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S SPECTRAL THEORY OF MULTIVALUED LINEAR OPERATOR IN BANACH SPACES

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Abstract. In this paper, we begin with the definition of the S-resolvent set of a linear relation. Throughout this paper, X will denote a normed linear space over the complex field $\mathbb{C}$. Operator S plays the role of a transition multivalued linear operator from X. It is the main goal of the present note to study the basic spectral properties of T linked to the transition multivalued linear operator S.

1. Introduction

Let $T, S : X \to Y$. For all $\lambda \in \mathbb{C}$, the map $P(\lambda) := T + \lambda S$ is called a linear bundle. It is know that many problems of mathematical physics (for example, quantum theory...) are reduced to some study of certain reversibility conditions of operators $P(\lambda)$. In light of this, this paper shares the same concern of these studies. In fact, it highlights some recent mathematical developments, including the spectral theory of multivalued linear operator bundle. The study of the spectral theory of linear operators, including the study of S-essential spectrum, has recently been the core subject of several works.

Most importantly, A. Jeribi [13] has treated the notion of S-essential spectra in a comprehensive account of the known definitions of essential spectra (see Chapter 9). In [2] T. Alvarez, A. Ammar and A. Jeribi have characterized some S-essential spectra of a closed linear relation in terms of certain linear relations of semi-Fredholm type. In [3], they studied the decomposition of Frobenius-Schur in order to determine the essential spectrum of a matrix multivalued linear operator.

Most importantly in [4] A. Ammar, A. Jeribi and B. Saadaoui considered a 2×2 block multivalued linear operator matrices and described its essential pseudospectrum.

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Among the most important works on the spectral theory of block linear relation matrix, we mention [5,6] where the authors proposed the development of the essential spectra of a 2×2 block linear relation matrix. Recently, B. Saadaoui [15] continued this study to investigate the (P,Q)-Outer Generalized inverses and their stability of pseudo spectrum. In [7, 16] B. Saadaoui generalized some known results in the condition S-spectrum of a compact operator in a right quaternionic Hilbert space. Several problems can be described by systems of mixed order linear differential equations in mathematical physics. The localization of the essential spectra gave substantial physical information. The findings acquired in this memorandum provides through the conformable ones in [8], and they are robustly affined to notions from different spectral problems in applied sciences (for concerning works see, for sample, [11,12,17]). In specific, the research of different sort of degenerate equations on Banach spaces could be done using the connotations and outcomes gained in the current note, cf. [11]. Paradigms to uncover the applicability of our theoretical processing will be extended in [5]. More punctually, the main outcomes of this matter will be exercised to research different perturbations of multivalued linear operator in Banach spaces in the spirit of the results obtained in [9, 14].

The purpose of this paper is to extend the results of [10] to the case of multivalued linear operators. It consists of three sections. In Section 2, we present some basic notations and results connected to the main body of the work. In Section 3 we give a characterization of the S-resolvent set of a closed multivalued linear operator T.

2. Preliminary Results

This section contains some definitions and auxiliary results which will be needed in the rest of this paper. We adhere with the notation and the terminology of the book [10]. Let X, Y and Z be infinite dimensional vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. T multivalued linear operator or simply a linear relation $T : X \to Y$ is a mapping from a subspace $\mathcal{D}(T)$ of X, called the domain of T, into $P(Y)\backslash\{\emptyset\}$ (the collection of non-empty subsets of Y) such that $T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$ for all non-zero scalars $\alpha, \beta \in \mathbb{K}$ and $x_1, x_2 \in \mathcal{D}(T)$. If T maps the points of its domain to singletons, then T is said to be a single valued linear operator or simply an operator, which is equivalent to $T(0) = \{0\}$. We denote by $\mathcal{LR}(X, Y)$ the class of linear relations everywhere defined and we write $\mathcal{LR}(X) := \mathcal{LR}(X, X)$. $T \in \mathcal{LR}(X, Y)$ is uniquely determined by its graph $G(T)$, which is defined by:

$$G(T) := \Big\{(x, y) \in X \times Y : x \in \mathcal{D}(T), y \in Tx\Big\},$$

so that we can identify T with $G(T)$. The closure of T, denoted by $\overline{T}$, is the linear relation defined by

$$G(\overline{T}) := \overline{G(T)}.$$

We denote by $\mathcal{CR}(X, Y)$ the class of all closed linear relations from X to Y. If $X = Y$, we take $\mathcal{CR}(X, X) := \mathcal{CR}(X)$.

The inverse of T is a linear relation T^{-1} given by:

$$G(T^{-1}) := \Big\{(y,x) \in X \times Y : (x,y) \in G(T)\Big\}.$$

If $G(T)$ is closed, then T is said to be closed. We design by $R(T) = T(\mathcal{D}(T))$ the range of T. T is called surjective if $R(T) = Y$. The subspace $N(T) := T^{-1}(0)$ is called the null space of T. T is called injective if $N(T) = \{0\}$, that is, if T^{-1} is a single valued linear operator.

Notice that when $x \in \mathcal{D}(T)$, $y \in Tx$ if and only if $Tx = y + T(0)$.

For $T, S \in \mathcal{LR}(X,Y)$, the notation $T \subset S$ means that $G(T) \subset G(S)$. The linear relation $T + S$ is defined by:

$$G(T+S) := \Big\{(x,y) \in X \times Y : y = u + v \text{ with } (x,u) \in G(T), (x,v) \in G(S)\Big\}.$$

Let $T \in \mathcal{LR}(X,Y)$ and $S \in \mathcal{LR}(Y,Z)$ where $R(T) \cap \mathcal{D}(S) \neq \emptyset$. The product of ST is defined by:

$$G(ST) := \Big\{(x,z) \in X \times Z : (x,u) \in G(T) \text{ and } (u,z) \in G(S) \text{ for some } u \in Y\Big\}.$$

Let Q_T denote the quotient map from X onto $X/\overline{T(0)}$. We shall denote $Q_{\overline{T(0)}}$ by Q_T. Clearly, $Q_T T$ is a single valued operator and the norm of T is defined by $\|T\| := \|Q_T T\|$. We say that T is continuous if for each neighborhood V in $R(T)$, $T^{-1}(V)$ is a neighborhood in $\mathcal{D}(T)$ (equivalently $\|T\| < +\infty$); bounded if it is continuous with $\mathcal{D}(T) = X$; open if T^{-1} is continuous equivalently $\gamma(T) > 0$ where $\gamma(T)$ is the minimum modulus of T defined by

$$\gamma(T) := \sup \Big\{\lambda \geq 0 : \lambda d(x, N(T)) \leq \|Tx\| \text{ for } x \in \mathcal{D}(T)\Big\},$$

where $d(x, N(T))$ is the distance between x and $N(T)$. Continuous linear relations defined everywhere on X are referred to as bounded linear relations. The class of such relations is denoted by $\mathcal{BR}(X,Y)$. If $X = Y$, we take $\mathcal{BR}(X,X) := \mathcal{BR}(X)$.

If M and N are subspaces of X and of the dual space X', respectively, then

$$M^{\perp} := \Big\{x' \in X' : x'(x) = 0 \text{ for all } x \in M\Big\}$$

and

$$N^{\top} := \Big\{x \in X : x'(x) = 0 \text{ for all } x' \in N\Big\}.$$

The conjugate of $T \in \mathcal{LR}(X,Y)$ is the linear relation $T^{'}$ defined by

$$G(T^{'}) := G(-T^{-1})^{\perp} \subset Y^{'} \times X^{'},$$

so that $(y^{'}, x^{'}) \in G(T^{'})$ if and only if $y^{'}(y) = x^{'}(x)$ for all $(x,y) \in G(T)$.

Lemma 2.1 ([10])**.** *Let X and Y be two vector spaces and let $T \in L\mathcal{R}(X,Y)$. Then,*

(a) *$\mathcal{D}(T^{-1}) = R(T)$; $\mathcal{D}(T) = R(T^{-1})$;*
(b) *T is injective if and only if $T^{-1}T = I_{\mathcal{D}(T)}$;*
(c) *T is single valued if and only if $T(0) = \{0\}$;*
(d) *$TT^{-1}y = y + T(0)$, $y \in R(T)$, and $T^{-1}Tx = x + T^{-1}(0)$.*

Proposition 2.1 ([10]). *Let $R, S, T \in L\mathcal{R}(X)$. Then,*

(a) *$(R+S)T \subset RT + ST$ with equality if T is single valued;*
(b) *$T(R+S)$ is an extension of $TR+TS$ and $TR+TS = T(R+S)$ if $\mathcal{D}(T)$ is the whole space.*

Lemma 2.2. *Let X and Y be two vector spaces. Let $T, S \in \mathcal{LR}(X,Y)$.*

(a) ([10, Exercise I.2.14 (b)]) *If $\mathcal{D}(T) = \mathcal{D}(S)$ and $T(0) = S(0)$, then $T = S$ or the graphs of T and S are incomparable.*
(b) ([10, Definition II.5.1 (2) and, Proposition II.5.3]) *T^{-1} is closed if, and only if, T is closed if, and only if, $Q_T T$ is closed single valued and $T(0)$ is a closed space.*
(c) ([10, Definition II.5.1 (6)]) *If T is continuous, $\mathcal{D}(T)$ and $T(0)$ are closed, then T is closed.*

Lemma 2.3. ([2, Lemma 2.2 (ii)] *and* [1, Lemma 3.1])

(a) *Let $S, T \in \mathcal{LR}(X,Y)$. If $S(0) \subset T(0)$ and $\mathcal{D}(T) \subset \mathcal{D}(S)$, then $Q_{T+S} = Q_T$ and $T - S + S = T$.*
(b) *If $T \in \mathcal{LR}(X,Y)$ and $S \in \mathcal{LR}(Y,Z)$ are closed with $\alpha(S) < +\infty$ and $R(S)$ closed, then $ST \in \mathcal{CR}(X,Z)$.*
(c) *If $T(0) \subset N(S)$ or $T(0) \subset N(R)$, then $(R+S)T = RT + ST$.*

Lemma 2.4. ([10, Exercise II.5.18]) *Let $T \in \mathcal{BR}(X,Y)$ be single valued and $S \in \mathcal{CR}(Y,Z)$. Then, $ST \in \mathcal{CR}(X,Z)$.*

Lemma 2.5. ([10, Theorem II.3.11 and Corollary III.7.7]) *Let $S, T \in \mathcal{LR}(Y,Z)$.*

(a) *If $N(S) \subset R(T)$, then $\gamma(ST) \geq \gamma(S)\gamma(T)$.*
(b) *Let T be open and injective with dense range. Then, for any relation S such that $S(0) \subset \overline{T(0)}$, $\mathcal{D}(S) \supset \mathcal{D}(T)$ and $\|S\| < \gamma(T)$, we have $T+S$ is open and injective with dense range.*

Proposition 2.2. ([10, Proposition VI.5.2]) *Let X be complete, and let $S, T \in \mathcal{CR}(X)$ be bijective. Then, ST has the same properties.*

Theorem 2.1. ([10, Theorem III.4.2]) *Let $T \in \mathcal{LR}(X,Y)$ be closed. Then,*

(a) *T is continuous if and only if $\mathcal{D}(T)$ is closed;*
(b) *T is open if and only if $R(T)$ is closed.*

3. S-spectral Theory of Multivalued Linear Operators in Banach Spaces

Definition 3.1. Let $S, T \in \mathcal{LR}(X)$ with $\|S\| \neq 0$. We define the S-resolvent set of T by:

$$\rho_S(T) := \Big\{\lambda \in \mathbb{C} : \lambda S - T \text{ is injective, open with dense range}\Big\}.$$

The spectra of T is the set $\sigma_S(T) := \mathbb{C} \backslash \rho_S(T)$.

Lemma 3.1. *Let $S \in \mathcal{LR}(X)$ be continuous. For $T \in \mathcal{CR}(X)$ such that $\|S\| \neq 0$, $S(0) \subset T(0)$ and $\overline{D(T)} \subset \mathcal{D}(S)$, we have that instead of we define the S-resolvent set of T by:*

$$\rho_S(T) := \Big\{\lambda \in \mathbb{C} : (\lambda S - T)^{-1} \text{ is single valued and everywhere defined}\Big\}.$$

Proof. We remark that if X is Banach, then the Lemma 3.1 is true.

Indeed, let $\lambda \in \rho_S(T)$. Then, $\lambda S - T$ is closed. Indeed, $\lambda S - T$ is closed if, and only if, $Q_{S-T}(\lambda S - T)$ is closed and $(\lambda S - T)(0)$ is closed [2, Proposition II.5.3]. But, $(\lambda S - T)(0) = \lambda S(0) - T(0) = T(0)$ closed (as $S(0) \subset T(0)$ and T closed) and $Q_{S-T}(\lambda S - T) = \lambda Q_T S - Q_T T = \lambda Q_A Q_S S - Q_T T$ (where $A := T(0)/\overline{S(0)}$ by virtue of [2, Lemma IV.5.2]) which is closed by [2, Exercise II.5.16]. Now, the result follows immediately from Theorem 2.1 and Lemma 2.3. Let $\lambda \in \rho_S(T)$, then $\lambda S - T$ is injective, open with dense range . Furthermore, $N(\lambda S - T) = (\lambda S - T)^{-1}(0) = \{0\}$, then $(\lambda S - T)^{-1}$ is single valued.

On the other hand, $\lambda S - T$ is open, then applying Theorem 2.1 we obtain $R(\lambda S - T)$ is closed. In short, $\overline{R(\lambda S - T)} = R(\lambda S - T) = X$. □

Example 3.1. We see the following examples, where $\sigma_S(T)$ can be discrete or the whole complex plane.

(a) Let $T = \begin{pmatrix} 3 & 2 \\ 0 & 5 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then, $\sigma_S(T) = \{3\}$.

(b) Let $T = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$. Then, $\sigma_S(T) = \mathbb{C}$.

(c) Let $T = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then, $\sigma_S(T) = \emptyset$.

Remark 3.1. (a) It follows immediately from Definition 3.1 that, for $S = I$ we have

$$\rho_I(T) := \Big\{\lambda \in \mathbb{C} : \lambda - T \text{ is injective, open with dense range}\Big\}.$$

(b) If X is a finite dimension space and S is an invertible operator such that $\sigma(S)\backslash\{1\}$ is not empty, then $\sigma_S(S) = \{1\}$, which implies that $\sigma(S) \neq \sigma_S(S)$.

Theorem 3.1. *Let $T \in \mathcal{CR}(X)$ and $S \in \mathcal{LR}(X)$ be continuous such that $S(0) \subset T(0)$, $\|S\| \neq 0$, $S \neq T$ and $\mathcal{D}(T) \subset \mathcal{D}(S)$. Let $\lambda \in \rho_S(T)$, and let $\mu \in \mathbb{C}$ such that*

$$|\mu - \lambda| < \frac{\gamma(\lambda S - T)}{\|S\|}.$$

Then, $\mu \in \rho_S(T)$ and

$$\|(\mu S - T)^{-1}\| \leq \frac{\|(\lambda S - T)^{-1}\|}{1 - |\mu - \lambda| \cdot \|(\lambda S - T)^{-1}\| \cdot \|S\|}.$$

In particular, $\rho_S(T)$ is open.

Proof. Let $\lambda \in \rho_S(T)$, and let $(x,y) \in G(T)$. Since $(\lambda S - T)^{-1}$ is a bounded operator, it follows from the identity $(\lambda S - T)^{-1}(\lambda S - T)x = x$ that

$$\|(\lambda S - T)^{-1}\| \cdot \|(\lambda S - T)x\| \geq \|x\|.$$

For each $\mu \in \mathbb{C}$ one has

$$\begin{aligned}\|(\mu S - T)x\| =& \|(\lambda S - T)x - (\mu - \lambda)Sx\| \\ \geq& \|(\lambda S - T)x\| - |\mu - \lambda| \cdot \|Sx\| \\ \geq& \|(\lambda S - T)x\| - |\mu - \lambda| \cdot \|S\| \cdot \|x\|,\end{aligned}$$

which leads to

$$\begin{aligned}\|(\lambda S - T)^{-1}\| \cdot \|(\mu S - T)x\| \geq& \|(\lambda S - T)^{-1}\| \cdot \|(\lambda S - T)x\| \\ &- |\mu - \lambda| \cdot \|(\lambda S - T)^{-1}\| \cdot \|S\| \cdot \|x\| \\ \geq& \|x\| - |\mu - \lambda| \cdot \|(\lambda S - T)^{-1}\| \cdot \|S\| \cdot \|x\| \\ =& (1 - |\mu - \lambda| \cdot \|(\lambda S - T)^{-1}\| \cdot \|S\|) \cdot \|x\|.\end{aligned}$$

This inequality shows that $(\mu S - T)^{-1}$ is a bounded operator. This leads to

$$\|(\mu S - T)^{-1}\| \leq \frac{\|(\lambda S - T)^{-1}\|}{1 - |\mu - \lambda| \cdot \|(\lambda S - T)^{-1}\| \cdot \|S\|}.$$

□

Lemma 3.2. *Let $T \in \mathcal{CR}(X)$ and $S \in \mathcal{LR}(X)$ be continuous such that $S(0) \subset T(0)$, $\|S\| \neq 0$, $S \neq T$ and $\mathcal{D}(T) \subset \mathcal{D}(S)$. If $\lambda \in \rho_S(T)$ and $|\mu - \lambda| < \frac{\gamma(\lambda S - T)}{\|S\|}$, then $\rho_S(T)$ is an open set of $\mathbb{C}$.*

Proof. Since $\lambda \in \rho_S(T)$, one has $\mu \in \rho_S(T)$ and $R(\lambda S - T) = X$, then $\gamma(\lambda S - T) > 0$. Let $|\mu - \lambda| < \frac{\gamma(\lambda S - T)}{\|S\|}$. Then by Lemma 2.3 1. and Lemma 2.5 2., we have $\mu S - T = (\mu - \lambda)S + (\lambda S - T)$ is injective and open with dense range. Therefore, $\mu \in \rho_S(T)$. In particular, $\rho_S(T)$ is open. □

Theorem 3.2. *Let X be a Banach space. Let $T \in \mathcal{CR}(X)$ and $S \in \mathcal{BR}(X)$ with $\mathcal{D}(T) = X$ such that $S(0) \subset T(0)$ and $\dim S(0) < +\infty$. If $0 \in \rho_I(S)$, then*

$$\rho_S(T) = \rho_I(S^{-1}T) \cap \rho_I(TS^{-1}).$$

Proof. First of all, it should be mentioned that $\lambda I - S^{-1}T$ is closed. In fact, $\mathcal{D}(S) = X$ is closed, $S(0)$ is closed (since $\dim S(0) < +\infty$) and S is continuous so that by Lemma 2.2 (c) S is closed. Then by Lemma 2.2 (b), S^{-1} is closed. Further $R(S^{-1}) = \mathcal{D}(S) = X$, $\dim N(S^{-1}) = \dim S(0) < +\infty$ and T is closed and thus by Lemma 2.3 (b), $S^{-1}T$ is closed. Now, applying Lemma 2.3, we get $\lambda I - S^{-1}T$ is closed.

Moreover, $\lambda I - TS^{-1}$ is closed. Indeed, since $0 \in \rho_I(S)$, then S^{-1} is single valued (since S is injective). We get S^{-1} is closed, yet $\mathcal{D}(S^{-1}) = R(S) = X$, then S^{-1} is a bounded single valued. Thus, by Lemma 2.4, we have TS^{-1} is closed, which implies by Lemma 2.3, that $\lambda I - TS^{-1}$ is closed.

We can say that $S(0) \subset ST^{-1}(0) = (TS^{-1})^{-1}(0) = N(TS^{-1})$. So, using Lemma 2.3, we infer that $\lambda S - T = (\lambda I - TS^{-1})S$. Let $x \in \mathcal{D}(S^{-1}) = R(S) = X$, then there is $a \in X$ such that $(x, a) \in G(S^{-1})$. This is equivalent to $(a, x) \in G(S)$. Therefore, $(x, x) \in G(SS^{-1})$. We have $I \subset SS^{-1}$ and consequently $\lambda S - T \subset SS^{-1}(\lambda S - T) = S(\lambda I - S^{-1}T)$. So, we get $\lambda S - T \subset S(\lambda I - S^{-1}T)$. It is easy to notice that

$$S(\lambda I - S^{-1}T)(0) = S(\lambda(0) - S^{-1}T(0)) = SS^{-1}T(0).$$

By using Lemma 2.1 (d) we get $SS^{-1}T(0) = T(0) + S(0) = (\lambda S - T)(0)$, additionally $\mathcal{D}(\lambda S - T) = \mathcal{D}(\lambda S) \cap \mathcal{D}(T) = \mathcal{D}(T)$ and

$$\mathcal{D}(S(\lambda I - S^{-1}T)) := \left\{ x \in \mathcal{D}(\lambda I - S^{-1}T) : \begin{array}{c} (\lambda I - S^{-1}T)x \cap \mathcal{D}(S) \\ = (\lambda I - S^{-1}T)x \cap X \neq \emptyset \end{array} \right\}.$$

Equivalently,

$$\begin{aligned} \mathcal{D}(\lambda I - S^{-1}T) =& \mathcal{D}(\lambda) \cap \mathcal{D}(S^{-1}T) = D(S^{-1}T) \\ =& \Big\{ x \in \mathcal{D}(T) : Tx \cap \mathcal{D}(S^{-1}) = Tx \cap R(S) \neq \emptyset \Big\} \\ =& \mathcal{D}(T) = \mathcal{D}(\lambda S - T). \end{aligned}$$

Then, by Lemma 2.2 (a), we infer that $\lambda S - T = S(\lambda I - S^{-1}T)$. Let $\lambda \in \rho_S(T)$ such that $\lambda S - T$ is closed and bijective. First of all, it should be mentioned that $\lambda I - S^{-1}T$ is bijective. Indeed,

$$R(\lambda I - S^{-1}T) = R(S^{-1}(\lambda S - T)) = R(S^{-1}) = \mathcal{D}(S) = X,$$

and let $x \in N(\lambda I - S^{-1}T)$ if and only if

$$x \in \mathcal{D}(S^{-1}T) \text{ and } (\lambda I - S^{-1}T)x = (\lambda I - S^{-1}T)(0)$$

if and only if $S(\lambda I - S^{-1}T)x = S(\lambda I - S^{-1}T)(0)$ if and only if $(\lambda S - T)x = (\lambda S - T)(0)$ if and only if $x = \{0\}$ (since $\lambda S - T$ is injective).

Then, $\lambda I - S^{-1}T$ is bijective. In the same manner we can prove that $\lambda I - TS^{-1}$ is bijective. Accordingly, $\lambda I - TS^{-1}$ and $\lambda I - S^{-1}T$ are bijective and closed, then

$$\rho_S(T) \subset \rho(TS^{-1}) \cap \rho(S^{-1}T). \tag{3.1}$$

Conversely, let $\lambda \in \rho_I(S^{-1}T) \cap \rho_I(TS^{-1})$. Then, $\lambda I - S^{-1}T$ and $\lambda I - TS^{-1}$ are bijective. Hence, S is surjective, then $R(\lambda S - T) = SR(\lambda I - S^{-1}T) = R(S) = X$ and $N(\lambda S - T) \subset N(S^{-1}(\lambda S - T)) = N(\lambda I - S^{-1}T) = \{0\}$. Then,

$$\rho(TS^{-1}) \cap \rho(S^{-1}T) \subset \rho_S(T). \tag{3.2}$$

By (3.1) and (3.2) we conclude that $\rho(TS^{-1}) \cap \rho(S^{-1}T) = \rho_S(T)$. □

Remark 3.2. If T and S are single valued with S is invertible, then

$$\rho_S(T) = \rho(TS^{-1}) = \rho(S^{-1}T).$$

Proposition 3.1. *Let X be a Banach space. Let $T \in \mathcal{CR}(X)$ and $S \in \mathcal{BR}(X)$ be single valued with $\mathcal{D}(T) = X$. If $0 \in \rho_I(S)$, then*

$$\rho_S(T) \backslash \{0\} = \rho_I(S^{-1}T) \backslash \{0\} = \rho_I(TS^{-1}) \backslash \{0\}.$$

Proof. Let $0 \in \rho_I(S^{-1})$, then $N(S^{-1}) = S(0) = \{0\}$ and $R(S^{-1}) = \mathcal{D}(S) = X$. Consequently, S^{-1} is single valued, hence $S^{-1}(0) = N(S) = \{0\}$. Since S^{-1} is closed (since S is closed), $\mathcal{D}(S^{-1}) = R(S) = X$, and S^{-1} is single valued, then S^{-1} is bounded.

Therefore, by Proposition 3.2 we get $\rho_S(T)\backslash\{0\} = \rho_I(S^{-1}T)\backslash\{0\} = \rho_I(TS^{-1})\backslash\{0\}$. □

Definition 3.2. Let $S, T \in \mathcal{LR}(X)$ and $\lambda \in \mathbb{C}$. We call the S-resolvent of T in λ the operator defined by:

$$R_S(\lambda, T) := (\lambda S - T)^{-1}.$$

Theorem 3.3. *Let $T \in \mathcal{CR}(X)$ and let $S \in \mathcal{LR}(X)$ be continuous such that $S(0) \subset T(0)$, $\|S\| \neq 0$, $S \neq T$ and $\mathcal{D}(T) \subset \mathcal{D}(S)$. Let $\lambda, \mu \in \rho_S(T)$. Then, the following hold.*

(a) $R_S(\mu, T) - R_S(\lambda, T) = (\lambda - \mu)R_S(\mu, T)SR_S(\lambda, T)$.

(b) *If $\mu \in \rho_S(T)$ and $|\lambda - \mu| \cdot \|S\| \leq \gamma(\mu S - T)$, then*

$$R_S(\lambda, T) = \sum_{i=0}^{+\infty} (\lambda - \mu)^i R_S(\mu, T)(SR_S(\mu, T))^i.$$

Proof. (a) Let $(x, y) \in G(R_S(\mu, T) - R_S(\lambda, T))$, so that $(x, y_1) \in G(R_S(\mu, T))$ and $(x, y_2) \in G(R_S(\lambda, T))$ with $y = y_1 + y_2$. One has $(y_1, x) \in G(\mu S - T)$ and $(y_2, x) \in G(\lambda S - T) = G(\mu S - T + (\lambda - \mu)S)$. Then, $x \in (\mu S - T + (\lambda - \mu)S)y_2 = (\mu S - T)y_2 + (\lambda - \mu)Sy_2$. By Lemma 2.3 we get $x - (\lambda - \mu)Sy_2 \in (\mu S - T)y_2$. Hence, $(y_2, x - (\lambda - \mu)Sy_2) \in G(\mu S - T)$, so that

$$(y, (\lambda - \mu)Sy_2) = (y_1, x) - (y_2, x - (\lambda - \mu)Sy_2) \in G(\lambda S - T).$$

Implies that $((\lambda - \mu)Sy_2, y) \in G(R_S(\lambda, T))$. By Lemma 2.3 which shows that $(y_2, y) \in (\lambda - \mu)R_S(\lambda, T)S$. Then,

$$(x, y) \in G((\lambda - \mu)R_S(\lambda, T)SR_S(\mu, T)),$$

which leads to $R_S(\mu, T) - R_S(\lambda, T) \subseteq (\lambda - \mu)R_S(\mu, T)SR_S(\lambda, T)$.

Conversely, let $(x, y) \in G((\lambda - \mu)R_S(\mu, T)SR_S(\lambda, T))$. Then, $(x, z) \in G(R_S(\lambda, T))$, $(z, w) \in G((\lambda - \mu)S)$ and $(w, y) \in G(R_S(\mu, T))$ for some $y, z \in X$. It follows from $(x, z) \in G(R_S(\lambda, T))$ that $(z, x) \in G(\lambda S - T)$. So that,

$$x \in (\lambda S - T)z = (\mu S - T)z + (\lambda - \mu)Sz.$$

By using Lemma 2.3 we get $x + (\mu - \lambda)Sz \in (\mu S - T)z$. Hence,

$$(z, x + (\mu - \lambda)Sz) \in G(\mu S - T).$$

Since $w \in (\mu - \lambda)Sz$, then $(z, x - w) \in (z, x + (\mu - \lambda)Sz) \in G(\mu S - T)$. Consequently,

$$(x, z + y) = (x - w, z) + (w, y) \in G(R_S(\mu, T)).$$

Finally,

$$(x, y) = (x, z + y) - (x, z) \in G(R_S(\mu, T)) - G(R_S(\lambda, T)) \subset G(R_S(\mu, T) - R_S(\lambda, T)).$$

We deduce that,

$$(\lambda-\mu)R_S(\mu,T)SR_S(\lambda,T)\subseteq R_S(\mu,T)-R_S(\lambda,T).$$

So, $(\lambda-\mu)R_S(\mu,T)SR_S(\lambda,T)=R_S(\mu,T)-R_S(\lambda,T)$.

(b) Since $(\lambda-\mu)R_S(\mu,T)SR_S(\lambda,T)=R_S(\mu,T)-R_S(\lambda,T)$, then

$$R_S(\lambda,T)=\sum_{i=0}^{n}(\lambda-\mu)^iR_S(\mu,T)(SR_S(\mu))^i+(\lambda-\mu)^{n+1}R_S(\lambda,T)(SR_S(\mu,T))^{n+1}.$$

From the estimation

$$\|(\lambda-\mu)^{n+1}R_S(\lambda,T)(SR_S(\mu,T))^{n+1}\|\leq\|R_S(\lambda,T)\|(|\lambda-\mu|\cdot\|R_S(\mu,T)\|\cdot\|S\|)^{n+1},$$

and the inequality $|\lambda-\mu|\cdot\|R_S(\mu,T)\|\cdot\|S\|<1$, it follows that the rest term

$$(\lambda-\mu)^{n+1}R_S(\lambda,T)(SR_S(\mu,T))^{n+1}$$

tends to 0 as $n\to+\infty$. This completes the proof. □

Theorem 3.4. *Let $T\in\mathcal{CR}(X)$ and $S\in\mathcal{LR}(X)$, which is continuous, satisfy $S(0)\subset T(0)$, $\|S\|\neq0$, $S\neq T$ and $\mathcal{D}(T)\subset\mathcal{D}(S)$. For all $\lambda\in\rho_S(T)$, we have $R_S(\lambda,T)$ commutes with S. Then,*

$$\lim_{\mu\to\lambda}(\mu-\lambda)^{-1}(R_S(\mu,T)-R_S(\lambda,T))=-R_S(\lambda,T)SR_S(\lambda,T).$$

Proof. Let $\lambda\in\rho_S(T)$. We have from Theorem 3.3 that

$$\begin{aligned}&(\mu-\lambda)^{-1}(R_S(\mu,T)-R_S(\lambda,T))+R_S(\lambda,T)SR_S(\lambda,T)\\=&-R_S(\mu,T)SR_S(\lambda,T)+R_S(\lambda,T)SR_S(\lambda,T)\\=&(R_S(\lambda,T)S-R_S(\mu,T)S)R_S(\lambda,T).\end{aligned}$$

Hence, we can write

$$\begin{aligned}&\|(\mu-\lambda)^{-1}(R_S(\mu,T)-R_S(\lambda,T)+R_S(\lambda,T)SR_S(\lambda,T)\|\\\leq&\|R_S(\lambda,T)-R_S(\mu,T)\|\cdot\|S\|\cdot\|R_S(\lambda,T)\|.\end{aligned}$$

Now we take $|\mu-\lambda|\cdot\|S\|<\|R_S(\lambda,T)\|^{-1}$. Then,

$$\sum_{n\geq0}|\mu-\lambda|^n\cdot\|S\|^n\cdot\|R_S(\lambda,T)\|^n<+\infty,$$

with $(I-(\lambda-\mu)SR_S(\lambda,T))^{-1}=\sum_{n\geq0}(\mu-\lambda)^nS^nR_S(\lambda,T)^n$. Since $R_S(\lambda,T)$ and S commute, then for $x\in\mathcal{D}(T)$ we have

$$(\lambda-\mu)R_S(\lambda,T)^{-1}SR_S(\lambda,T)x=(\lambda-\mu)R_S(\lambda,T)^{-1}R_S(\lambda,T)Sx.$$

By Lemme 2.1 4., we get

$$\begin{aligned}(\lambda-\mu)R_S(\lambda,T)^{-1}SR_S(\lambda,T)x=&(\lambda-\mu)Sx+T(0)\\=&R_S(\lambda,T)^{-1}x-R_S(\mu,T)^{-1}.\end{aligned}$$

From the above we find

$$\begin{aligned}
&R_S(\lambda,T)^{-1}(I-(\lambda-\mu)SR_S(\lambda,T))x\\
=&R_S(\lambda,T)^{-1}x+(\mu-\lambda)R_S(\lambda,T)^{-1}SR_S(\lambda,T))x\\
=&R_S(\lambda,T)^{-1}x-R_S(\lambda,T)^{-1}x+R_S(\mu,T)^{-1}x\\
=&R_S(\lambda,T)^{-1}(0)+R_S(\mu,T)^{-1}x.
\end{aligned}$$

Since $R_S(\lambda,T)^{-1}(0)=(\lambda S-T)(0)=T(0)$ and $R_S(\mu,T)^{-1}x=(\lambda S-T)x$, then

$$R_S(\lambda,T)^{-1}(I-(\lambda-\mu)SR_S(\lambda,T))x=R_S(\mu,T)^{-1}x.$$

This implies that $R_S(\mu,T)=(I-(\lambda-\mu)SR_S(\lambda,T))^{-1}R_S(\lambda,T)$. So, in the first place, we have

$$R_S(\mu,T)=\sum_{n\geq 0}(\lambda-\mu)^nS^nR_S(\lambda,T)^{n+1}.$$

In the second place, we have

$$\begin{aligned}
R_S(\mu,T)-R_S(\lambda,T)=&\sum_{n\geq 0}(\lambda-\mu)^nS^nR_S(\lambda,T)^{n+1}-R_S(\lambda,T)\\
=&R_S(\lambda,T)\sum_{n\geq 1}(\lambda-\mu)^nS^nR_S(\lambda,T)^n,
\end{aligned}$$

and we conclude

$$\|R_S(\mu,T)-R_S(\lambda,T)\|\leq\|R_S(\lambda,T)\|\sum_{n\geq 1}|(\lambda-\mu)|^n\cdot\|S\|^n\cdot\|R_S(\lambda,T)\|^n.$$

Arguing as above we conclude that

$$\lim_{\mu\to\lambda}(\mu-\lambda)^{-1}(R_S(\mu,T)-R_S(\lambda,T))=-R_S(\lambda,T)SR_S(\lambda,T).$$

□

Corollary 3.1. *Let $T\in\mathcal{CR}(X)$ and $S\in\mathcal{LR}(X)$, which is continuous, satisfy $S(0)\subset T(0)$, $\|S\|\neq 0$, $S\neq T$ and $\mathcal{D}(T)\subset\mathcal{D}(S)$. For all $\lambda\in\rho_S(T)$, we have $R_S(\lambda,T)$ commutes with S.*

The function $\varphi:\lambda\to R_S(\lambda,T)$ is holomorphic for all $\lambda\in\rho_S(T)$.

Proof. If λ and $\mu\in\rho_S(T)$, then

$$\lim_{\mu\to\lambda}\frac{\varphi(\mu)-\varphi(\lambda)}{\mu-\lambda}=-R_S(\lambda,T)SR_S(\lambda,T).$$

□

Proposition 3.2. *Let T be open and injective with dense range. Then, for any relation $\|S\|\neq 0$ such that $S(0)\subset\overline{T(0)}$ and $\mathcal{D}(S)\supset\mathcal{D}(T)$ we have*

$$\sigma_S(T)\subset\left\{\lambda\in\mathbb{C}:|\lambda|<\frac{\gamma(T)}{\|S\|}\right\}.$$

Proof. Choose λ such that $0 < |\lambda| < \frac{\gamma(T)}{\|S\|}$. Then, by Lemma 2.5 2., we have $\lambda S - T$ is open, injective and has dense range. Thus, $\lambda \in \rho_S(T)$. $\square$

Proposition 3.3. *Let $T \in \mathcal{CR}(X)$ and $S \in \mathcal{LR}(X)$, which is continuous, satisfy $S(0) \subset T(0)$, $\|S\| \neq 0$, $S \neq T$ and $\mathcal{D}(T) \subset \mathcal{D}(S)$. For all $\lambda \in \rho_S(T)$, we have $R_S(\lambda, T)$ commutes with S, and $\rho_S(T)$ is nonempty and unbounded. Then,*

$$\lim_{|\lambda|\to+\infty} \|R_S(\lambda, T)\| = 0, \quad \textit{for all } \lambda \in \rho_S(T).$$

Proof. Let $\lambda, \mu \in \rho_S(T)$ and μ be fixed,

$$(R_S(\lambda, T) - R_S(\mu, T))(\mu S - T) = (\mu - \lambda)R_S(\lambda, T)SR_S(\mu, T)(\mu S - T).$$

This is equivalent to $R_S(\lambda, T)(\mu S - T) - I_{\mathcal{D}(T)} = (\mu - \lambda)R_S(\lambda, T)S$. It follows that, $|\mu - \lambda| \cdot \|R_S(\lambda, T)\| \cdot \|S\| \leq 1 + \|\mu S - T\| \cdot \|R_S(\lambda, T)\|$. Thus, since $\|\mu S - T\| < +\infty$,

$$\Big(|\mu - \lambda| \cdot \|S\| - \|\mu S - T\|\Big)\|R_S(\lambda, T)\| < 1.$$

Then, $\|R_S(\lambda, T)\| < \frac{1}{|\mu-\lambda|\cdot\|S\|-\|\mu S-T\|}$. Consequently, $\lim_{|\lambda|\to+\infty} \|R_S(\lambda, T)\| = 0$. $\square$

Theorem 3.5. *Let $T \in \mathcal{CR}(X)$ and $S \in \mathcal{LR}(X)$, which is continuous, satisfy $S(0) \subset T(0)$, $\|S\| \neq 0$, $S \neq T$ and $\mathcal{D}(T) \subset \mathcal{D}(S)$. For all $\lambda \in \rho_S(T)$, we have $R_S(\lambda, T)$ commutes with S, and $\rho_S(T)$ is non-empty and unbounded. Then, the S-spectrum of T is non-empty.*

Proof. Suppose that $\rho_S(T) = \mathbb{C}$. We have for $x \in X$ and $x^{'} \in X^{'}$,

$$\lim_{\mu\to\lambda} \frac{x'R_S(\lambda, T)x - x'R_S(\mu, T)x}{\lambda - \mu} = x'R_S(\lambda, T)SR_S(\lambda, T)x.$$

Thus, the (single valued) function $f(\lambda) = x^{'}R_S(\lambda, T)SR_S(\lambda, T)x$ is an entire analytic function.

Moreover, $f(\lambda) \leq \|x'\| \cdot \|R_S(\lambda, T)\| \cdot \|S\| \cdot \|R_S(\lambda, T)\| \cdot \|x\|$. By Proposition 3.3, it is clear that $\lim_{|\lambda|\to\infty} \|R_S(\lambda, T)\| = 0$, then we have $f(\lambda) = 0$ for all λ. Since x' is arbitrary, we have $R_S(\lambda, T)x = 0$ for all $x \in X$. Thus, $X = N(R_S(\lambda, T)) = (\lambda S - T)(0) = T(0)$, and hence,

$$0 = R_S(\lambda, T)(\lambda S - T)(0) = R_S(\lambda, T)(\lambda S - T)x = x, \quad \text{for all } x \in X$$

(since $R_S(\lambda, T)$ is injective), which contradicts with our assumption that X is non-trivial. $\square$

Definition 3.3. The augmented S-spectrum of T is the set

$$\overline{\sigma}_S(T) = \begin{cases} \sigma_S(T) \cup \{+\infty\}, & \text{if } 0 \in \sigma_S(T^{-1}), \\ \sigma_S(T), & \text{otherwise.} \end{cases}$$

The augmented S-spectrum is non-empty (if $X \neq \{0\}$) since if $\{\infty\} \nsubseteq \overline{\sigma}_S(T)$ then $\overline{\sigma}_S(T) = \sigma_S(T)$ is non-empty by Theorem 3.5.

The Möbius transformation $\eta(\lambda) = (\mu - \lambda)^{-1}$, where μ is a fixed point of $\mathbb{C}$, and a topological homeomorphism from $\mathbb{C}_\infty$ onto itself.

Theorem 3.6. *Let $S \in \mathcal{LR}(X,Y)$ be continuous and $T \in \mathcal{CR}(X,Y)$ such that $S(0) \subset \overline{T(0)}$ and $\{0\} \neq \mathcal{D}(T) \supset \mathcal{D}(S)$ with $\mu \in \rho_S(T)$. Then,*

$$\eta(\overline{\sigma_S}(T)) = \sigma(SR_S(\mu, T)).$$

Proof. Without loss of generality, we assume that X is complete, T is closed and S is continuous. Let $\lambda \in \mathbb{C}$, $\lambda \neq \mu$, and let $A := (\mu - \lambda)((\mu - \lambda)^{-1} - SR_S(\mu, T))$. Then,

$$\begin{aligned}\lambda S - T =&(\mu S - T) - (\mu - \lambda)S \quad (\text{since } S(0) \subset T(0))\\ =&(I - (\mu - \lambda)SR_S(\lambda, T))(\mu S - T)\\ =&A(\mu S - T).\end{aligned}$$

We shall verify that A is injective. Suppose that $\lambda \in \rho_S(T)$. For $x \in X$ we have $A(x) = A(0) = S(0)$, then $(\mu - \lambda)((\mu - \lambda)^{-1} - SR_S(\mu, T))x = S(0) \subset (\lambda S - T)(0)$, which give $R_S(\lambda, T)(\mu - \lambda)((\mu - \lambda)^{-1} - SR_S(\mu, T))x \subset R_S(\lambda, T)(\lambda S - T)(0) = 0$. This implies that $(R_S(\lambda, T) - R_S(\lambda, T) + R_S(\mu, T))x = 0$, equivalent to $R_S(\mu, T)x = 0$. Therefore, $(\mu S - T)R_S(\mu, T)x = (\mu S - T)(0) = T(0)$. By using Lemma 2.1 (d) we get $x + (\mu S - T)(0) = x + T(0) = T(0)$, so we conclude that $x = 0$. Then, A is injective. Next we have $X = R(\lambda S - T) = R(A(\mu S - T)) \subset R(A)$. Hence, S is surjective. As required, since S is both bijective and open, it follows that $(\mu - \lambda)^{-1} \in \rho_I(SR_S(\lambda, T))$.

Conversely, let $(\mu - \lambda)^{-1} \in \rho_I(SR_S(\lambda, T))$. For $x \in \mathcal{D}(T)$, we have

$$\|(\lambda S - T)x\| = \|A(\mu S - T)x\| \geq \gamma(A(\mu S - T))d(x, R_S(\lambda, T)A^{-1}(0)).$$

Since S is injective, then $R_S(\lambda, T)A^{-1}(0) = R_S(\lambda, T)(0) = 0$. Therefore,

$$\|(\lambda S - T)x\| \geq \gamma(A(\mu S - T))\|x\|.$$

By Theorem 2.5 (a) as S is injective we have $\gamma(A(\mu S - T)) \geq \gamma(A)\gamma(\mu S - T)$. Hence, $\gamma(A) > 0$ from the hypothesis, and $\gamma(\mu S - T) > 0$ since $\mu \in \rho_S(T)$. Hence, it is easy to verify that $\lambda S - T$ is injective and open, and we have $\lambda S - T$ is surjective. We deduce that $\lambda \in \rho_S(T)$. □

Proposition 3.4. *Let the relations S and T commute such that $\overline{\mathcal{D}(T)} \subset \mathcal{D}(S)$. Then, $(\lambda S - T)$ and $(\mu S - T)$ have the same properties.*

Proof. It is clear that the domains of $(\mu S - T)(\lambda S - T)$ and $(\lambda S - T)(\mu S - T)$ are each equal to $\mathcal{D}(T^2)$. We have for $x \in \mathcal{D}(T^2)$

$$\begin{aligned}(\lambda S - T)(\mu S - T)x =&(\lambda S - T)(\mu Sx - Tx)\\ =&(\lambda S - T)\{\mu y_1 - y_2 : y_1 \in Sx \text{ and } y_2 \in Tx, y_1, y_2 \in \mathcal{D}(T)\}\\ =&\{\lambda\mu Sy_1 - \lambda Sy_2 - \mu Ty_1 + Ty_2 : y_1 \in Sx \text{ and } y_2 \in Tx,\\ &y_1, y_2 \in \mathcal{D}(T)\}\\ =&\lambda\mu S^2x - \lambda STx - \mu TSx + T^2x.\end{aligned}$$

Since T and S commute, then

$$\begin{aligned}(\lambda S-T)(\mu S-T)x =& \lambda\mu S^2x-\lambda TSx-\mu STx+T^2x\\ =&(\mu S-T)(\lambda S-T)x.\end{aligned}$$

□

Proposition 3.5. *Let $\mu\in\rho_S(T)$ such that $T\in\mathcal{CR}(X)$ and $S\in\mathcal{LR}(X)$ be continuous with $S(0)\subset T(0)$, $\mathcal{D}(T)\subset\mathcal{D}(S)$ and let $\lambda\neq\mu$. Then,*

$$N(\lambda S-T)=N((\mu-\lambda)^{-1}-(\mu S-T)^{-1}S).$$

Proof. Let $\mu\in\rho_S(T)$ and $\lambda\in\mathbb{C}$ such that $\mu\neq\lambda$.

(a) $T(0)+(\mu-\lambda)Sx=(\mu S-T)x$. Indeed, let $(x,y)\in G(T-T+(\mu-\lambda)S)$, then $x\in\mathcal{D}(T)$ and $y\in(T-T+(\mu-\lambda)S)x$, so that, $y\in T(0)+(\mu-\lambda)Sx$.

From another angle, $T(0)=S(0)+T(0)\subset(\lambda S-T)x$, we obtain that

$$\begin{aligned}y\in T(0)+(\mu-\lambda)Sx\subset&(\lambda S-T)x+(\mu-\lambda)Sx\\ =&(\mu S-T)x \quad \text{(by Lemma 2.3 1.)}.\end{aligned}$$

We infer that $G(T-T+(\mu-\lambda)S)\subset G(\mu S-T)$. Furthermore, $(T-T+(\mu-\lambda)S)(0)=T(0)=(\mu S-T)(0)$ and clearly $\mathcal{D}(T-T+(\mu-\lambda)S)=\mathcal{D}(T)\cap\mathcal{D}(S)=\mathcal{D}(T)=\mathcal{D}(\mu S-T)$. So, by Lemma 2.2 (a) and Lemma 2.3 (a), we have

$$T(0)+(\mu-\lambda)Sx=(\lambda S-T)x+(\mu-\lambda)Sx=(\mu S-T)x.$$

(b) $N(\lambda S-T)=N((\mu-\lambda)^{-1}-(\mu S-T)^{-1}S)$. Indeed, we first note that $x\in N((\mu-\lambda)^{-1}-(\mu S-T)^{-1}S)$ if, and only if, $x=(\mu-\lambda)(\mu S-T)^{-1}Sx$, because

$$\begin{aligned}(I-(\mu-\lambda)(\mu S-T)^{-1}S)(0)=&(\mu-\lambda)(\mu S-T)^{-1}(\mu S-T)(0)\\ =&(\mu S-T)^{-1}(0)=0.\end{aligned}$$

The latter implies that $x\in R((\mu S-T)^{-1})=\mathcal{D}(\mu S-T)$ and, hence, it suffices to consider the case $x\in\mathcal{D}(T),\ x\neq0$: $x\in N(\lambda S-T)$ if and only if

$$x\in\mathcal{D}(\lambda S-T)=\mathcal{D}(T)\quad\text{and}\quad(\lambda S-T)x=(\lambda S-T)(0)=T(0)$$

if and only if

$$(\mu S-T)x=(\lambda S-T)x+(\mu-\lambda)Sx=T(0)-\lambda Sx+\mu Sx\quad(\text{by}(a))$$

if and only if

$$(\mu S-T)^{-1}(\mu S-T)x=(\mu S-T)^{-1}((\mu-\lambda)Sx+T(0))$$

if and only if

$$x+(\mu S-T)^{-1}(0)=(\mu-\lambda)(\mu S-T)^{-1}Sx+(\mu S-T)^{-1}(\mu S-T)(0)$$

if and only if

$$0=(I-(\mu-\lambda)(\mu S-T)^{-1}S)x.$$

Then, $N(\lambda S-T)=N((\mu-\lambda)^{-1}-(\mu S-T)^{-1}S)$. □

Theorem 3.7. *Let $T \in \mathcal{CR}(X)$ and let $S \in \mathcal{BR}(X)$ be single valued such that S and T commute, with $\mathcal{D}(T) = X$ and $0 \in \rho(S)$. Then, for any complex polynomial P, we have*

$$\sigma(P(S^{-1}T)) \cup \sigma(P(TS^{-1})) = P(\sigma_S(T)).$$

Proof. Let $\lambda \in \mathbb{C}$, and let

$$\mu I - P(\lambda) := c \prod_{j=1}^{n} S(\alpha_j - \lambda).$$

Then,

$$\mu I - P(S^{-1}T) := c \prod_{j=1}^{n} S(\alpha_j I - S^{-1}T)$$

and

$$\mu I - P(TS^{-1}) := c \prod_{j=1}^{n} S(\alpha_j I - TS^{-1}).$$

Without loss of generality, we assume that X is complete with $S^{-1}T$ and TS^{-1} closed.

Let $\mu \in \sigma(P(S^{-1}T)) \cup \sigma(P(TS^{-1}))$. If $\alpha_j \in \rho_S(T) = \rho(TS^{-1}) \cap \rho(TS^{-1})$ for all $1 \leq j \leq n$, then $(\alpha_j I - TS^{-1})$ and $(\alpha_j I - S^{-1}T)$ would be bijective for all $1 \leq j \leq n$. Since $0 \in \rho(S)$, then S is bijective. Hence, by Proposition 2.2, $\mu I - P(S^{-1}T)$ and $\mu I - P(TS^{-1})$ are bijective. Contradicting the assumption that $\mu \in \sigma(P(S^{-1}T)) \cup \sigma(P(TS^{-1}))$. Thus exists j, $1 \leq j \leq n$ such that $\alpha_j \in \sigma_S(T)$. Since $P(\alpha_j) = \mu$, it follows that $\mu \in P(\sigma_S(T))$.

Conversely, suppose that $\mu \in P(\sigma_S(T))$. Then, $\mu = P(\lambda)$ for some $\lambda \in \sigma_S(T)$. Thus, $\lambda S = \alpha_j S$ since S is single valued, then $\lambda = \lambda S S^{-1} = \alpha_j S S^{-1} = \alpha_j$ for some j such that $1 \leq j \leq n$. Since the factors commute (see Proposition 2.2), we may assume that $j = 1$. We notice the existence of two cases.

<u>1st case</u>. Suppose that $\alpha_1 S - T$ is injective. Then it cannot be surjective ($\alpha_1 = \lambda \in \sigma_S(T)$). Consequently, $\mu I - P(S^{-1}T)$ or $\mu I - P(TS^{-1})$ cannot be surjective. Thus, $\mu \in \sigma(P(S^{-1}T))$.

<u>2nd case</u>. $\alpha_j S - T$ is surjective for every j, $1 \leq j \leq n$. Since $\alpha_1 \in \sigma_S(T)$, $\alpha_1 S - T$ cannot be injective. Thus, $\mu I - P(S^{-1}T)$ or $\mu I - P(TS^{-1})$ is not injective. It follows that $\mu \in \sigma(P(S^{-1}T)) \cup \sigma(P(TS^{-1}))$. □

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