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FURTHER IMPROVEMENTS OF HERMITE-HADAMARD INTEGRAL INEQUALITY

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ABSTRACT. We give here improvements of Hermite-Hadamard inequality by an arbitrary mean value. In particular, improvements involving well known classes of quasi-arithmetic, integral and Lagrange means are considered.

1. INTRODUCTION

A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on an non-empty interval I if the inequality

$$f(px + qy) \le pf(x) + qf(y) \tag{1.1}$$

holds for all $x, y \in I$ and all non-negative *weights* $p, q; p + q = 1$.

If the inequality (1.1) reverses, then f is said to be concave on I [1].

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on an interval I and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}. \tag{1.2}$$

This double inequality is well known in the literature as Hermite-Hadamard (HH) integral inequality for convex functions. See, for example, [3] and references therein.

There is a number of refinements and possible generalizations of HH inequality. Some recent trends can be found in [2] and [5].

If f is concave, both inequalities in (1.2) hold in the reversed direction.

Recall that $M(a, b)$ is a mean on I if the inequality

$$\min\{a, b\} \le M(a, b) \le \max\{a, b\},$$

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holds for each $a, b \in I$.

Most known ordered family of means on $I = \mathbb{R}^+$ is the following family Δ_0 of elementary means,

$$\Delta_0 : \ H \leq G \leq L \leq I \leq A \leq S,$$

where

$$H = H(a,b) =: 2(1/a + 1/b)^{-1}, \quad G = G(a,b) =: \sqrt{ab}, \quad L = L(a,b) =: \frac{b-a}{\log b - \log a},$$

$$I = I(a,b) =: \frac{(b^b/a^a)^{1/(b-a)}}{e}, \quad A = A(a,b) =: \frac{a+b}{2}, \quad S = S(a,b) =: a^{\frac{a}{a+b}} b^{\frac{b}{a+b}},$$

are the harmonic, geometric, logarithmic, identric, arithmetic and Gini mean, respectively.

Most known families of functional means are: quasi-arithmetic mean $\mathcal{A}_f = \mathcal{A}_f(a,b) =: f^{-1}\left(\frac{f(a)+f(b)}{2}\right)$, integral mean $\mathcal{I}_f = \mathcal{I}_f(a,b) =: f^{-1}\left(\frac{1}{b-a}\int_a^b f(t)dt\right)$, and Lagrange mean $\mathcal{L}_f = \mathcal{L}_f(a,b) =: (f')^{-1}\left(\frac{f(b)-f(a)}{b-a}\right)$, where it is supposed that the function f is invertible on I.

Our goal in this paper is to improve the inequality (1.2) by an arbitrary mean $M(a,b)$ defined on I.

2. Results and Proofs

We shall give improvements of this kind for both sides of Hermite-Hadamard inequality. The result for right-hand side follows.

Theorem 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on an interval I and $M = M(a,b)$ be a mean on I. Then*

$$\frac{1}{b-a}\int_a^b f(t)dt \leq \frac{1}{2}f(M) + \frac{1}{2(b-a)}[(M-a)f(a) + (b-M)f(b)]. \tag{2.1}$$

Proof. We shall derive the proof by Hermite-Hadamard inequality itself. Indeed, applying twice the right part of this inequality, we get

$$\frac{1}{M-a}\int_a^M f(t)dt \leq \frac{1}{2}(f(a) + f(M))$$

and

$$\frac{1}{b-M}\int_M^b f(t)dt \leq \frac{1}{2}(f(M) + f(b)).$$

Utilizing mean property $a \leq M(a,b) \leq b$, we obtain

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(t)dt =& \frac{1}{b-a}\int_a^M f(t)dt + \frac{1}{b-a}\int_M^b f(t)dt \\
\leq& \frac{1}{2(b-a)}[(M-a)(f(M)+f(a)) + (b-M)(f(M)+f(b))] \\
=& \frac{1}{2}f(M) + \frac{1}{2(b-a)}[(M-a)f(a) + (b-M)f(b)]. \qquad \square
\end{aligned}$$

The next assertion gives a meaning to the whole paper.

Theorem 2.2. *For any mean M the approximation* (2.1) *is better than original one.*

Proof. We need the following well known assertion.

Lemma 2.1. [4] *If f is convex on I and $s, t \in I$, then the ratio*

$$\frac{f(s)-f(t)}{s-t}$$

is monotone increasing in both variables.

Now, denote

$$F_f(M) = F_f(a, b; M) =: \frac{1}{2} f(M) + \frac{1}{2(b-a)}[(M-a)f(a) + (b-M)f(b)].$$

It could be easily checked that $F_f(M)$ can be written in the form

$$F_f(M) = \frac{f(a)+f(b)}{2} - \frac{M-a}{2}\left[\frac{f(b)-f(a)}{b-a} - \frac{f(M)-f(a)}{M-a}\right].$$

Since $a \le M \le b$, by Lemma 2.1 we get

$$\frac{f(b)-f(a)}{b-a} \ge \frac{f(M)-f(a)}{M-a}.$$

Hence,

$$\frac{1}{b-a}\int_a^b f(t)dt \le F_f(M) \le \frac{f(a)+f(b)}{2}.$$ □

A possible application involving functional means defined above, yields the next improvements of HH inequality.

Theorem 2.3. *Let f be convex and invertible on I. Then for any $a, b \in I$ we have*

$$\frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2} - \frac{1}{2}(\mathcal{A}_f(a,b) - A(a,b))\frac{f(b)-f(a)}{b-a} \tag{2.2}$$

and

$$\frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2} - (\mathcal{I}_f(a,b) - A(a,b))\frac{f(b)-f(a)}{b-a}, \tag{2.3}$$

where $\mathcal{A}$, $\mathcal{I}$ and A denotes quasi-arithmetic, integral and arithmetic means, respectively.

Proof. Note that for $M = \mathcal{A}_f$ we have $f(M) = \frac{f(a)+f(b)}{2}$ and, analogously, for $M = \mathcal{I}_f$, $f(M) = \frac{1}{b-a}\int_a^b f(t)dt$. Putting this in (2.1), after some calculation, the result appears. □

Remark 2.1. There is a natural question which of those two approximations is better. Evidently, the answer depends on the inequality

$$\mathcal{I}_f \gtrless \frac{A + \mathcal{A}_f}{2}. \tag{2.4}$$

An interesting fact is that its counterpart, the inequality

$$\frac{1}{b-a}\int_a^b f(t)dt \le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]$$

is valid for any convex f and represents an improvement of Hermite-Hadamard inequality [4].

Nevertheless, closer examination shows that (2.4) is not true in general. For example, $f(x)=1/x$ gives

$$L(a,b) \gtrless \frac{A(a,b)+H(a,b)}{2}, \tag{2.5}$$

and neither of these inequalities is valid for all $a,b\in\mathbb{R}^+$. Therefore, estimations (2.2) and (2.3) are not comparable.

Anyway, the question which mean M gives best possible approximation of the form (2.1) is answered in the following

Theorem 2.4. *The best possible approximation* (2.1) *is reached by Lagrange mean* $\mathcal{L}_f$.

Proof. Let $M(a,b)=c$, where $c\in(a,b)$ is arbitrary. Then

$$F_f(M)=F_f(c)=\frac{1}{2}f(c)+\frac{1}{2(b-a)}[(c-a)f(a)+(b-c)f(b)].$$

Since $f(c)$ is a convex function in c, the same holds for $F_f(c)$. Therefore, there exists an unique minimum which is given by the equation $F_f'(c)=0$, i.e.,

$$f'(c)=\frac{f(b)-f(a)}{b-a},\quad c=(f')^{-1}\left(\frac{f(b)-f(a)}{b-a}\right)=\mathcal{L}_f(a,b). \qquad \square$$

Remark 2.2. A number of interesting inequalities with means can be obtained from the above assertions. For example, $f(x)=-\log x$, $x\in\mathbb{R}^+$, gives $\mathcal{A}=G$, $\mathcal{I}=I$, $\mathcal{L}=L$. This is left to the readers.

Note that the inequality (2.2) can be generalized by the mean

$$\mathcal{A}_f^{p,q}(a,b)=f^{-1}(pf(a)+qf(b)),$$

where p and q are arbitrary weights.

Theorem 2.5. *Let f be convex and invertible on I. Then for any $a,b\in I$ we have*

$$\frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}-\frac{1}{2}(\mathcal{A}_f^{p,q}(a,b)-A^{p,q}(a,b))\frac{f(b)-f(a)}{b-a}, \tag{2.6}$$

where $A^{p,q}(a,b)=pa+qb$ is the weighted arithmetic mean.

As an illustration we give a new inequality between the difference and the ratio of weighted arithmetic and geometric means.

Theorem 2.6. *For any $a, b \in \mathbb{R}^+$ and arbitrary weights p and q, we have*

$$0 \leq pa + qb - a^p b^q \leq 2(A(a,b) - L(a,b))$$

and

$$1 \leq \frac{pa+qb}{a^p b^q} \leq \left(\frac{I(a,b)}{G(a,b)}\right)^2.$$

As a consequence we get the inequality

$$I \geq \sqrt{HS}.$$

Proof. Let $f(x) = -\log x$. Then $\mathcal{A}_f^{p,q}(a,b) = G^{p,q}(a,b) = a^p b^q$ and (2.6) gives

$$-\log I(a,b) \leq -\log G(a,b) + \frac{1}{2L(a,b)}(G^{p,q}(a,b) - A^{p,q}(a,b)).$$

Now, the identity

$$\frac{A}{L} - \log\frac{I}{G} = 1,$$

yields the result. The left-hand side of this inequality is obvious.

For the second inequality, let $f(x) = e^x$. Then $\mathcal{A}_f^{p,q}(a,b) = \log(pe^a + qe^b)$ and the relation (2.6) gives

$$\frac{e^b - e^a}{b-a} \leq \frac{e^a + e^b}{2} - \frac{1}{2}(\log(pe^a + qe^b) - (pa+qb))\frac{e^b - e^a}{b-a}.$$

Now, by changing variables $a \to \log a, b \to \log b$, we get

$$L(a,b) \leq A(a,b) - \frac{1}{2}(\log A^{p,q}(a,b) - \log G^{p,q}(a,b))L(a,b),$$

that is,

$$\log\frac{A^{p,q}(a,b)}{G^{p,q}(a,b)} \leq 2\left(\frac{A(a,b)}{L(a,b)} - 1\right) = 2\log\frac{I(a,b)}{G(a,b)},$$

and the proof is done.

Finally, putting

$$p = \frac{b}{a+b}, \quad q = \frac{a}{a+b},$$

we obtain

$$pa + qb = \frac{2ab}{a+b} = H(a,b),$$

and

$$a^p b^q = a^{\frac{b}{a+b}} b^{\frac{a}{a+b}} = \frac{G^2(a,b)}{S(a,b)}.$$

Therefore, applying the last inequality, we get $H(a,b)S(a,b) \leq I^2(a,b)$. □

In an analogous way, we obtain improvement of the left-hand side of HH inequality.

Theorem 2.7. *Let f be a convex function on an interval I and $N = N(a,b)$ be a mean on I. Then*

$$\frac{1}{b-a}\int_a^b f(t)dt \geq \frac{1}{b-a}\left[(N-a)f\left(\frac{a+N}{2}\right)+(b-N)f\left(\frac{N+b}{2}\right)\right]. \tag{2.7}$$

Proof. Applying the left part of Hermite-Hadamard inequality, we get

$$\frac{1}{N-a}\int_a^N f(t)dt \geq f\left(\frac{a+N}{2}\right),$$

and

$$\frac{1}{b-N}\int_N^b f(t)dt \geq f\left(\frac{N+b}{2}\right).$$

Hence,

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(t)dt =&\frac{1}{b-a}\int_a^N f(t)dt + \frac{1}{b-a}\int_N^b f(t)dt\\ \geq&\frac{1}{b-a}\left[(N-a)f\left(\frac{a+N}{2}\right)+(b-N)f\left(\frac{N+b}{2}\right)\right].\end{aligned}$$ □

Theorem 2.8. *For any mean N the approximation* (2.7) *is better than the original one.*

Proof. Denote

$$G_f(N) = G_f(a,b;N) =: \frac{1}{b-a}\left[(N-a)f\left(\frac{a+N}{2}\right)+(b-N)f\left(\frac{N+b}{2}\right)\right].$$

Since f is a convex function, applying its definition (1.1) with

$$p=\frac{N-a}{b-a},\quad q=\frac{b-N}{b-a},\quad x=\frac{a+N}{2},\quad y=\frac{N+b}{2},$$

we get

$$\begin{aligned}G_f(N) =&\frac{1}{b-a}\left[(N-a)f\left(\frac{a+N}{2}\right)+(b-N)f\left(\frac{N+b}{2}\right)\right]\\ \geq& f\left(\frac{N-a}{b-a}\frac{a+N}{2}+\frac{b-N}{b-a}\frac{N+b}{2}\right) = f\left(\frac{a+b}{2}\right).\end{aligned}$$

Hence,

$$\frac{1}{b-a}\int_a^b f(t)dt \geq G_f(N) \geq f\left(\frac{a+b}{2}\right).$$ □

Problem of best possible approximation of the form (2.7) is somewhat ambiguous. For example, for the function $f(x) = 1/x$ best possible choice is given by $N = G$ and this yields the inequality

$$L(a,b) \leq \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 = A_{1/2}(a,b).$$

In general case we propose the following.

Open question. *Determine the mean* $N^* = N_f^*(a, b)$ *which gives best possible approximation of the form* (2.7).

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