_s),Tr(SV_s))$.

2. Results

Proposition 2.1. *If $c\in\mathbb{S}$, $d\in\mathbb{S}$ and $c\overset{tr}{\sim}d$ as $n\to+\infty$ holds, then $c\in Tr(R_{\infty,s})$ and $d\in Tr(R_{\infty,s})$.*

Proof. Let $c,d\in\mathbb{S}$ and $c\overset{tr}{\sim}d$ as $n\to+\infty$ hold. Therefore, for $\lambda=1$, it holds $\lim_{n\to+\infty}\frac{c_{n+1}}{d_n}=+\infty$ and $\lim_{n\to+\infty}\frac{d_{n+1}}{c_n}=+\infty$. For $\lambda\geqslant 1$ it holds $\lim_{n\to+\infty}\frac{c_{[\lambda+n]}}{c_n}=\lim_{n\to+\infty}\left(\frac{c_{[\lambda]+n}}{d_{[\lambda]+n-1}}\cdot\frac{d_{[\lambda]+n-1}}{c_{[\lambda]+n-2}}\cdots\frac{d_{n+1}}{c_n}\right)=+\infty$ for each $\lambda\in[k,k+1)$, $k=2s$, $s\in\mathbb{N}$. It means, for $\lambda=2$, $\lim_{n\to+\infty}\frac{c_{n+2}}{c_n}=\lim_{n\to+\infty}\left(\frac{c_{n+2}}{d_{n+1}}\cdot\frac{d_{n+1}}{c_n}\right)=+\infty$. Therefore, $+\infty=\lim_{n\to+\infty}\left(\frac{c_{n+2}}{c_{n+1}}\cdot\frac{c_{n+1}}{c_n}\right)=\lim_{s\to+\infty}\left(\frac{c_{s+1}}{c_s}\right)^2=\left(\lim_{s\to+\infty}\frac{c_{s+1}}{c_s}\right)^2$. Thus, $\lim_{s\to+\infty}\frac{c_{s+1}}{c_s}=+\infty$, so for each $\lambda\geqslant 1$, $\lim_{s\to+\infty}\frac{c_{[\lambda+s]}}{c_s}=+\infty$ holds. Therefore, $c\in Tr(R_{\infty,s})$. Analogously we prove that $d\in Tr(R_{\infty,s})$. ☐

Proposition 2.2. *The relation $\overset{tr}{\sim}$ is a reflexive, symmetric and nontransitive relation in $Tr(R_{\infty,s})$.*

Proof. 1. (Reflexivity) According to Proposition 2.1, from $c\overset{tr}{\sim}d$ as $n\to+\infty$ it follows $c,d\in Tr(R_{\infty,s})$. The asymptotic relation $\lim_{n\to+\infty}\frac{c_{[\lambda+n]}}{c_n}=+\infty$ holds for each $\lambda\geqslant 1$ in the class $Tr(R_{\infty,s})$, thus $c\overset{tr}{\sim}c$ as $n\to+\infty$.

2. (Symmetry) According to the definition of $\overset{tr}{\sim}$, symmetry holds.

3. (Nontransitivity) The following example shows that the relation is not transitive. Consider the sequences $c_n = (n-1)!\ln(n+1)$, $d_n = n!$ and $e_n = \frac{(n+1)!}{\ln(n+1)}$, $n \in \mathbb{N}$. It holds $c \overset{tr}{\sim} d$, $d \overset{tr}{\sim} e$ as $n \to +\infty$, but $c \overset{tr}{\sim} e$ does not hold as $n \to +\infty$. $\square$

Proposition 2.3. *Let $c, d \in \mathbb{S}$. If $c \overset{tr}{\sim} d$, then $c \overset{r}{\sim} d$ as $n \to +\infty$.*

Proof. Let $c, d \in \mathbb{S}$ and $c \overset{tr}{\sim} d$ as $n \to +\infty$. According to Proposition 2.1 it follows $c, d \in Tr(R_{\infty,s}) \subsetneq R_{\infty,s}$. Therefore, $\lim_{n\to+\infty} \frac{c_{n+1}}{d_n} = \lim_{n\to+\infty} \frac{d_{n+1}}{c_n} = +\infty$ holds. It follows $\lim_{n\to+\infty} \frac{c_{[\lambda n]}}{d_n} = \lim_{n\to+\infty} \left(\frac{c_{[\lambda n]}}{c_{[\lambda n]-1}} \cdot \frac{c_{[\lambda n]-1}}{c_{[\lambda n]-2}} \cdots \frac{c_{n+1}}{d_n} \right) = +\infty$ for $\lambda > 1$. Analogously it can be proved that $\lim_{n\to+\infty} \frac{d_{[\lambda n]}}{c_n} = +\infty$ holds for each $\lambda > 1$, thus $c \overset{r}{\sim} d$ as $n \to +\infty$ holds. $\square$

Proposition 2.4. *Let $TS = Tr(SV_s)$, $x \in \mathcal{M}_c^{TS}$. Then it holds $x \sim c$ as $n \to +\infty$ ($\sim$ is the relation defined by $\lim_{n\to+\infty} \frac{x_n}{c_n} = 1$). Also, $\mathcal{M}_c^{TS} \subsetneq Tr(SV_s)$ holds.*

Proof. Let $x \in \mathcal{M}_c^{TS}$. Therefore, it holds $c_n \leqslant x_n \leqslant c_{n+1}$ for each $n \in \mathbb{N}$. It means that $1 \leqslant \lim_{n\to+\infty} \frac{x_n}{c_n} \leqslant \lim_{n\to+\infty} \frac{c_{n+1}}{c_n} = 1$, thus $c \sim x$ as $n \to +\infty$. Thus, $\mathcal{M}_c^{TS} \subsetneq [c]_\sim$ ($[c]_\sim$ is the class of strong asymptotic equivalence, generated by the sequence c). It follows $c \in \mathcal{M}_c^{TS}$ holds ($c \in Tr(SV_s)$). So, if $x \in \mathcal{M}_c^{TS}$, then $x \in [c]_\sim$ and thus $x_n = h_n \cdot c_n$, where for the sequence $h = (h_n)$, $n \in \mathbb{N}$, $h \to 1$ holds as $n \to +\infty$. Therefore, it holds $\lim_{n\to+\infty} \frac{x_{n+1}}{x_n} = 1$, which means $x \in Tr(SV_s)$.

The sequence $d = (d_n)$, $n \in \mathbb{N}$, given by $d_n = c_{n+1} + \frac{1}{n}$ as $n \to +\infty$, belongs to the class $Tr(SV_s)$ and it does not belong to the class $\mathcal{M}_c^{TS}$. It holds also $d \in [c]_\sim$. It means that $\mathcal{M}_c^{TS} \subsetneq [c]_\sim \subsetneq Tr(SV_s)$ holds. $\square$

Proposition 2.5. *The player* II *has a winning strategy in the game $G_1(\mathcal{M}_c^{TS}, \mathcal{M}_c^{TS})$.*

Proof. Let $m \in \mathbb{N}$. In m^{th} round the player I chooses an element $A^m \in \mathcal{M}_c^{TS}$. Then II chooses an element $y_m \in A^m$, $m \in \mathbb{N}$. It holds $c_m \leqslant y_m \leqslant c_{m+1} \leqslant y_{m+1} \leqslant c_{m+2}$, for $m \in \mathbb{N}$. Therefore, $1 \leqslant \frac{y_{m+1}}{y_m} \leqslant \frac{c_{m+2}}{c_m} = \frac{c_{m+2}}{c_{m+1}} \cdot \frac{c_{m+1}}{c_m}$ and $\lim_{n\to+\infty} \frac{y_{m+1}}{y_m} = 1$ hold. Hence, $y \in Tr(SV_s)$ and it holds $c_m \leqslant y_m \leqslant c_{m+1}$, so $y \in \mathcal{M}_c^{TS}$. $\square$

Corollary 2.1. *The selection principle $S_1(\mathcal{M}_c^{TS}, \mathcal{M}_c^{TS})$ holds.*

Proposition 2.6. *The player* II *has a winning strategy in the game $G_{\alpha_2}(\mathcal{M}_c^{TS}, \mathcal{M}_c^{TS})$.*

Proof. (m^{th} round, $m \geqslant 1$) Take a sequence $p_1 < p_2 < \cdots$ of prime numbers. In m^{th} round the player I chooses the sequence $A^m \in \mathcal{M}_c^{TS}$ and the player II chooses a subsequence $A^{k_m(n)}$ of the sequence A^m, so that $k_m(n) = p_m^n$ for $n \in \mathbb{N}$. Consider the set $Y = \bigcup_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} A^{k_m(n)}$ of positive real numbers. We can consider this set as the subsequence of the sequence $y = (y_i)$, $i \in \mathbb{N}$, given by

$$y_i = \begin{cases} A^{k_m(n)}, & \text{if } i = k_m(n) \text{ for some } m, n \in \mathbb{N}, \\ c_i, & \text{otherwise.} \end{cases}$$

By the construction of the sequence y, we have that $y \in \mathbb{S}$, $y \sim c$ as $i \to +\infty$, $c_i \leqslant y_i \leqslant c_{i+1}$ for $i \in \mathbb{N}$. Therefore, $y \in \mathcal{M}_c^{TS}$ Also, $y \cap A^m$ has infinitely many elements for each $m \in \mathbb{N}$. This means that II wins the play $A^1, A^{k_1(n)}; A^2, A^{k_2(n)}, \ldots$, i.e., II has a winning strategy in the game $G_{\alpha_2}(\mathcal{M}_c^{TS}, \mathcal{M}_c^{TS})$. □

Corollary 2.2. *The selection principle $\alpha_2(\mathcal{M}_c^{TS}, \mathcal{M}_c^{TS})$ holds.*

Consider now an important subclass of $R_{\infty,s}$.

Let $c \in R_{\infty,s}$. Therefore, it holds $\underline{\lim}_{n\to+\infty} \frac{c_{n+1}}{c_n} = A \geqslant 1$. It follows from (1.1), because $\frac{c_{[\lambda n]}}{c_n} = \frac{c_{[\lambda n]}}{c_{[\lambda n]-1}} \cdots \frac{c_{n+1}}{c_n}$ holds for $n \in \mathbb{N}$ large enough. On the right side there are $[\lambda n] - n$, $n \in \mathbb{N}$, factors which tend to $+\infty$ as $n \to +\infty$.

The class of rapidly varying sequences which satisfy the relation $\lim_{n\to+\infty} \frac{c_{n+1}}{c_n} = A > 1$, $A \in \mathbb{R}$, we will denote by $R_{\infty,s}^{TR}$ and the class of rapidly varying sequences which satisfy the relation $\lim_{n\to+\infty} \frac{c_{n+1}}{c_n} = 1$ by $R_{\infty,s}^{TS}$. We see that

$$R_{\infty,s}^{TR} \cup R_{\infty,s}^{TS} \subsetneq R_{\infty,s}, \quad R_{\infty,s}^{TS} \subsetneq Tr(SV_s) \quad \text{and} \quad R_{\infty,s}^{TR} \subsetneq Tr(RV_s),$$

where $Tr(RV_s)$ is the class of translationally regularly varying sequences in the sense of Karamata (see, e.g., [5]).

Example 2.1. The sequence $(c_n) = (e^n)$, $n \in \mathbb{N}$, is an element of the class $R_{\infty,s}^{TR}$, and the sequence $(d_n) = (e^{\sqrt{n}})$, $n \in \mathbb{N}$, is an element of the class $R_{\infty,s}^{TS}$.

Proposition 2.7. *Let $TRV = R_{\infty,s}^{TR}$, $x = (x_n)$, $n \in \mathbb{N}$, and $x \in \mathcal{M}_c^{TRV}$. Then $x_n \asymp c_n$ as $n \to +\infty$ ($\asymp$ is the relation defined by $0 < \liminf_{n\to+\infty} \frac{x_n}{c_n} \leqslant \limsup \frac{x_n}{c_n} < +\infty$). Also, $\mathcal{M}_c^{TRV} \subsetneq R_{\infty,s}$.*

Proof. Let $c \in R_{\infty,s}^{TR} = TRV$, $c \in \mathcal{M}_c^{TRV}$ and for the sequence x it holds $c_n \leqslant x_n \leqslant c_{n+1}$ for $n \in \mathbb{N}$. It means that for some $A \in \mathbb{R}$, it holds

$$1 \leqslant \underline{\lim}_{n\to+\infty} \frac{x_n}{c_n} \leqslant \overline{\lim}_{n\to+\infty} \frac{x_n}{c_n} \leqslant \lim_{n\to+\infty} \frac{c_{n+1}}{c_n} = A < +\infty.$$

Hence, $c \asymp x$ as $n \to +\infty$. Thus, $\mathcal{M}_c^{TRV} \subsetneq [c]_\asymp$ ($[c]_\asymp$ is the class of weak asymptotic equivalence generated by the sequence c). It holds that $c \in \mathcal{M}_c^{TRV}$, $c \in R_{\infty,s}$. If $x \in \mathcal{M}_c^{TRV}$, then $x \in [c]_\asymp$ and $x_n = h_n \cdot c_n$, and for the sequence $h = (h_n)$, $n \in \mathbb{N}$, it holds $1 \leqslant \underline{\lim}_{n\to+\infty} h_n \leqslant \overline{\lim}_{n\to+\infty} h_n \leqslant A < +\infty$. Thus, for $\lambda > 1$,

$$\underline{\lim}_{n\to+\infty} \frac{x_{[\lambda n]}}{x_n} \geqslant \underline{\lim}_{n\to+\infty} \frac{h_{[\lambda n]}}{h_n} \cdot \underline{\lim}_{n\to+\infty} \frac{c_{[\lambda n]}}{c_n} = \frac{1}{A} \cdot (+\infty) = +\infty$$

holds. The last means that $x \in R_{\infty,s}$ so $\mathcal{M}_c^{TRV} \subsetneq \{c\}_\asymp \subsetneq R_{\infty,s}$. □

Proposition 2.8. *The player* II *has a winning strategy in the game $G_1(\mathcal{M}_c^{TRV}, \mathcal{M}_c^{TRV})$.*

Proof. Let $m \in \mathbb{N}$. In m^{th} round I chooses an element $A^m \in \mathcal{M}_c^{TRV}$. II chooses an element $y_m \in A^m$, $m \in \mathbb{N}$. Thus, we get the sequence (y_m). Therefore, for each $m \in \mathbb{N}$, $c_m \leqslant y_m \leqslant c_{m+1} \leqslant y_{m+1} \leqslant c_{m+2}$, so $1 \leqslant \frac{y_{m+1}}{y_m} \leqslant \frac{c_{m+2}}{c_m}$. It follows $1 \leqslant \underline{\lim}_{n\to+\infty} \frac{y_{m+1}}{y_m} \leqslant \overline{\lim}_{n\to+\infty} \frac{y_{m+1}}{y_m} \leqslant \overline{\lim}_{n\to+\infty} \frac{c_{m+2}}{c_{m+1}} \cdot \overline{\lim}_{n\to+\infty} \frac{c_{m+1}}{c_m} = A \cdot A = A^2$ and for each $m \in \mathbb{N}$, $c_m \leqslant y_m \leqslant c_{m+1}$. Hence, $y \in \mathcal{M}_c^{TRV}$. □

Corollary 2.3. *The selection principle* $S_1(\mathcal{M}_c^{TRV}, \mathcal{M}_c^{TRV})$ *holds.*

Proposition 2.9. *The player* II *has a winning strategy in the game* $G_{\alpha_2}(\mathcal{M}_c^{TRV}, \mathcal{M}_c^{TRV})$.

Proof. (m^{th} round, $m \geqslant 1$) Let $p_1 < p_2 < \cdots$ be a sequence of prime numbers. In m^{th} round I chooses the sequence $A^m \in \mathcal{M}_c^{TRV}$, and II chooses a subsequence $A^{k_m(n)}$ of the sequence A^m, so that $k_m(n) = p_m^n$ for $n \in \mathbb{N}$. Consider the set $Y = \bigcup_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} A^{k_m(n)}$ of positive real numbers. This set we can consider as the subsequence of the sequence $y = (y_i)$, $i \in \mathbb{N}$, given by

$$y_i = \begin{cases} A^{k_m(n)}, & \text{if } i = k_m(n) \text{ for some } m, n \in \mathbb{N}, \\ c_i, & \text{otherwise.} \end{cases}$$

By the construction of the sequence y, we have that $y \in \mathbb{S}$, $y_i \asymp c_i$ as $i \to +\infty$, $c_i \leqslant y_i \leqslant c_{i+1}$ for $i \in \mathbb{N}$. Therefore, $y \in \mathcal{M}_c^{TRV}$. Also, $y \cap A^m$ has infinitely many elements for each $m \in \mathbb{N}$. This means that II wins the play $A^1, A^{k_1(n)}$; $A^2, A^{k_2(n)}; \dots; A^m, A^{k_m(n)}; \dots$ In other words, II has a winning strategy in the game $G_{\alpha_2}(\mathcal{M}_c^{TRV}, \mathcal{M}_c^{TRV})$. □

Corollary 2.4. *The selection principle* $\alpha_2(\mathcal{M}_c^{TRV}, \mathcal{M}_c^{TRV})$ *holds.*

Remark 2.1. In Propositions 2.8 and 2.9, and in Corollaries 2.3 and 2.4, improvements of some results from [3] are given.

Remark 2.2. Propositions 2.5, 2.6, Corollaries 2.1 and 2.2 hold also for the class $R_{\infty,s}^{TS} \subsetneq Tr(SV_s)$.

A sequence $x = (x_n) \in \mathbb{S}$ is said to be *logarithmic rapidly varying, with base* 2, if $(\log_2 x_n)$, $n \in \mathbb{N}$, is an element of the class $R_{\infty,s}$ (see, e.g., [6]). The class of all logarithmic rapidly varying sequences is denoted by $L_2(R_{\infty,s})$. It holds $L_2(R_{\infty,s}) \subsetneq R_{\infty,s}$.

Proposition 2.10. *Let* $x, y \in \mathbb{S}_1$ *and* $x \overset{r}{\sim} y$ *as* $n \to +\infty$. *If* $x \in L_2(R_{\infty,s})$ *holds, then* $y \in L_2(R_{\infty,s})$.

Proof. Let sequences $x, y \in \mathbb{S}_1$ be given, and let the sequence $(\log_2 x_n)$, $n \in \mathbb{N}$, be rapidly varying. Define the functions $f(t) = x_{[t]}$ and $g(t) = y_{[t]}$, $t \geqslant 1$. Therefore, it holds $f(t) \overset{r}{\sim} g(t)$ as $t \to +\infty$, and $\log_2 f(t)$ is rapidly varying function. The functions f and g are also nondecreasing. It holds $\frac{\log_2 g(\lambda t)}{\log_2 g(t)} \geqslant \frac{\log_2(f(\lambda^{\frac{2}{3}} \cdot t))}{\log_2(f(\lambda^{\frac{1}{3}} \cdot t))} \to +\infty$ as $t \to +\infty$, for each $\lambda > 1$. For t large enough, $g(t) < f(\lambda^{\frac{1}{3}} \cdot t)$ and $f(\lambda^{\frac{2}{3}} \cdot t) < g(\lambda t)$ hold for $\lambda > 1$. Therefore, $\log_2 g(t) = h(t)$, $t \geqslant 1$, belongs to the class $R_{\infty,f}$ and hence $(\log_2 y_n) \in R_{\infty,s}$. □

Corollary 2.5. *Proposition* 2.10 *holds when* $x_n \overset{tr}{\sim} y_n$ *as* $n \to +\infty$.

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