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HARTOGS-BOCHNER EXTENSION THEOREM FOR CR-FORMS ON UNBOUNDED DOMAINS

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Abstract. We establish some generalizations of the Hartogs-Bochner phenomenon for CR-forms on the boundary of unbounded domains in complex manifolds. These extension results are characterized in terms of the vanishing of the associated $\overline{\partial}$-cohomology groups with prescribed supports, considered in various function spaces, including $\mathcal{C}^\infty$, L^p_{loc}, and $\mathcal{C}^k$.

1. Introduction

A characteristic feature of the theory of functions of several complex variables is the phenomenon of holomorphic extension from a given domain to a larger one. This phenomenon was stated at the beginning of the twentieth century and was later formulated in terms of CR functions, in the 1940s. The first example illustrating this phenomenon is the well-known Hartogs theorem: any function that is holomorphic on a neighborhood of the boundary of a ball in $\mathbb{C}^n$, $n \geq 2$, admits a holomorphic extension to the interior of the ball. A stronger result is the so-called Hartogs–Bochner extension theorem, established by Bochner [5], which asserts that if Ω is a bounded domain in $\mathbb{C}^n$, $n \geq 2$, with $\mathcal{C}^\infty$ connected boundary $b\Omega$, then any function $f \in \mathcal{C}^\infty(b\Omega)$ satisfying $\bar{\partial}_b f = 0$ extends holomorphically to Ω and smoothly to $\overline{\Omega}$. Since that time many versions and generalizations of the phenomenon have appeared either for bounded domains in Stein manifolds or for unbounded domains in complex manifolds, see e.g., [7, 12, 13, 18]. For the latter kind of domains, the phenomenon is not always

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true and a geometric characterization of the obstacles has been studied. It follows that the vanishing of groups of the Dolbeault cohomology is a crucial element.

In [22] and [23], Lupacciolu linked the problem of extending CR functions to the vanishing of the $\overline{\partial}$-cohomology group of $(0,1)$-forms with supports in a paracompactifying family Φ in a non-compact connected complex manifold X, denoted by $H^{0,1}_{\Phi}(X)$. This family generalizes the phenomenon to not necessarily compact domains in X. Under the assumption $H^{0,1}_{\Phi}(X)=0$, the author proved the Hartogs-Bochner extension phenomenon for a domain D with $\mathcal{C}^{\infty}$ connected boundary in X such that $D \in \Phi$ for smooth CR-functions.

This result was later proved by Khidr and Sambou [15], by using a slightly different approach inspired by ideas of Chirka and Stout [10], in the case where Φ is a cofinal paracompactifying family of supports. In that paper, geometric conditions ensuring the vanishing of $H^{0,1}_{\Phi}(X)$ were established, in particular for Stein manifolds. Particularly, it was shown that the Hartogs-Bochner extension phenomenon for smooth CR functions is equivalent to the vanishing of $H^{0,1}_{\Phi}(X)$ (see [15, Proposition 2.6, Theorems 2.8 and 2.9]). These results have been generalized to L^2_{loc} CR-functions in [16].

For the case of CR-forms, the existence of $\overline{\partial}$-closed extension depends on the q-convexity of the domain, as shown by Kohn and Rossi [17] and by Andreotti and Hill [2,3]. In particular, Kohn and Rossi [17] established the existence of such $\overline{\partial}$-closed extensions for domains whose boundaries satisfy the condition $Z(n-q-1)$, with $q<n-1$. Their technique relies on solving the $\overline{\partial}$-Neumann problem and on boundary regularity of its solutions.

Thus, it is natural to ask whether our results in [15] and [16] can be generalized to CR-forms of higher degree. In the present paper, we address this question. More precisely, our main objective is not only to establish an isomorphism between the cohomology groups $H^{l,r}_{\Phi,L^p_{\mathrm{loc}}}(X)$ and $H^{l,r}_{\Phi,\mathrm{cur}}(X)$, but also to provide geometric characterizations of X and Φ that ensure the $\overline{\partial}$-closed extension of CR (l,r)-forms, either of class $\mathcal{C}^{\infty}$ or with L^p_{loc}-coefficients. The method we use to prove the $\overline{\partial}$-closed extension of CR-forms is essentially the same as that we employed in [15] and [16] to establish holomorphic extension of CR-functions. However, for $r>1$, there is no longer a direct equivalence between the $\overline{\partial}$-closed extension of CR $(l,r-1)$-forms and the vanishing of the $\overline{\partial}$-cohomology group of (l,r)-forms with supports in Φ. To obtain the desired equivalence, additional assumptions are required, such as cohomological invariance. In particular, we show in Theorem 1.1 below that this equivalence still holds provided we further assume that the $\overline{\partial}$-cohomology group of $(l,r-1)$-forms on the complement of the closure of the domain is trivial. We also give a geometric characterization, stating geometric assumptions for domains having $H^{l,r}_{\Phi}(X)=0$, see Theorems 1.2 and 1.3. These results generalize the Hartogs-Bochner phenomenon from functions to higher-degree CR-forms, providing a general cohomological characterization.

We begin with recalling the definitions and fixing the notations. Let X be a complex manifold of complex dimension n. Following [8], we recall that a family Φ of closed subsets of X is called a *family of supports* if it satisfies the following conditions.

(1) If $A \in \Phi$, $A_1 \subset A$, and A_1 is closed in X, then $A_1 \in \Phi$.
(2) If $A_1, A_2 \in \Phi$, then $A_1 \cup A_2 \in \Phi$.

A family of supports Φ is said to be *paracompactifying* if, in addition, it satisfies the following two conditions.

(3) If $A \in \Phi$, then A is paracompact.
(4) If $A \in \Phi$, then A admits a closed neighborhood that also belongs to Φ.

Most important examples of paracompactifying families of supports are the collection of all compact subsets of X, the family of all closed subsets of X, and, if Y is a closed subset of X, the set of all closed subsets contained in $X \setminus Y$. All these families are cofinal in the sense of Chirka and Stout [10], i.e., there exists a sequence $C_j \in \Phi$ such that each $C \in \Phi$ is contained in some C_j. To keep the nice properties of the considered topological vector spaces, we restrict our attention to paracompactifying families with cofinal sequences. In fact, by considering cohomology groups with supports in a given family, we can pass from cohomology groups theory on X to cohomology groups theory on subsets of X, with suitably related families of supports.

Let $0 \leq l \leq n$ and $1 \leq r \leq n$. Denote by $\mathcal{E}^{l,r}(X)$ the space of (l,r)-forms of class $\mathcal{C}^\infty$ on X. The associated $\bar\partial$-cohomology group is denoted by $H^{l,r}(X)$. $\mathcal{E}^{l,r}_\Phi(X)$ stands for the space of $\mathcal{C}^\infty$-forms of type (l,r) on X with supports in Φ. For $\phi \in \Phi$, we denote by $\mathcal{E}^{l,r}_\phi(X)$ the subspace of $\mathcal{E}^{l,r}(X)$ consisting of (l,r)-forms supported in ϕ. Then,

$$\mathcal{E}^{l,r}_\Phi(X) = \bigcup_{\phi\in\Phi} \mathcal{E}^{l,r}_\phi(X).$$

Each of the spaces $\mathcal{E}^{l,r}_\phi(X)$ is closed in $\mathcal{E}^{l,r}(X)$, they are then Fréchet spaces and the topology on $\mathcal{E}^{l,r}_\Phi(X)$ is the finest topology for which the inclusions $\mathcal{E}^{l,r}_{\phi_j}(X) \hookrightarrow \mathcal{E}^{l,r}_\Phi(X)$ are all continuous. In this way, $\mathcal{E}^{l,r}_\Phi(X)$ is exhibited as the strict inductive limit of the sequence $\{\mathcal{E}^{l,r}_{\phi_j}(X)\}_{j=1}^\infty$ of Fréchet spaces. The space $\mathcal{E}^{l,r}_\Phi(X)$ is a locally convex, Hausdorff, complete topological vector space. In general, it is not metrizable.

The $\bar\partial$-operator carries $\mathcal{E}^{l,r}_\Phi(X)$ continuously to $\mathcal{E}^{l,r+1}_\Phi(X)$. Let $\mathcal{Z}^{l,r}_\Phi(X)$ be the subspace of all $\bar\partial$-closed forms in $\mathcal{E}^{l,r}_\Phi(X)$. The $\bar\partial$-cohomology group with support in Φ is then defined as the quotient space

$$H^{l,r}_\Phi(X) = \mathcal{Z}^{l,r}_\Phi(X)/\bar\partial\mathcal{E}^{l,r-1}_\Phi(X).$$

This group is equipped with the quotient topology which in general is not Hausdorff. Cohomology group for (l,r)-forms with coefficients in $\mathcal{C}^k$ is defined similarly and denoted by $H^{l,r}_{\Phi,\mathcal{C}^k}(X)$. The cohomology group of forms in $\mathcal{E}^{l,r}(X)$ supported in $Y \in \Phi$ is denoted by $H^{l,r}_Y(X)$. The space of currents of bidegree (l,r) on X is denoted as usual by $\mathcal{D}'^{l,r}(X)$. We denote by $\mathcal{D}'^{l,r}_\Phi(X)$ the subspace of $\mathcal{D}'^{l,r}(X)$ consisting of currents whose supports belong to Φ. By duality $\bar\partial$ extends to a map from $\mathcal{D}'^{l,r}_{\Phi}(X)$ to $\mathcal{D}'^{l,r+1}_{\Phi}(X)$. Cohomology groups for currents with supports in Φ is defined as the quotient space

$$H^{l,r}_{\Phi,cur}(X) = \mathcal{Z}'^{l,r}_{\Phi}(X)/\bar\partial\mathcal{D}'^{l,r-1}_{\Phi}(X).$$

Let $\Omega \subset X$ be a domain with smooth boundary $b\Omega$. For $p \geq 1$, denote by $(L^p_{loc})^{l,r}(\Omega)$ the subspace of $\mathcal{D}'^{l,r}(\Omega)$ consisting of (l,r)-currents with coefficients in $L^p_{loc}(\Omega)$ endowed with the topology of L^p-convergence on compact subsets of Ω. The associated Dolbeault cohomology groups with supports in Φ is denoted by $H^{l,r}_{\Phi,L^p_{loc}}(X)$.

We now state the main results, which relate the vanishing of $\bar\partial$-cohomology groups with prescribed supports to Hartogs-Bochner type extension phenomena for CR-forms.

Theorem 1.1. *Let X be a non-compact complex manifold of complex dimension $n \geq 2$ such that $H^{l,r}(X) = 0$ for $0 \leq l \leq n$ and $1 \leq r \leq n$. Let Φ be a paracompactifying family of closed subsets of X that does not contain X. Let $D \subset X$ be a domain with $\mathcal{C}^\infty$-smooth, connected boundary so that $\overline{D} \in \Phi$. Suppose further that $H^{l,r-1}(X \setminus \overline{D}) = 0$. Then, the Hartogs-Bochner phenomenon for CR $(l, r-1)$-forms of class $\mathcal{C}^\infty$ is equivalent to the vanishing*

$$H^{l,r}_{\Phi}(X) = 0.$$

Theorem 1.2. *Let X be a Stein manifold of complex dimension $n \geq 2$. Let Φ be a paracompactifying family of closed subsets of X that does not contain X. Assume that Φ satisfies the following condition: for all $K \in \Phi$, exists $\tilde{K} \in \Phi$ with $K \subset \tilde{K}$ such that X is a generalized q-concave extension of $X \setminus \widetilde{K}$ and $H^{l,r}(X\setminus \overset{\circ}{\widetilde{K}}) = 0$. Then, $H^{l,r}_{\Phi}(X) = 0$ for all $1 \leq r \leq q-1$.*

Theorem 1.3. *Let X and Φ be as in Theorem 1.2. Let $D \subset X$ be a domain with a $\mathcal{C}^\infty$-connected boundary bD such that $\overline{D} \in \Phi$. Assume that $H^{l,r-1}(X \setminus \overline{D}) = 0$, $0 \leq l \leq n$, $1 \leq r \leq n$, and for any form $f \in \ker(\bar\partial_b) \cap \mathcal{E}^{l,r-1}(bD)$, with prescribed support in Φ, there is a form $F \in \ker(\bar\partial) \cap \mathcal{E}^{l,r-1}(\overline{D})$ such that $F_{|bD} = f$, then $H^{l,r}_{\Phi}(X) = 0$.*

Theorem 1.4 (Hartogs-Bochner extension theorem)**.** *Let X and Φ be as in Theorem 1.2. Assume moreover that X is a generalized q-concave extension of $X \setminus K$ for all $K \in \Phi$. Then, the Hartogs-Bochner extension phenomenon holds true for $(l, r-1)$-forms with L^p_{loc}-coefficients for all $1 \leq r \leq q-1$.*

2. Isomorphism Results

We know that there is a natural isomorphism between $H^{l,r}(X)$ and $H^{l,r}_{cur}(X)$ called the Dolbeault isomorphism. What can be said about the natural map between $H^{l,r}_{\Phi,L^p_{loc}}(X)$ and $H^{l,r}_{\Phi,cur}(X)$? It was indicated in [21] that the cohomology groups $H^{l,r}_{\Phi}(X)$, $H^{l,r}_{\Phi,L^p_{loc}}(X)$, $H^{l,r}_{\Phi,cur}(X)$ and $H^{l,r}_{\Phi,\mathcal{C}^k}$ are isomorphic. Our first objective is to give a clear proof of this result, by using Chirka's theory [9].

Theorem 2.1. *Let X be a complex manifold of complex dimension n. Then, for all $0 \le l \le n$, $1 \le r \le n-1$, $1 \le p \le +\infty$, the natural map*

$$H^{l,r}_{\Phi,L^p_{\mathrm{loc}}}(X) \to H^{l,r}_{\Phi,cur}(X)$$

is an isomorphism.

Proof. For each $\varepsilon > 0$, Chirka [9] defined the following linear regularizing operators

$$R_\varepsilon : \mathcal{D}'^{l,r}(X) \to \mathcal{E}^{l,r}(X), \quad A_\varepsilon : \mathcal{D}'^{l,r}(X) \to \mathcal{D}'^{l,r-1}(X), \quad 0 \le l \le n,\, 1 \le r \le n,$$

with the following properties.

(1) The supports of $R_\varepsilon T$ and $A_\varepsilon T$ are contained in some ε-neighborhoods of the support of T for all $T \in \mathcal{D}'^{p,q}(X)$.
(2) $R_\varepsilon T \to T$ and $A_\varepsilon T \to 0$ weakly as $\varepsilon \to 0$.
(3) $R_\varepsilon T \in \mathcal{E}^{l,r}(X)$. A_ε maps continuously $(L^p_{loc}(X))^{l,r}$ into $(L^p_{\mathrm{loc}}(X))^{l,r-1}$ and $\mathcal{E}^{l,r}(X)$ into $\mathcal{E}^{l,r-1}(X)$.
(4) For all $T \in \mathcal{D}'^{p,q}(X)$, we have the $\bar\partial$-homotopy relation

$$T - R_\varepsilon T = \bar\partial A_\varepsilon T + A_\varepsilon \bar\partial T. \tag{2.1}$$

(5) $\bar\partial R_\varepsilon T = R_\varepsilon \bar\partial T$.

We proceed now to show that the natural map is an isomorphism.

Injectivity. Let $[h] \in H^{l,r}_{\Phi,L^p_{\mathrm{loc}}}(X)$ be such that its range belongs to the null class of $H^{l,r}_{\Phi,cur}(X)$. Then there is a $(l, r-1)$-current S supported in Φ so that $\bar\partial S = h$. From (2.1) and by continuity of A_ε on $(L^p_{\mathrm{loc}})^{l,r}(X)$, we deduce that $h = \bar\partial(R_\varepsilon S + A_\varepsilon h)$ and $(R_\varepsilon S + A_\varepsilon h) \in (L^p_{\mathrm{loc}})^{l,r}(X)$ with support in an ε-neighborhood of the support of h, for $\varepsilon > 0$ small enough. By definition, the support of $(R_\varepsilon S + A_\varepsilon h)$ belongs to Φ. Therefore, $[h] = [0]$ in $H^{l,r}_{\Phi,L^p_{\mathrm{loc}}}(X)$. This proves the injectivity of the natural map.

Surjectivity. Let $[h] \in H^{l,r}_{\Phi,cur}(X)$, we have $\bar\partial h = 0$. Then, by (2.1), we have $h - R_\varepsilon h = \bar\partial A_\varepsilon h$, where $R_\varepsilon h \in \mathcal{E}^{l,r}(X) \subset (L^p_{\mathrm{loc}})^{(l,r)}(X)$, $A_\varepsilon h \in \mathcal{D}'^{l,r-1}(X)$ and their supports are in some ε-neighborhood of the support of h. For $\varepsilon > 0$ small enough, the supports of $R_\varepsilon h$ and of $A_\varepsilon h$ are in Φ. Then, $[h] = [R_\varepsilon h]$ in $H^{l,r}_{\Phi,L^p_{\mathrm{loc}}}(X)$. Thus, the natural map is surjective. □

Remark 2.1. Analogously, one can show that $H^{l,r}_{\Phi,cur}(X)$ is isomorphic to $H^{l,r}_{\Phi,\mathcal{C}^k}(X)$ using $\mathcal{C}^k$-estimates for $\bar\partial$, see e.g., [4] or [14].

For an open set $\Omega \subset X$, the space $\check{H}^{l,r}(\Omega)$ denotes the $\bar\partial$-cohomology group of extendable (l, r)-currents on Ω.

Corollary 2.1. *Let X be a complex manifold of complex dimension $n \ge 2$. Let $D \subset X$ be a domain so that $\overline{D} \in \Phi$. Assume for all $0 \le l \le n$ and $1 \le r \le n-1$ that $H^{l,r}_\Phi(X) = 0$ and $H^{l,r-1}(X \setminus D) = 0$. Then,*

$$H^{l,r}_{\overline{D}}(X) = 0, \quad H^{l,r}_{\overline{D},L^p_{\mathrm{loc}}}(X) = 0 \quad \textit{and} \quad H^{l,r}_{\overline{D},\mathcal{C}^k}(X) = 0.$$

In addition, if $D \subset\subset X$, $\overset{\circ}{\overline{D}} = D$, *then* $\check{H}^{l,r}(D) = 0$.

Proof. Let $f \in \mathcal{E}^{l,r}_{\overline{D}}(X)$ with $\bar\partial f = 0$. Since $H^{l,r}_{\Phi}(X) = 0$, there is a form $g \in \mathcal{E}^{l,r-1}_{\Phi}(X)$ such that $\bar\partial g = f$. Then, $\mathrm{supp}(f) \subset \mathrm{supp}(g)$ and $\bar\partial g_{|X\setminus \mathrm{supp}(g)} = 0$. If $\mathrm{supp}(g) \subseteq \overline{D}$, then $H^{l,r}_{\overline{D}}(X) = 0$. Otherwise, we have $\bar\partial g_{|X\setminus\overline{D}} = 0$ leads to $\bar\partial g_{|X\setminus D} = 0$. If $r = 1$, we have $H^{l,1}_{\overline{D}}(X) = 0$ by [16, Corollary 1]. For $r > 1$, since $H^{l,r-1}(X \setminus D) = 0$, then there exists a form $u \in \mathcal{E}^{l,r-2}(X \setminus D)$ such that $\bar\partial u = g$. Extend u to $\hat u$ on D. Thereby,

$$\tilde g = \begin{cases} \bar\partial \hat u, & \text{on } D, \\ g, & \text{on } X \setminus D, \end{cases}$$

is $\bar\partial$-closed on D and the form $\hat g = g - \tilde g$ is supported in $\overline{D}$. We further have $\bar\partial \hat g = f$ in X, and hence $H^{l,r}_{\overline{D}}(X) = 0$.

By analogy, we can show that

$$H^{l,r}_{\overline{D},L^p_{\mathrm{loc}}}(X) = 0 \quad \text{and} \quad H^{l,r}_{\overline{D},\mathcal{C}^k}(X) = 0.$$

The additional statement follows as in [24]. □

3. Extension of CR-differential Forms

In this section, we give the proof of Theorem 1.1, and then we give some consequences. We first recall the notion of CR-forms.

Definition 3.1. Let S be a real hypersurface of an n-dimensional complex manifold X. Let $l \in [0, n]$ and $r \in [0, n-1]$. A form $f \in \mathcal{E}^{l,r}(S)$ is called CR-form if $r = n-1$ or if $r \in [0, n-2]$ and

$$\int_S f \wedge \bar\partial \varphi = 0,$$

for all $(n-l, n-r-2)$-form φ of class $\mathcal{C}^\infty(X)$ such that $S \cap \mathrm{supp}(\varphi)$ is compact. This is equivalent to $\bar\partial_b f = 0$ on S (see [1]).

Remark 3.1. Thanks to [6, p. 147, Theorem 2], Definition 3.1 is also equivalent to saying that a form $f \in \mathcal{E}^{l,r}(S)$ is called CR if it admits a $\mathcal{C}^\infty$ extension $\tilde f$ to a neighborhood of S such that $\bar\partial \tilde f$ vanishes to infinite order on S.

Definition 3.2. Let $D \subset X$ be a domain in a complex manifold X. The pair (X, D) is said to have the $\mathcal{C}^\infty$-Hartogs-Bochner extension property for (l, r)-forms if: for any CR-form $f \in \mathcal{E}^{l,r}(bD)$ there exists a form F in $\mathcal{E}^{l,r}(\overline{D})$, $\bar\partial$-closed in D, and $F_{|bD} = f$.

Examples of manifolds enjoying the Hartogs-Bochner extension property are $(n-1)$-complete manifolds and $(n-1)$-strictly-hyperconvex Kähler manifolds, $n \geq 2$, see e.g., [2,3] and [11].

We now turn to the proof of Theorem 1.1.

Proof of Theorem 1.1. Let f be a CR $(l, r-1)$-form of class $\mathcal{C}^\infty$ on bD. By definition, f can be extended to a $\mathcal{C}^\infty$ $(l, r-1)$-form $\tilde{f}$ on X such that $\bar{\partial}\tilde{f}$ vanishes to infinite order on bD. Let $\chi_{\overline{D}}$ be a characteristic function of $\overline{D}$ and set $g = \chi_{\overline{D}}\bar{\partial}\tilde{f}$. Then, g is a $\bar{\partial}$-closed (l, r)-form of class $\mathcal{C}^\infty$ with support in $\overline{D}$. Suppose $H^{l,r}_\Phi(X) = 0$, since $\overline{D} \in \Phi$, there is a $(l, r-1)$-form u of class $\mathcal{C}^\infty$ on X such that $\bar{\partial}u = g$ and $K = \mathrm{supp}(u) \in \Phi$. We have $\overline{D} \subset K$ which leads to $X \setminus K \subset X \setminus \overline{D}$. We have $\bar{\partial}u_{|X\setminus\overline{D}} = 0$. For $r = 1$, f is a CR function of class $\mathcal{C}^\infty$ on bD and we assume that its support is not compact, then according to [15, Proposition 2.6], there exists a function $F \in \mathcal{C}^\infty(\overline{D}) \cap \mathcal{O}(D)$ such that $F_{|bD} = f$.

For $r > 1$, since $H^{l,r-1}(X \setminus \overline{D}) = 0$, then there exists a form h in $\mathcal{E}^{l,r-2}(X \setminus \overline{D})$ so that $\bar{\partial}h = u$. Let $\hat{h}$ be an extension of class $\mathcal{C}^\infty$ of h to $\overline{D}$. Put $\tilde{u} = u - \bar{\partial}\hat{h}$, we then have $\tilde{u}_{|bD} = 0$. Take $F = \tilde{f} - \tilde{u}$. It is a $(l, r-1)$-form, $\bar{\partial}$-closed on D, of class $\mathcal{C}^\infty$ on $\overline{D}$, and $F_{|bD} = f$. The smooth extension therefore takes place for any unbounded domain D with boundary of class $\mathcal{C}^\infty$ such that $\overline{D} \in \Phi$.

Suppose now that the smooth extension takes place for any unbounded domain D with boundary of class $\mathcal{C}^\infty$ such that $\overline{D} \in \Phi$. Let $[h] \in H^{l,r}_\Phi(X)$. Since $H^{l,r}(X) = 0$, by assumption, there is then a form $g \in \mathcal{E}^{l,r-1}(X)$ such that $\bar{\partial}g = h$. As $\mathrm{supp}(h) \in \Phi$, by definition, there is an open neighborhood, denoted also by D, of $\mathrm{supp}(h)$ with smooth boundary bD of class $\mathcal{C}^\infty$ such that $\overline{D} \in \Phi$. Since $\mathrm{supp}(h) \subset \overline{D}$, we have $\bar{\partial}g_{X\setminus\overline{D}} = h_{X\setminus\overline{D}} = 0$ and hence $\bar{\partial}_b g = 0$ on bD. Therefore, by hypothesis, there exists a $(l, r-1)$-form G of class $\mathcal{C}^\infty$ on $\overline{D}$, $\bar{\partial}$-closed in D, such that $G_{|bD} = g$. For $r = 1$, due to [15, Theorem 2.9], we get $H^{l,1}_\Phi(X) = 0$. For $r > 1$, since $H^{l,r-1}(X \setminus \overline{D}) = 0$, there exists a form $u \in \mathcal{E}^{l,r-2}(X \setminus \overline{D})$ such that $\bar{\partial}u = g$ in $X \setminus \overline{D}$. So, the form

$$\tilde{g} = \begin{cases} G, & \text{on } \overline{D}, \\ \bar{\partial}u, & \text{on } X \setminus \overline{D}, \end{cases}$$

is a $\bar{\partial}$-closed $(l, r-1)$-form on X and the form $\hat{g} = g - \tilde{g}$ is supported in $\overline{D}$. We then have $\bar{\partial}\hat{g} = \bar{\partial}g = h$ on X. This means that $[h] = 0$ in $H^{l,r}_\Phi(X)$. We therefore have $H^{l,r}_\Phi(X) = 0$. □

As a consequence of Theorem 1.1, we have the following corollary.

Corollary 3.1. *Under assumptions of Theorem* 1.1*, the Hartogs-Bochner extension phenomenon for CR-smooth forms is equivalent to*

$$H^{l,r}_{\Phi,\mathcal{C}^k}(X) = H^{l,r}_{\Phi,L^p_{\mathrm{loc}}}(X) = H^{l,r}_{\Phi,cur}(X) = 0.$$

Proof. From Theorem 1.1, we see that the Hartogs-Bochner extension phenomenon for CR-forms is equivalent to $H^{l,r}_\Phi(X) = 0$. It follows from Proposition 1.2 in [20] that the cohomology groups $H^{l,r}_\Phi(X)$ and $H^{l,r}_{\Phi,L^p_{loc}}(X)$ are isomorphic to each other. In addition, from Theorem 2.1, the cohomology groups $H^{l,r}_{\Phi,\mathcal{C}^k}(X)$, $H^{l,r}_{\Phi,L^p_{\mathrm{loc}}}(X)$ and $H^{l,r}_{\Phi,cur}(X)$ are isomorphic. Therefore, the Hartogs-Bochner phenomenon for CR-forms is equivalent to $H^{l,r}_{\Phi,\mathcal{C}^k}(X) = H^{l,r}_{\Phi,L^p_{\mathrm{loc}}}(X) = H^{l,r}_{\Phi,cur}(X) = 0$. □

Remark 3.2. Theorem 1.1 remains valid if X is a Stein manifold.

Proceeding as in the proof of Theorem 1.1 and using auxiliary results from [16] and [20], we state the $L^p_{loc}(X)$-version of Theorem 1.1 as follows.

Theorem 3.1. *Let X, Φ and D be as in Theorem* 1.1*. Assume that $H^{l,r-1}_{L^p_{\mathrm{loc}}}(X \setminus \overline{D}) = 0$ for all $0 \leq l \leq n$, $1 \leq r \leq n$, $p \geq 1$. Then, the Hartogs-Bochner extension phenomenon for CR-forms with coefficients in $L^p_{\mathrm{loc}}(bD)$ is equivalent to $H^{l,r}_{\Phi,L^p_{\mathrm{loc}}}(X) = 0$.*

In a similar manner, we have a $\mathcal{C}^k$-version of Theorem 1.1.

Theorem 3.2. *Let X, Φ, and D be as in Theorem* 1.1*. Suppose that $H^{l,r-1}_{\mathcal{C}^k}(X\setminus\overline{D}) = 0$, $0 \leq l \leq n$, $1 \leq r \leq n$ and $k \geq 0$. Then, the Hartogs-Bochner extension phenomenon for CR-forms with coefficients in $\mathcal{C}^k(bD)$ is equivalent to $H^{l,r}_{\Phi,\mathcal{C}^k}(X) = 0$.*

4. Vanishing Conditions of $H^{l,r}_{\Phi}(X)$

If X is a Stein manifold, one can ask whether there exists a paracompactifying family Φ of closed subsets of X that does not contain X such that $H^{l,r}_{\Phi}(X) = 0$?.The answer is yes, thanks to the following example.

Example 4.1 (see [24]). Let X be a Stein manifold and ψ be a strictly q-convex exhaustion function. Take Φ a paracompactifying family containing all the closed subsets $D_c = \{x \in X \mid \psi(z) < c, c \in \mathbb{R}\}$ without X, then $H^{l,r}_{D_c}(X) = 0$ for all $1 \leq r \leq q$.

In this example, the elements of Φ are either the closure of the sublevel $\overline{D}_c$ or their closed subsets. If f is a $\mathcal{C}^\infty$ differential (l,r)-form with support in $\overline{D} \in \Phi$, then f has a support in some $\overline{D}_c$. As X is Stein, there is a $\mathcal{C}^\infty$ a $(l, r-1)$-form g on X such that $\overline{\partial} g = f$. The form g is $\overline{\partial}$-closed on $X \setminus \overline{D}_c$ and by Hartogs' phenomenon it is the restriction of a $\overline{\partial}$-closed-form $\tilde{g}$ on X. The form $g - \tilde{g}$ is then $\mathcal{C}^\infty$ and moreover solves the $\overline{\partial}$-equation with support in $\overline{D}_c$, i.e., $\mathrm{supp}(g - \tilde{g}) \in \Phi$. This proves the vanishing of the cohomology group $H^{l,r}_{\Phi}(X)$.

Proof of Theorem 1.2. Let $[f] \in H^{l,r}_{\Phi}(X)$, then $\bar{\partial} f = 0$, so there is $u \in \mathcal{E}^{l,r-1}(X)$ such that $\bar{\partial} u = f$. Let $K = \mathrm{supp}(f)$, then there exists $\tilde{K} \in \Phi$ with $K \subset \tilde{K}$ such that X is a generalized q-concave extension of $X \setminus \tilde{K}$. According to [19], the restriction map $H^{l,r}(X) \to H^{l,r}(X \setminus \tilde{K})$ is an isomorphism for $0 \leq r \leq q-1$. Since $\bar{\partial} u_{|X\setminus\tilde{K}} = 0$, we have $\bar{\partial} u_{|X\setminus\overset{\circ}{\tilde{K}}} = 0$. If $r = 1$, then thanks to Theorem 3.2 in [15], we have $H^{l,1}_{\Phi}(X) = 0$. If $r > 1$, since $H^{l,r-1}(X\setminus \overset{\circ}{\tilde{K}}) = 0$, then there exists a $(l, r-2)$-form h of class $\mathcal{C}^\infty$ defined on $X\setminus \overset{\circ}{\tilde{K}}$ such that $\bar{\partial} h = u$. Let $\hat{h}$ be an extension of class $\mathcal{C}^\infty$ of h to $\overset{\circ}{\tilde{K}}$. Let $\tilde{u} = u - \bar{\partial}\hat{h}$. This is a $(l, r-1)$-form of class $\mathcal{C}^\infty$ on X and $\mathrm{supp}(\tilde{u}) \subset \tilde{K}$, so $\mathrm{supp}(\tilde{u}) \in \Phi$. Also, $\bar{\partial}\tilde{u} = f$. Thus, $H^{l,r}_{\Phi}(X) = 0$ for all $1 \leq r \leq q-1$. □

Corollary 4.1. *Under assumptions of Theorem* 1.2, *we have*

$$H^{l,r}_{\Phi,\mathcal{C}^k}(X) = H^{l,r}_{\Phi,L^p_{\text{loc}}}(X) = H^{l,r}_{\Phi,cur}(X) = 0.$$

Proof. By Theorem 2.1, the cohomology groups $H^{l,r}_{\Phi}(X)$, $H^{l,r}_{\Phi,\mathcal{C}^k}(X)$, $H^{l,r}_{\Phi,L^p_{\text{loc}}}(X)$ and $H^{l,r}_{\Phi,cur}(X)$ are isomorphic. Further, by Theorem 1.2, we have $H^{l,r}_{\Phi}(X) = 0$. Then, we obtain $H^{l,r}_{\Phi,\mathcal{C}^k}(X) = H^{l,r}_{\Phi,L^p_{\text{loc}}}(X) = H^{l,r}_{\Phi,cur}(X) = 0$. □

Proof of Theorem 1.3. Let $[f] \in H^{l,r}_{\Phi}(X)$, then $[f] \in H^{l,r}(X) = 0$. This means that there is a form $g \in \mathcal{E}^{l,r-1}(X)$ that solves the equation $\bar\partial g = f$. Let $K = \text{supp}(f)$, we have $\bar\partial g_{X\setminus K} = 0$. By the definition of Φ, there exists D_1 an open neighborhood of K such that $\overline{D}_1 \in \Phi$. Choose D_2 an unbounded domain with boundary of class $\mathcal{C}^\infty$ such that $\overline{D}_2 \in \Phi$ and $\overline{D}_1 \subset D_2$. We have $X \setminus \overline{D}_2 \neq \emptyset$ because X does not belong to Φ. We have $\bar\partial g_{|X\setminus\overline{D}_2} = 0$ because $X \setminus \overline{D}_2 \subset X \setminus K$ and therefore $\bar\partial_b g = 0$ on bD_2. By hypothesis, there is $G \in \mathcal{E}^{l,r-1}(\overline{D}_2)$, $\bar\partial$-closed in D_2 such that $G_{|bD_2} = g$. For $r = 1$, f is a CR function of class $\mathcal{C}^\infty$ on bD whose support is not compact, there is a function $F \in \mathcal{O}(D) \cap \mathcal{C}^\infty(\overline{D})$ such that $F_{|bD} = f$. Then, according to Proposition 3.4 in [15], we have $H^{l,1}_{\Phi}(X) = 0$. For $r > 1$, since $H^{l,r-1}(X \setminus \overline{D}_2) = 0$, there exists a $(l, r-2)$-form u of class $\mathcal{C}^\infty$ defined on $X \setminus \overline{D}_2$ such that $\bar\partial u = g$. Let us pose

$$\hat{g} = \begin{cases} \bar\partial u, & \text{on } X \setminus \overline{D}_2, \\ G, & \text{on } \overline{D}_2, \end{cases}$$

and $\tilde g = g - \hat g$. Thus, $\tilde g$ is a $(l, r-1)$-form of class $\mathcal{C}^\infty$ with support in $\overline{D}_2 \in \Phi$. Moreover $\bar\partial \tilde g = f$ on X. Therefore we have $H^{l,r}_{\Phi}(X) = 0$. □

Proof of Theorem 1.4. By virtue of Theorem 1.2, we have $H^{l,r}_{\Phi}(X) = 0$, and hence $H^{l,r}_{\Phi,L^p_{\text{loc}}}(X) = 0$. Let $D \subset X$ be a domain with connected boundary such as $\overline{D} \in \Phi$. Let f be a CR-form of bidegree $(l, r-1)$ with coefficients in $L^p_{\text{loc}}(bD)$. Let $\tilde f$ be an extension of f with coefficients in $L^p_{\text{loc}}(\overline{D})$ such that $\overline{\partial}\tilde f$ is a form of bidegree (l, r) with coefficients also in $L^p_{\text{loc}}(X)$. Let us pose $g = \chi(\overline{D})\bar\partial\tilde f$, g is a (l, r)-form with coefficients in $L^p_{\text{loc}}(X)$ and with support $\overline{D} \in \Phi$ and $\bar\partial g = 0$. Since we have $H^{l,r}_{\Phi,L^p_{\text{loc}}}(X) = 0$, there exists a $(l, r-1)$-form η with coefficients in $L^p_{\text{loc}}(X)$ such that $\bar\partial\eta = g$ and $\text{supp}(\eta) \in \Phi$. We have $\bar\partial\eta_{|X\setminus\overline{D}} = 0$. For $r = 1$, f is a CR function with coefficients in $L^p_{loc}(bD)$ and we assume that its support is not compact, by Theorem 3 in [16], we can find a function $F \in \mathcal{O}(D) \cap L^p_{\text{loc}}(\overline{D})$ such that $F_{|bD} = f$. Suppose now $r > 1$. From Proposition 1.1 in [20], $H^{l,r}(X)$ is isomorphic to $H^{l,r}_{L^p_{\text{loc}}}(X)$, therefore $H^{l,r}(X\setminus\overline{D})$ is isomorphic to $H^{l,r}_{L^p_{\text{loc}}}(X\setminus\overline{D})$. Since X is a generalized q-concave extension of $X \setminus \overline{D}$, then the restriction map $H^{l,r}_{L^p_{\text{loc}}}(X) \to H^{l,r}_{L^p_{\text{loc}}}(X \setminus \overline{D})$ is an isomorphism for $0 \le r \le q-1$ and hence $H^{l,r}_{\Phi,L^p_{\text{loc}}}(X\setminus\overline{D}) = 0$. Since $\bar\partial\eta_{|X\setminus\overline{D}} = 0$, there is a $(l, r-2)$-form

λ with coefficients in $L^p_{\text{loc}}(X \setminus \overline{D})$ such that $\bar{\partial}\lambda = \eta$. Let

$$\hat{\lambda} = \begin{cases} \lambda, & \text{on } X \setminus \overline{D}, \\ 0, & \text{on } \overline{D}, \end{cases}$$

and $\tilde{\eta} = \eta - \bar{\partial}\hat{\lambda}$, we have $\tilde{\eta}_{|bD} = 0$. Let $F = \tilde{f} - \tilde{\eta}$. It is a $(l, r-1)$-form with coefficients in $L^p_{loc}(\overline{D})$, $\bar{\partial}$-closed on D and $F_{|bD} = f$. □

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