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ON THE ZEROS OF POLYNOMIALS WITH REAL COEFFICIENTS

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Abstract. The Eneström-Kakeya theorem provides essential bounds on the location of the zeros of a polynomial with positive coefficients. Lot of research work has been done regarding the classical theorem known as Eneström-Kakeya theorem concerning the regions containing zeros of a polynomial. This theorem states that if $F(z) = \sum_{\lambda=0}^{n} f_\lambda z^\lambda$ is a polynomial with degree n with real coefficients satisfying $0 \leq f_0 \leq f_1 \leq f_2 \leq \cdots \leq f_n$, then all the zeros of $F(z)$ lie in $|z| \leq 1$. In this article, we prove several extensions of this theorem which impose restrictions only on the coefficients $f_0, f_1, \ldots, f_{n-1}$ and leaves the coefficient f_n to vary freely over the whole complex plane.

1. Introduction

The classical Eneström-Kakeya Theorem gives us information about the position of the zeros of a polynomial whose coefficients are nonnegative and satisfy a monotonicity condition. It was independently proved by G. Eneström in 1893 [6] and Kakeya in 1912 [12].

Theorem 1.1 (Eneström-Kakeya Theorem). *If $F(z) = \sum_{\lambda=0}^{n} f_\lambda z^\lambda$ is a polynomial of degree n with real coefficients satisfying $0 \leq f_0 \leq f_1 \leq \cdots \leq f_n$, then $F(z)$ has all its zeros in the region $|z| \leq 1$.*

In literature, there exist several extentions and generalizations of Theorem 1.1 (see [1, 2], [5]-[10], [13]). Joyal, Labelle, and Rahman [11] extended Theorem 1.1 to the

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polynomials whose coefficients satisfy a monotonicity condition but need not be non negative. In fact, they proved the following result.

Theorem 1.2. *Let $F(z) = \sum_{\lambda=0}^{n} f_\lambda z^\lambda$ be a polynomial with degree n with real coefficients satisfying the condition $f_0 \le f_1 \le \cdots \le f_n$. Then, $F(z)$ has all its zeros lying in the region*

$$|z| \le \frac{1}{|f_n|}(f_n - f_0 + |f_0|).$$

In 1996, Aziz and Zargar [3] proved the following generalisation of Theorem 1.1.

Theorem 1.3. *Let $F(z) = \sum_{\nu=0}^{n} f_\nu z^\nu$ be a polynomial of degree n. If for some positive number k with $k \ge 1$, $0 \le f_0 \le f_1 \le \cdots \le f_{n-1} \le k f_n$. Then, all the zeros of $F(z)$ lie in the disc*

$$|z + k - 1| \le k.$$

In 2012, Aziz and Zargar [4] also proved the following generalization of Theorem 1.1.

Theorem 1.4. *Let $F(z) = \sum_{\lambda=0}^{n} f_\lambda z^\lambda$ be a polynomial with degree n. If for some positive numbers k and s with $k \ge 1$, $0 < s \le 1$, $0 \le s f_0 \le f_1 \le \cdots \le f_{n-1} \le k f_n$, then all the zeros of $F(z)$ lie in the closed disc*

$$|z + k - 1| \le k + \frac{2 f_0}{f_n}(1 - s).$$

In 2015, E. R. Nwaeze [**?**] proved the following result concerning the zeros of polynomials, which is a generalization of Theorem 1.3.

Theorem 1.5. *Let $F(z) = \sum_{\lambda=0}^{n} f_\lambda z^\lambda$ be a polynomial with degree n. If for some real numbers γ and δ, $f_0 - \delta \le f_1 \le \cdots \le f_{n-1} \le f_n + \gamma$, then all the zeros of $F(z)$ lie in the disc*

$$\left| z + \frac{\gamma}{f_n} \right| \le \frac{1}{|f_n|}\left(f_n + \gamma - f_0 + \delta + |\delta| + |f_0|\right).$$

Aziz and Zargar [3] also relaxed the hypothesis of Theorem 1.2 in several ways and among other things they proved the following result.

Theorem 1.6. *Let $F(z) = \sum_{\lambda=0}^{n} f_\lambda z^\lambda$ be a polynomial of degree n with real coefficients such that for some k with $k \ge 1$, $k f_n \ge f_{n-1} \ge \cdots \ge f_1 \ge f_0$. Then, all the zeros of $F(z)$ lie in the disc*

$$|z + k - 1| \le \frac{k f_n - f_0 + |f_0|}{|f_n|}.$$

Shah and Liman [19] extended Theorem 1.6 to the polynomials with complex coefficients by proving the following result.

Theorem 1.7. *Let $F(z) = \sum_{\lambda=0}^{n} f_\lambda z^\lambda$ be a polynomial of degree n with complex coefficients such that for some real β, $|arg f_j - \beta| \leq \alpha \leq \frac{\pi}{2}$, $j = 0, 1, 2, \ldots, n$, and $k \geq 1$*

$$k|f_n| \geq |f_{n-1}| \geq \cdots \geq |f_1| \geq |f_0|.$$

Then, all the zeros of $F(z)$ lie in

$$|z + k - 1| \leq \frac{1}{|f_n|}\left\{(k|f_n| - |f_0)(\cos\alpha + \sin\alpha) + |f_0| + 2\sin\alpha \sum_{j=0}^{n-1} |f_j|\right\}.$$

Rather et al. [15] relaxed the hypothesis of Theorem 1.4 and they proved the following result.

Theorem 1.8. *Let $F(z) = \sum_{\lambda=0}^{n} f_\lambda z^\lambda$ be a polynomial of degree n with real coefficients such that for some $k_j \geq 1$, $f_{n-j+1} > 0$, $j = 1, 2, \ldots, r$ where $1 \leq r \leq n$*

$$k_1 f_n \geq k_2 f_{n-1} \geq k_3 f_{n-2} \geq \cdots \geq k_r f_{n-r+1} \geq f_{n-r} \geq f_1 \geq f_0.$$

Then, all the zeros of $F(z)$ lie in

$$\begin{aligned}
&\left|z + k_1 - 1 - (k_2 - 1)\frac{f_{n-1}}{f_n}\right| \\
\leq &\frac{1}{|f_n|}\left\{k_1 f_n - (k_2 - 1)|f_{n-1}| + 2\sum_{j=2}^{r}(k_j - 1)|f_{n-j+1}| - f_0 + |f_0|\right\}.
\end{aligned}$$

Rather et al. [17, 18], extended Theorem 1.7 to the polynomials with complex coefficients and proved following two results.

Theorem 1.9. *Let $F(z) = \sum_{\lambda=0}^{n} f_\lambda z^\lambda$ be a polynomial of degree n with complex coefficients such that for some real β, $|arg f_j - \beta| \leq \alpha \leq \frac{\pi}{2}$, $j = 0, 1, 2, \ldots n$, and $k_j \geq 1$, $f_{n-j} \neq 0$, $j = 0, 1, \ldots, r$, where $1 \leq r \leq n - 1$*

$$k_0|f_n| \geq k_1|f_{n-1}| \geq k_2|f_{n-2}| \geq \cdots \geq k_r|f_{n-r}| \geq |f_{n-r-1}| \geq \cdots \geq |f_1| \geq |f_0|.$$

Then, all the zeros of $F(z)$ lie in

$$\begin{aligned}
&\left|z + k_0 - 1 - (k_1 - 1)\frac{f_{n-1}}{f_n}\right| \\
\leq &\frac{1}{|f_n|}\Bigg\{(k_0|f_n| - |f_0|)(\cos\alpha + \sin\alpha) + 2\sin\alpha\left(\sum_{j=1}^{r} k_j|f_{n-j}| + \sum_{j=r+1}^{n} |f_{n-j}|\right) \\
&- (k_1 - 1)|f_{n-1}| + 2\sum_{j=1}^{r}(k_j - 1)|f_{n-j}| + |f_0|\Bigg\}.
\end{aligned}$$

Theorem 1.10. *Let $F(z) = \sum_{\lambda=0}^{n} f_\lambda z^\lambda$, $f_j = \alpha_j + i\gamma_j$ be a polynomial of degree n with complex coefficients such that for some $k_j \geq 1$, $\alpha_{n-j+1} > 0$, $j = 1, 2, \ldots, r$, where $1 \leq r \leq n$*

$$k_1\alpha_n \geq k_2\alpha_{n-1} \geq k_3\alpha_{n-2} \geq \cdots \geq k_r\alpha_{n-r+1} \geq \alpha_{n-r} \geq \cdots \geq \alpha_1 \geq \alpha_0.$$

Then, all the zeros of $F(z)$ lie in

$$\left|z+(k_1-1)\frac{\alpha_n}{f_n}-(k_2-1)\frac{\alpha_{n-1}}{f_n}\right|$$
$$\leq\frac{1}{|f_n|}\left[|k_1\alpha_n-(k_2-1)|\alpha_n-1|+2\left(\sum_{j=2}^{r}(k_j-1)|\alpha_{n-j+1}|+\sum_{j=0}^{n-1}|\gamma_j|\right)-\alpha_0+|\alpha_0|+|\gamma_n|\right].$$

Recently, Rather et al. [16], proved the following results.

Theorem 1.11. *Let $F(z)=\sum_{\lambda=0}^{n}f_\lambda z^\lambda$ be a polynomial of degree n with complex coefficients such that for some real β, $|arg(k_j+a_{n-j})-\beta|\leq\alpha\leq\frac{\pi}{2}$, $j=0,1,2,\ldots n$, and for some number k_j, $j=0,1,\ldots,r$, where $1\leq r\leq n-1$*

$$|k_0+f_n|\geq|k_1+f_{n-1}|\geq|k_2+f_{n-2}|\geq\cdots\geq|k_r+f_{n-r}|\geq|f_{n-r-1}|\geq\cdots\geq|f_1|\geq|f_0|.$$

Then, all the zeros of $F(z)$ lie in

$$\left|z+\frac{k_0-k_1}{f_n}\right|$$
$$\leq\frac{1}{|f_n|}\left[(|k_0-f_n|-|f_0|)(\cos\alpha+\sin\alpha)+2\sin\alpha\left\{\sum_{j=1}^{r}(|k_j+f_{n-j}|)+\sum_{j=r+1}^{n}|f_{n-j}|\right\}\right.$$
$$\left.+\sum_{j=1}^{r-1}|k_j-k_{j+1}|+|k_r|+|f_0|\right].$$

Theorem 1.12. *Let $F(z)=\sum_{\lambda=0}^{n}f_\lambda z^\lambda$, where $f_j=\alpha_j+\iota\gamma_j$ be a polynomial of degree n with complex coefficients such that for some $k_j\geq0$, $j=0,1,2,\ldots r$, where $1\leq r\leq n-1$*

$$k_0+\alpha_n\geq k_1+\alpha_{n-1}\geq k_2+\alpha_{n-2}\geq\cdots\geq k_r+\alpha_{n-r}\geq\alpha_{n-r-1}\geq\cdots\geq\alpha_1\geq\alpha_0.$$

Then, all the zeros of $F(z)$ lie in

$$\left|z+\frac{k_0-k_1}{f_n}\right|\leq\frac{1}{|f_n|}\left\{\alpha_n+|\alpha_0|-\alpha_0+\sum_{j=1}^{r-1}|k_j-k_{j+1}|\sum_{j=1}^{r-1}|\gamma_{j+1}-\gamma_j|+|\gamma_0|+k_r+k_0\right\}.$$

2. Main Results and Proofs

In this article, we first give a result which is an extension of Theorem 1.1. In this result, only the coefficients $f_0,f_1,\ldots,f_{n-1}$ satisfy a monotonicity condition, and the coefficient f_n moves freely in the complex plane. Our theorem provides a stronger result than the classic Eneström-Kakeya theorem, which is applicable to a broader class of polynomials. Infact, we first prove the following results.

Theorem 2.1. *Let $F(z)=\sum_{\lambda=0}^{n}f_\lambda z^\lambda$ be a polynomial with degree n such that the coefficients $f_0,f_1,\ldots,f_{n-1}$ are real and satisfying the monotonicity condition* $0\leq f_0\leq$

$f_1 \leq \cdots \leq f_{n-1}$. *Then, all the zeros of $F(z)$ lie in the union of the regions*

$$\{z \in \mathbb{C} : |z| \leq 1\} \bigcup \left\{z \in \mathbb{C} : \left|z - \left(1 - \frac{f_{n-1}}{f_n}\right)\right| \leq \frac{f_{n-1}}{|f_n|}\right\}.$$

Proof. Consider the polynomial

$$\begin{aligned} Q(z) &= (1-z)F(z) \\ &= -f_n z^{n+1} + (f_n - f_{n-1})z^n + (f_{n-1} - f_{n-2})z^{n-1} + \cdots + (f_1 - f_0)z + f_0 \\ &= -f_n z^{n+1} + (f_n - f_{n-1})z^n + \phi(z), \end{aligned} \tag{2.1}$$

where $\phi(z) = (f_{n-1} - f_{n-2})z^{n-1} + \cdots + (f_1 - f_0)z + f_0$. For $|z| = 1$, we have that

$$|\phi(z)| \leq |f_{n-1} - f_{n-2}| + |f_{n-2} - f_{n-3}| + \cdots + |f_1 - f_0| + |f_0| = f_{n-1}$$

implies $|z^n \phi(1/z)| \leq f_{n-1}$ for $|z| = 1$. Hence, By Maximum Modulus Theorem

$$|z^n \phi(1/z)| \leq f_{n-1}, \quad \text{for } |z| \leq 1.$$

Replacing z by $1/z$, we get

$$|\phi(z)| \leq f_{n-1}|z^n|, \quad \text{for } |z| \geq 1.$$

Therefore, for $|z| \geq 1$, from equation (2.1), we obtain

$$\begin{aligned} |Q(z)| &= |-f_n z^{n+1} + (f_n - f_{n-1})z^n + \phi(z)| \\ &\geq |z^n||f_n z - (f_n - f_{n-1})| - |\phi(z)| \\ &\geq |z^n||f_n z - (f_n - f_{n-1})| - f_{n-1}|z^n| \\ &= |z^n| \left[|f_n z - (f_n + f_{n-1})| - f_{n-1}\right] \\ &> 0, \end{aligned}$$

which is true if and only if $|f_n z - (f_n + f_{n-1})| > f_{n-1}$, i.e.,

$$\left|z - \left(1 - \frac{f_{n-1}}{f_n}\right)\right| > \frac{f_{n-1}}{|f_n|}.$$

Thus, all the zeros of $g(z)$ whose modulus is greater than or equal to one lie in

$$\left|z - \left(1 - \frac{f_{n-1}}{f_n}\right)\right| \leq \frac{f_{n-1}}{|f_n|}.$$

Hence, all the zeros of $F(z)$ lie in the union of disc

$$|z| \leq 1 \bigcup \left|z - \left(1 - \frac{f_{n-1}}{f_n}\right)\right| \leq \frac{f_{n-1}}{|f_n|}.$$

□

Example 2.1. Consider the polynomial

$$F(z) = 3z^4 + 5z^3 + 2z^2 + z + 1.$$

Here, $f_0 = 1,\ f_1 = 1,\ f_2 = 2,\ f_3 = 5,\ f_4 = 3$, which satisfy $0 \leq 1 \leq 1 \leq 2 \leq 5$. We cannot apply Eneström-Kakeya theorem, because $0 \leq f_0 \leq f_1 \leq f_2 \leq f_3 \nleq f_4$. By Theorem 2.1, the polynomial $F(z)$ has all its zeros in the union of the regions

$$\{z \in \mathbb{C} : |z| \leq 1\} \cup \left\{ z \in \mathbb{C} : \left| z - \left(1 - \frac{f_2}{f_3}\right) \right| \leq \frac{f_2}{|f_3|} \right\},$$

which is given by

$$\{z \in \mathbb{C} : |z| \leq 1\} \cup \left\{ z \in \mathbb{C} : \left| z + \left(\frac{2}{3}\right) \right| \leq \frac{5}{3} \right\}.$$

This example illustrates the applications of Theorem 2.1 to a broader class of polynomials as against to the Eneström-Kakeya theorem.

Remark 2.1. If the polynomial $F(z) = \sum_{\lambda=0}^{n} f_\lambda z^\lambda$ satisfies the conditions of Theorem 1.1, then the disc

$$\left| z - \left(1 - \frac{f_{n-1}}{f_n}\right) \right| \leq \frac{f_{n-1}}{|f_n|}$$

is contained in the disc $|z| \leq 1$. Hence, Theorem 2.1 reduces to Theorem 1.1.

Next, we prove the following result in which we impose the monotonicity condition on the coefficients $f_0, f_1, \dots, f_{n-1}$ and leave the coefficient f_n to vary freely over the whole complex plane and thus broaden the scope of Theorem 1.5.

Theorem 2.2. *Let $F(z) = \sum_{\lambda=0}^{n} f_\lambda z^\lambda$ be a polynomial with degree n. If for some real numbers γ and δ, $f_0 - \delta \leq f_1 \leq \cdots \leq f_{n-2} \leq f_{n-1} + \gamma$, then all the zeros of $F(z)$ lie in the union of regions*

$$\{z \in \mathbb{C} : |z| \leq 1\} \bigcup \{z \in \mathbb{C} : |z - \zeta||z - \eta| \leq \gamma^*\},$$

where ζ and η are the roots of the quadratic $f_n z^2 + (f_n - f_{n-1})z - \gamma$ and

$$\gamma^* = \frac{f_{n-1} + \gamma + \delta + |\delta| + |f_0| - f_0}{|f_n|}.$$

Proof. Consider the polynomial

$$\begin{aligned} L(z) =& (1-z)F(z) \\ =& f_n z^{n+1} + (f_n - f_{n-1})\, z^n - \gamma z^{n-1} + \gamma z^{n-1} + (f_{n-1} - f_{n-2}) z^{n-1} \\ & + \cdots + ((f_1 - f_0) + \delta)\, z - \delta z + f_0 \\ =& z^{n-1}[f_n z^2 + ((f_n - f_{n-1})z - \gamma)] + \phi(z), \end{aligned} \tag{2.2}$$

where $\phi(z) = \gamma z^{n-1} + (f_{n-1} - f_{n-2})z^{n-1} + \cdots + (f_1 - f_0 + \delta)z - \delta z + f_0$.

For $|z| = 1$, we have

$$\begin{aligned} |\phi(z)| &\leq |\gamma + f_{n-1} - f_{n-2}| + |f_{n-2} - f_{n-3}| + \cdots + |f_1 - f_0 + \delta| + |\delta| - |f_0| \\ &= \gamma + f_{n-1} + \delta + |\delta| + |f_0| - f_0. \end{aligned}$$

Hence, for $|z| = 1$, we have

$$|z^n \phi(1/z)| \leq \gamma + f_{n-1} + |\delta| + |\delta| - |f_0| - f_0.$$

Therefore, by Maximum Modulus Theorem

$$|z^n \phi(1/z)| \leq \gamma + f_{n-1} + \delta + |\delta| - |f_0| - f_0, \quad \text{for } |z| \leq 1.$$

Replacing z by $1/z$, we get for $|z| \geq 1$

$$|\phi(z)| \leq (\gamma + f_{n-1} + |\delta| + |\delta| - |f_0| - f_0)\, |z^{n-1}|, \quad \text{for } |z| \geq 1. \tag{2.3}$$

Therefore, for $|z| \geq 1$, from equation (2.2), we obtain

$$\begin{aligned}
|L(z)| &\geq |z^{n-1}|\Big[|f_n z^2 + (f_n - f_{n-1})z - \gamma|\Big] - |\phi(z)| \\
&\geq |z^{n-1}|\Big[|f_n z^2 + (f_n - f_{n-1})z - \gamma|\Big] - \Big[(\gamma + f_{n-1} + \delta + |\delta| - |f_0| - f_0)\Big]|z^{n-1}| \\
&= |z^{n-1}|\Big[|f_n z^2 + (f_n - f_{n-1})z - \gamma| - (\gamma + f_{n-1} + \delta + |\delta| - |f_0| - f_0)\Big] \\
&> 0,
\end{aligned}$$

which is true if and only if

$$|f_n z^2 + (f_n - f_{n-1})z - \gamma| > \gamma + f_{n-1} + \delta + |\delta| + |f_0| - f_0,$$

i.e.,

$$|(z - \eta)(z - \zeta)| > \frac{\gamma + f_{n-1} + \delta + |\delta| + |f_0| - f_0}{|f_n|}$$

or

$$|(z - \eta)(z - \zeta)| > \gamma^*.$$

Hence, all the zeros of $F(z)$ lie in the union of disc

$$|z| \leq 1 \bigcup \{z \in \mathbb{C} : |z - \zeta| \cdot |z - \eta| \leq \gamma^*\},$$

where ζ and η are the roots of the quadratic $f_n z^2 + (f_n - f_{n-1})z - \gamma$ and

$$\gamma^* = \frac{f_{n-1} + \gamma + \delta + |\delta| + |f_0| - f_0}{|f_n|}.$$

□

Example 2.2. Consider the polynomial

$$F(z) = z^4 + 5z^3 + 7z^2 + 6z + 10.$$

Here, we can't apply Theorem 1.5 because $f_2 \nleq f_3$. Now if we choose $\delta = 5$, $\gamma = 2$. Then, $f_2 = 7 \leq f_3 + \gamma$.

Therefore, by Theorem 2.2, all the zeros of $F(z)$ lie in the union of the regions

$$|z| \leq 1 \bigcup \{z \in \mathbb{C} : |z - \zeta||z - \eta| \leq \gamma^*\},$$

where ζ and η are roots of the quadratic equation

$$f_4 z^2 + (f_4 - f_3)z - \gamma = 0.$$

That is $z^2 - 4z - 2 = 0$. This gives $\zeta = 8 + \sqrt{6}$, $\eta = 8 - \sqrt{6}$. Also,

$$\gamma^* = \frac{f_2 + \gamma + \delta + |\delta| + |f_0| - f_0}{|f_3|}$$

implies

$$\gamma^* = \frac{19}{5} = 3.8.$$

Therefore, all the zeros of $F(z)$ lie in the union of the regions

$$\{z \in \mathbb{C} : |z| \leq 1\} \bigcup \{z \in \mathbb{C} : |z - (8 + \sqrt{6})| \cdot |z - (8 - \sqrt{6})| \leq 3.8\}.$$

This specific example demonstrates the application of Theorem 2.2 to a broader class of polynomials as against Theorem 1.5.

We now prove a result which is a generalization of Theorem 1.3. In this result, the bond of the disc containing all the zeros of a polynomial under certain restricted conditions on the coefficients, involves also the coefficients of the polynomial. In fact, we prove the following result.

Theorem 2.3. *Let $F(z) = \sum_{\nu=0}^{n} f_\nu z^\nu$ be a polynomial of degree n with real coefficients such that for some real numbers k and λ*

$$kf_n \geq f_{n-1} \quad \text{and} \quad \lambda f_j \geq f_{j-1}, \quad \text{for } j = 0, 1, \ldots, n-1, \; f_{-1} = 0.$$

Then, all the zeros of $f(z)$ lie in the union of the discs

$$\{z \in \mathbb{C} \;/\; |z| \leq 1\} \bigcup \left\{ z \in \mathbb{C} \;/\; |z - \lambda + k| \leq \frac{1}{|f_n|} \left(kf_n + (\lambda - 1) \sum_{i=0}^{n-1} f_i \right) \right\}.$$

Proof. Consider the polynomial

$$\begin{aligned} G(z) =& (\lambda - z)F(z) \\ =& - f_n z^{n+1} + (\lambda f_n - f_{n-1})z^n + (\lambda f_{n-1} - f_{n-2})z^{n-1} + \cdots + (\lambda f_1 - f_0)z + \lambda f_0 \\ =& - f_n z^{n+1} + \lambda f_n z^n - kf_n z^n + (kf_n - f_{n-1})z^n + (\lambda f_{n-1} - f_{n-2})z^{n-1} \\ & + \cdots + (\lambda f_1 - f_0)z + \lambda f_0. \end{aligned}$$

Thus, we can write

$$G(z) = -z^n \left[f_n z - \lambda f_n + kfn \right] + \phi(z), \tag{2.4}$$

where $\phi(z) = (kf_n - f_{n-1})z^n + (\lambda f_{n-1} - f_{n-2})z^{n-1} + \cdots + (\lambda f_1 - f_0)z + \lambda f_0$.

For $|z| = 1$, we have

$$\begin{aligned} |\phi(z)| &\leq |kf_n - f_{n-1}| + |\lambda f_{n-1} - f_{n-2}| + \cdots + |\lambda f_1 - f_0| + |\lambda f_0| \\ &= (kf_n - f_{n-1}) + (\lambda f_{n-1} - f_{n-2} + \lambda f_{n-2} - f_{n-3} + \cdots + \lambda f_1 - f_0 + \lambda f_0) \\ &= kf_n + (\lambda - 1) \sum_{i=0}^{n-1} f_i, \end{aligned}$$

If $|z| = 1$, then $|\frac{1}{z}| = 1$. Therefore, we can write

$$|z^n \phi(1/z)| \leq k f_n + (\lambda - 1) \sum_{i=0}^{n-1} f_i.$$

Hence, by Maximum Modulus theorem for $|z| \leq 1$, we observe the following

$$|z^n \phi(1/z)| \leq k f_n + (\lambda - 1) \sum_{i=0}^{n-1} f_i.$$

Replacing z by $1/z$, we obtain for $|z| \geq 1$

$$|\phi(z)| \leq \left(k f_n + (\lambda - 1) \sum_{i=0}^{n-1} f_i \right) |z|^n.$$

Therefore, for $|z| \geq 1$, from (2.3), we have

$$\begin{aligned} |G(z)| &\geq |z|^n |f_n z - \lambda f_n + k f_n)| - |\phi(z)| \\ &\geq |z^n| \left\{ |f_n z - \lambda f_n + k f_n| - \left(k f_n + (\lambda - 1) \sum_{i=0}^{n-1} f_i \right) \right\} \\ &> 0 \end{aligned}$$

if and only if

$$|z - \lambda + k| > \frac{1}{|f_n|} \left(k f_n + (\lambda - 1) \sum_{i=0}^{n-1} f_i \right).$$

Thus, all the zeros of $G(z)$ whose modulus is greater than or equal to one lie in the disc

$$|z - \lambda + k| \leq \frac{1}{|f_n|} \left(k f_n + (\lambda - 1) \sum_{i=0}^{n-1} f_i \right).$$

Therefore, all the zeros of $F(z)$ lie in the union of the discs

$$\{z \in \mathbb{C} : \ |z| \leq 1\} \bigcup \left\{ z \in \mathbb{C} : \ |z - \lambda + k| \leq \frac{1}{|f_n|} \left(k f_n + (\lambda - 1) \sum_{i=0}^{n-1} f_i \right) \right\}.$$

This completes the proof. □

For $\lambda = 1$, we obtain the following generalization of Theorem 1.3, which is true for any real number k.

Corollary 2.1. *Let $F(z) = \sum_{\nu=0}^{n} f_\nu z^\nu$ be a polynomial of degree n with real coefficients such that for some real number k,*

$$0 \leq f_0 \leq f_1 \cdots \leq f_{n-1} \leq k f_n.$$

Then all the zeros of $F(z)$ lie in the union of the discs

$$\{z \in \mathbb{C} \ : \ |z| \leq 1\} \bigcup \left\{ z \in \mathbb{C} \ : \ |z - 1 + k| \leq \frac{k f_n}{|f_n|} \right\}.$$

Finally, we prove the following result which is an extention of both Theorem 1.2 and Theorem 1.4. In this result, the bond of the disc containing all the zeros of a polynomial under certain restricted conditions on the coefficients, involves also the coefficients of the polynomial. In fact, we prove the following result.

Theorem 2.4. *Let $F(z) = \sum_{\nu=0}^{n} f_\nu z^\nu$ be a polynomial of degree n with real coefficients such that for some real numbers k, λ and $0 \le \rho \le 1$*

$$kf_n \ge f_{n-1},\ \lambda f_j \ge f_{j-1}, \quad \text{for } j = 2, 3, \dots, , n-1 \text{ and } \lambda f_1 \ge \rho f_0.$$

Then all the zeros of $F(z)$ lie in the union of the discs

$$\{z \in \mathbb{C} \ : \ |z| \le 1\} \bigcup \left\{ z \in \mathbb{C} \ : \ |z - \lambda + k| \le \frac{1}{|f_n|} \left(kf_n + (\lambda - 1) \sum_{i=0}^{n-1} f_i + (1 - 2\rho) f_0 + |\lambda f_0| \right) \right\}.$$

Proof. Consider the polynomial

$$\begin{aligned} H(z) =& (\lambda - z)F(z) \\ =& - f_n z^{n+1} + (\lambda f_n - f_{n-1}) z^n + (\lambda f_{n-1} - f_{n-2}) z^{n-1} + \cdots + (\lambda f_1 - f_0) z + \lambda f_0 \\ =& - f_n z^{n+1} + \lambda f_n z^n - k f_n z^n + (k f_n - f_{n-1}) z^n + (\lambda f_{n-1} - f_{n-2}) z^{n-1} \\ & + \cdots + (\lambda f_2 - f_1) z^2 \\ & + (\lambda f_1 - \rho f_0) z - (1 - \rho) f_0 z + \lambda f_0. \end{aligned}$$

Thus, we can write

$$H(z) = -z^n \left[f_n z - \lambda f_n + k f_n \right] + \phi(z), \tag{2.5}$$

where $\phi(z) = (kf_n - f_{n-1})z^n + (\lambda f_{n-1} - f_{n-2}) z^{n-1} + \cdots + (\lambda f_2 - f_1) z^2 + (\lambda f_1 - \rho f_0) z - (1 - \rho) f_0 z + \lambda f_0$. For $|z| = 1$, we have

$$\begin{aligned} |\phi(z)| \le& |kf_n - f_{n-1}| + |\lambda f_{n-1} - f_{n-2}| + \cdots + |\lambda f_2 - f_1| + |\lambda f_1 - \rho f_0| \\ & + (1 - \rho)|f_0| + |\lambda f_0| \\ =& (kf_n - f_{n-1}) + (\lambda f_{n-1} - f_{n-2}) + \cdots + (\lambda f_2 - f_1) + (\lambda f_1 - \rho f_0) \\ & + (1 - \rho)|f_0| + |\lambda f_0| \\ =& kf_n + (\lambda - 1) \left(f_{n-1} + f_{n-2} + f_{n-3} + \cdots + f_1 \right) - \rho f_0 + (1 - \rho)|f_0| + |\lambda f_0| \\ =& kf_n + (\lambda - 1) \sum_{i=1}^{n-1} a_i - \rho f_0 + (1 - \rho)|f_0| + |\lambda f_0|. \end{aligned}$$

If $|z| = 1$, then $|\frac{1}{z}| = 1$. Therefore, we can write

$$|z^n \phi(1/z)| \le kf_n + (\lambda - 1) \sum_{i=1}^{n-1} f_i - \rho f_0 + (1 - \rho)|f_0| + |\lambda f_0|$$

Hence, by Maximum Modulus theorem for $|z| \le 1$, we observe the following

$$|z^n \phi(1/z)| \le kf_n + (\lambda - 1) \sum_{i=1}^{n-1} f_i - \rho f_0 + (1 - \rho)|f_0| + |\lambda f_0|. \tag{2.6}$$

Replacing z by $1/z$, we obtain for $|z| \geq 1$

$$|\phi(z)| \leq \left(kf_n + (\lambda - 1)\sum_{i=1}^{n-1} f_i - \rho f_0 + (1-\rho)|f_0| + |\lambda f_0|\right)|z|^n.$$

Therefore, for $|z| \geq 1$, from equation (2.6), we get

$$\begin{aligned}|H(z)| &\geq |z|^n|f_n z - \lambda f_n + kf_n)| - |\phi(z)| \\ &\geq |z^n|\left\{|f_n z - \lambda f_n + kf_n| - \left(kf_n + (\lambda - 1)\sum_{i=1}^{n-1} f_i - \rho f_0 + (1-\rho)|f_0| + |\lambda f_0|\right)\right\} \\ &> 0,\end{aligned}$$

if and only if

$$|z - \lambda + k| > \frac{1}{|f_n|}\left(kf_n + (\lambda - 1)\sum_{i=1}^{n-1} f_i - \rho f_0 + (1-\rho)|f_0| + |\lambda f_0|\right).$$

Thus, all the zeros of $H(z)$ whose modulus is greater than or equal to one lie in the disc

$$|z - \lambda + k| \leq \frac{1}{|f_n|}\left(kf_n + (\lambda - 1)\sum_{i=1}^{n-1} f_i - \rho f_0 + (1-\rho)|f_0| + |\lambda f_0|\right).$$

Therefore, all the zeros of $F(z)$ lie in the union of discs

$$\{z \in \mathbb{C} : \ |z| \leq 1\} \bigcup \left\{z \in \mathbb{C} : \ |z - \lambda + k| \leq \frac{1}{|f_n|}\left(kf_n + (\lambda - 1)\sum_{i=1}^{n-1} f_i - \rho f_0 + (1-\rho)|f_0| + |\lambda f_0|\right)\right\}.$$

□

For $\lambda = 1$, we obtain the following generalization of Theorem 1.4, which is true for any positive real number k.

Corollary 2.2. *Let $F(z) = \sum_{\nu=0}^{n} f_\nu z^\nu$ is a polynomial of degree n. If for some positive numbers k and ρ such that $\rho \leq 1$, $0 \leq \rho f_0 \leq f_1 \leq \cdots \leq f_{n-1} \leq kf_n$, then all the zeros of $F(z)$ lie in the union of the discs*

$$\{z \in \mathbb{C} \ : \ |z| \leq 1\} \bigcup \left\{z \in \mathbb{C} \ : \ |z + k - 1| \leq \left(k + (1-\rho)\frac{2f_0}{f_n}\right)\right\}.$$

For $\lambda = 1$ and $\rho = 1$, we obtain the following generalization of Theorem 1.2.

Corollary 2.3. *Let $F(z) = \sum_{\nu=0}^{n} f_\nu z^\nu$ is a polynomial of degree n such that for some real number k, $f_0 \leq f_1 \leq \cdots \leq f_{n-1} \leq kf_n$. Then all the zeros of $F(z)$ lie in the union of the discs*

$$\{z \in \mathbb{C} \ / \ |z| \leq 1\} \bigcup \left\{z \in \mathbb{C} \ : \ |z| \leq \frac{1}{|f_n|}(kf_n - f_0 + |f_0|)\right\}.$$

3. Conclusion

In this paper, we have presented a significant generalization of the classical Eneström-Kakeya theorem, broadening its scope and applicability to a wider class of polynomials. By relaxing the traditional conditions on polynomial coefficients, we have established new results that constrain the zeros of these generalized polynomials within specific geometric regions in the complex plane.

Our work began with a review of the original Eneström-Kakeya theorem, highlighting its historical importance and the foundational role it plays in understanding polynomial zero distributions. Building upon this foundation, we introduced our generalized conditions and provided rigorous proofs to demonstrate the validity of our results. Through illustrative examples, we showcased the practical implications and advantages of our generalization.

The significance of our generalization lies not only in its theoretical contributions but also in its potential applications across various mathematical and engineering disciplines. By extending the Eneström-Kakeya theorem, we open new avenues for research in multivariate polynomials, numerical methods, and real-world problem-solving in fields such as control theory and signal processing.

Looking forward, we have identified several promising directions for future research. These include further relaxation of coefficient conditions, exploration of multivariate polynomials, and development of computational tools to apply our results to large-scale problems. Additionally, investigating the connections between our generalization and other polynomial theorems could yield deeper insights and more comprehensive understandings of polynomial behavior.

In conclusion, our generalization of the Eneström-Kakeya theorem represents a meaningful advancement in the study of polynomial zeros. By expanding the boundaries of this classical result, we contribute to a richer understanding of polynomial properties and lay the groundwork for future discoveries in both theoretical and applied mathematics.

4. Future Research Work

The generalization of the Eneström-Kakeya theorem presented in this paper opens several promising avenues for future research. As we extend the classical results to encompass broader classes of polynomials and other related functions, numerous questions and potential research directions emerge. Here, we outline some key areas that warrant further exploration:

1. Extensions to Multivariate Polynomials. While our generalization primarily addresses univariate polynomials, a natural progression is to investigate analogous results for multivariate polynomials. This would involve understanding the conditions under which the zeros of multivariate polynomials with specific coefficient constraints lie within certain geometric regions in higher-dimensional spaces.

2. Relaxation of Coefficient Conditions. Our current generalization relaxes the monotonicity conditions on polynomial coefficients. Further research could explore other types of conditions, such as boundedness, periodicity, or other functional forms. This would help to identify new classes of polynomials for which similar zero-constraining properties hold.

3. Connection with Other Polynomial Inequalities. Exploring the relationships between our generalized theorem and other known polynomial inequalities, such as the Gauss-Lucas theorem or the Laguerre-Pólya class, could yield deeper insights. Understanding these connections might lead to new results or more comprehensive theorems encompassing multiple aspects of polynomial zero behavior.

4. Applications in Control Theory and Signal Processing. The zeros of polynomials play a crucial role in control theory and signal processing. Investigating how our generalized results can be applied to the stability analysis of control systems or the design of filters in signal processing could have practical implications. This would involve translating theoretical findings into practical algorithms and techniques.

5. Numerical Methods and Algorithm Development. Developing efficient numerical methods and algorithms to test the conditions of our generalized theorem on large-scale polynomial datasets would be valuable. These computational tools could facilitate the application of our theoretical results to real-world problems, especially in fields that require handling polynomials with high degrees or complex coefficient structures.

6. Exploration of Polynomials with Random Coefficients. An interesting direction is to study polynomials with coefficients that are random variables following specific distributions. Analyzing the expected distribution of zeros for such random polynomials under the framework of our generalized theorem could provide insights into probabilistic aspects of polynomial zero distributions.

7. Generalization to Entire Functions. Since entire functions can be viewed as infinite-degree polynomials, extending the Eneström-Kakeya-type results to entire functions represents a challenging yet potentially rewarding endeavor. This would involve establishing conditions on the growth rates or other properties of the coefficients of entire functions to determine the regions where their zeros lie.

8. Impact on Polynomial Root-Finding Algorithms. Investigating how our generalization influences existing polynomial root-finding algorithms or inspires the development of new algorithms could have significant computational benefits. This line of research would aim to improve the efficiency and accuracy of locating polynomial zeros based on our theoretical findings.

By pursuing these research directions, we aim to deepen the understanding of polynomial zero behavior under more general conditions and to bridge the gap between theoretical advances and practical applications. The exploration of these avenues will not only enhance the theoretical landscape of polynomial analysis but also contribute to various applied fields that rely on polynomial properties.

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