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FIXED POINT THEOREMS UNDER ω-DISTANCE FUNCTIONS AND APPLICATIONS TO NONLINEAR INTEGRAL AND FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we utilize the family $\mathfrak{F}$ and the notion of ω-distance in an ordered $\mathcal{G}$-metric space and introduce (F,ω)-contractions in order to derive some fixed point results. We also discuss the problems of Ulam-Hyers stability, well-posedness and limit shadowing property. In order to illustrate the use of our results, we apply them to nonlinear integral equations, as well as to some three-point fractional integral boundary value problems, both with numerical examples.

1. Introduction

There are several generalizations of metric spaces that proved themselves useful in treating fixed point problems. One of them are $\mathcal{G}$-metric spaces introduced by Mustafa and Sims in 2006 [3] and there were a lot of results for mappings acting in such spaces that were obtained by several authors. Among them are some fixed point theorems achieved by Bose et al. [1] with the help of so-called ω-distance (see also [7, 9]).

Investigation of fixed point properties for monotone mappings acting in spaces with additional ordered structure have been done by a lot of researchers since the first results achieved by Ran and Reurings in 2004 [6].

Wardowski introduced in [11] a new kind of contractive conditions which he called F-contractions that were using a special family $\mathfrak{F}$ of auxiliary functions. They are in some cases weaker than standard ones, so better results can be obtained.

Key words and phrases. Fixed point, partially ordered set, $\mathcal{G}$-metric space, ω-distance function, Ulam-Hyers stability, fractional integral boundary value problem.

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In this paper, we utilize the family $\mathfrak{F}$ and the notion of ω-distance in an ordered $\mathcal{G}$-metric space and introduce (F,ω)-contractions in order to derive some fixed point results. We note that it was shown in [1] that this procedure produces better results than the ones obtained by known methods in standard metric spaces. In particular (see [1, Remark 2]), the method used in the papers [2, 8] of reducing such problems to problems in quasi-metric spaces may not be applicable. We also discuss the problems of Ulam-Hyers stability, well-posedness and limit shadowing property. We present an example showing the strength of our approach.

In order to illustrate the use of our results, we apply them in the last section to nonlinear integral equations, as well as to a three-point fractional integral boundary value problem. We illustrate both numerically through examples.

2. Preliminaries

For more details on the following definitions and results, we refer the reader to [3].

Definition 2.1. [3] Let $\mathcal{X}$ be a nonempty set and let $\mathcal{G}:\mathcal{X}\times\mathcal{X}\times\mathcal{X}\to\mathbb{R}^+$ be a function satisfying the following properties:

(G1) $\mathcal{G}(x,y,z)=0$ if $x=y=z$;
(G2) $0<\mathcal{G}(x,x,y)$, for all $x,y\in\mathcal{X}$ with $x\neq y$;
(G3) $\mathcal{G}(x,x,y)\leq\mathcal{G}(x,y,z)$, for all $x,y,z\in\mathcal{X}$ with $z\neq y$;
(G4) $\mathcal{G}(x,y,z)=\mathcal{G}(x,z,y)=\mathcal{G}(y,z,x)=\cdots$, (symmetry in all three variables);
(G5) $\mathcal{G}(x,y,z)\leq\mathcal{G}(x,a,a)+\mathcal{G}(a,y,z)$, for all $x,y,z,a\in\mathcal{X}$ (rectangle inequality).

Then the function $\mathcal{G}$ is called a $\mathcal{G}$-metric on $\mathcal{X}$ and the pair $(\mathcal{X},\mathcal{G})$ is called a $\mathcal{G}$-metric space.

Definition 2.2. [3] Let $(\mathcal{X},\mathcal{G})$ be a $\mathcal{G}$-metric space and let $\{x_n\}$ be a sequence of points in $\mathcal{X}$.

(a) A point $x\in\mathcal{X}$ is said to be the limit of $\{x_n\}$ if $\lim_{n,m\to\infty}\mathcal{G}(x,x_n,x_m)=0$, and one says that the sequence $\{x_n\}$ is $\mathcal{G}$-convergent to x.
(b) The sequence $\{x_n\}$ is said to be a $\mathcal{G}$-Cauchy sequence if, for every $\varepsilon>0$, there is a positive integer $\mathbb{N}$ such that $\mathcal{G}(x_n,x_m,x_l)<\varepsilon$, for all $n,m,l\geq\mathbb{N}$, that is, if $\mathcal{G}(x_n,x_m,x_l)\to 0$, as $n,m,l\to\infty$.
(c) $(\mathcal{X},\mathcal{G})$ is said to be $\mathcal{G}$-complete (or a complete $\mathcal{G}$-metric space) if every $\mathcal{G}$-Cauchy sequence in $(\mathcal{X},\mathcal{G})$ is $\mathcal{G}$-convergent in $\mathcal{X}$.

Thus, if $x_n\to x$ in a $\mathcal{G}$-metric space $(\mathcal{X},\mathcal{G})$, then for any $\varepsilon>0$, there exists a positive integer $\mathbb{N}$ such that $\mathcal{G}(x,x_n,x_m)<\varepsilon$, for all $n,m\geq\mathbb{N}$. It was shown in [3] that the $\mathcal{G}$-metric induces a Hausdorff topology and that the convergence, as described in the above definition, is relative to this topology. The topology being Hausdorff, a sequence can converge to at most one point.

We now recall the notion of ω-distance on $\mathcal{G}$-metric space.

Definition 2.3. [1] Let $(\mathcal{X},\mathcal{G})$ be a $\mathcal{G}-$metric space. Then a function $\omega:\mathcal{X}\times\mathcal{X}\times\mathcal{X}\to[0,\infty)$ is called an ω-distance on $\mathcal{X}$ if it satisfies the following properties.

(ω1) (a) $\omega(x, y, z) \leq \omega(x, a, a) + \omega(a, y, z)$;
(b) $\omega(x, y, z) \leq \omega(x, y, a) + \omega(a, a, z)$;
(c) $\omega(x, y, z) \leq \omega(x, a, z) + \omega(a, y, a)$, for all $x, y, z, a \in \mathcal{X}$.
(ω2) For each $\epsilon > 0$ there exists $\delta > 0$ such that $\omega(x, y, z) < \delta$ implies $\mathcal{G}(x, y, z) < \epsilon$ and vice versa.

Lemma 2.1. [1] *If ω is an ω-distance on $(\mathcal{X}, \mathcal{G})$, then for any $x, y \in \mathcal{X}$, $\omega(\cdot, x, y)$, $\omega(x, y, \cdot)$, $\omega(x, \cdot, y) : \mathcal{X} \to [0, \infty)$ are $\mathcal{G}$-continuous.*

Lemma 2.2. [1] *If for some sequences $(x_n), (y_n), (z_n)$ in $\mathcal{X}$, $\lim_{n\to\infty} \omega(x_n, y_n, z_n) = 0$ then $\lim_{n\to\infty} \omega(\pi(x_n, y_n, z_n)) = 0$ for any permutation π of (x_n, y_n, z_n).*

Example 2.1. [1] Let $\mathcal{X} = \mathbb{R}$ be endowed with usual metric. Then $\mathcal{G}(x, y, z) = \frac{1}{3}\Big\{|x - y| + |y - z| + |z - x|\Big\}$ is a $\mathcal{G}$-metric on $\mathcal{X}$ and $\omega(x, y, z) = \frac{1}{2}\Big\{|x - y| + |z - x|\Big\}$ is an ω-distance on $\mathcal{X}$, which is not a $\mathcal{G}$-metric on $\mathcal{X}$.

Lemma 2.3. [1] *A sequence $\{x_n\}$ in $(\mathcal{X}, \mathcal{G})$ is $\mathcal{G}$-Cauchy if and only if for each $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that $\omega(x_m, x_m, x_n) < \epsilon$, for all $n \geq m \geq N_\epsilon$.*

Wardowski introduced in [11] a new type of contractions which he called F-contractions. In Wardowski's approach, the following set of functions is used.

Definition 2.4. Denote by $\mathfrak{F}$ the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ with the following properties:
(F1) F is strictly increasing;
(F2) for each sequence $\{t_n\}$ of positive numbers,

$$\lim_{n\to\infty} t_n = 0 \text{ if and only if } \lim_{n\to\infty} F(t_n) = -\infty;$$

(F3) there exists $k \in (0, 1)$ such that $\lim_{t\to 0^+} t^k F(t) = 0$.

Example 2.2. [11] Let $F_i : \mathbb{R}^+ \to \mathbb{R}$, $i \in \{1, 2, 3, 4\}$, be defined by $F_1(t) = \ln t$, $F_2(t) = t + \ln t$, $F_3(t) = -1/\sqrt{t}$, $F_4(t) = \ln(t^2 + t)$. Then each F_i satisfies the properties (F1)-(F3).

3. Main Results

Let $\mathcal{X}$ be a nonempty set. The triple $(\mathcal{X}, \mathcal{G}, \preceq)$ will be called an ordered $\mathcal{G}$-metric space if:
(i) $(\mathcal{X}, \mathcal{G})$ is a $\mathcal{G}$-metric space, and
(ii) $(\mathcal{X}, \preceq)$ is a partially ordered set.

Definition 3.1. A partially ordered $\mathcal{G}$-metric space $(\mathcal{X}, \mathcal{G}, \preceq)$ is said to be regular if the following hypotheses hold:
(i) if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$ as $n \to \infty$, then $x_n \preceq x$, for all $n \in \mathbb{N}$;
(ii) if a non-increasing sequence $\{y_n\}$ is such that $y_n \to y$ as $n \to \infty$, then $y_n \succeq y$, for all $n \in \mathbb{N}$.

3.1. **Result-1.**

Definition 3.2. Let $(\mathcal{X}, \mathcal{G}, \preceq)$ be an ordered $\mathcal{G}$-metric space and ω be an ω-distance on $\mathcal{X}$. A self-mapping T on $\mathcal{X}$ is called an (F, ω)-contraction of type-I, if there exists $F \in \mathfrak{F}$ and $\tau \in \mathbb{R}^+$ such that

$$\begin{aligned} &x, y, z \in \mathcal{X} \text{ with } z \preceq y \preceq x, \text{ and } \omega(Tx, Ty, Tz) > 0 \text{ implies} \\ &\tau + F(\omega(Tx, Ty, Tz)) \leq F(\theta(x, y, z)), \end{aligned} \tag{3.1}$$

where

$$\theta(x, y, z) = \max \left\{ \begin{array}{c} \omega(x, y, z), \omega(x, y, Tx), \omega(x, Tx, z), \\ \omega(x, Tx, Tx), \omega(y, Ty, Ty), \omega(z, Tz, Tz), \\ \frac{1}{2}[\omega(x, Ty, Tz) + \omega(Tx, y, z)] \end{array} \right\}. \tag{3.2}$$

We denote by $\Lambda(F, \omega)$ the collection of all (F, ω)-contractions of type-I on $(\mathcal{X}, \mathcal{G}, \preceq)$.

If we take $F(t) = \ln t$ and $\tau = \ln\left(\frac{1}{\lambda}\right)$, where $\lambda = (0, 1)$, we see that every ω-contraction of type-I is also an (F, ω)-contraction of type-I in an ordered $\mathcal{G}$-metric space. However, for other functions $F \in \mathfrak{F}$, new conditions can be obtained (see further Remark 3.2).

Our first main result is the following.

Theorem 3.1. *Let $(\mathcal{X}, \mathcal{G}, \preceq)$ be a complete ordered $\mathcal{G}$-metric space and ω be an ω-distance on $\mathcal{X}$. Let $T : \mathcal{X} \to \mathcal{X}$ be a given mapping. Suppose that the following conditions hold:*

(I) *there exists $x_0 \in \mathcal{X}$ such that $x_0 \preceq Tx_0$;*
(II) *T is a non-decreasing mapping;*
(III) $T \in \Lambda(F, \omega)$;
(IV) (i) *T is a continuous mapping, or*
(ii) *$(\mathcal{X}, \mathcal{G}, \preceq)$ is regular, or*
(iii) *for every $x \in \mathcal{X}$ and every $y \in \mathcal{X}$ with $y \neq Ty$,*

$$\inf\{\omega(x, y, x) + \omega(x, y, Tx) + \omega(x, Tx, y) : x \preceq Tx\} > 0.$$

Then T has a fixed point $x^* \in \mathcal{X}$.

Proof. Starting from the given $x_0 \in \mathcal{X}$ satisfying $x_0 \preceq Tx_0$, construct the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for $n \in \mathbb{N}^* = \mathbb{N} \cup \{0\}$. Since T is a non-decreasing mapping, we have

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots.$$

If $x_{n_0} = x_{n_0+1}$ for some n_0, then $x_{n_0} \in \mathrm{Fix}(T)$ and hence the proof is completed. Thus, we will assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}^*$ and let $\sigma_n = \omega(x_n, x_{n+1}, x_{n+1})$ for $n \in \mathbb{N}^*$. Then $\mathcal{G}(x_n, x_{n+1}, x_{n+1}) \neq 0$ implies $\omega(x_n, x_{n+1}, x_{n+1}) \neq 0$ and therefore $\sigma_n > 0$, for all $n \in \mathbb{N}^*$. We will prove that $\lim_{n\to\infty} \sigma_n = 0$.

Since $x_n \preceq x_{n+1}$ and $T \in \Lambda(F,\omega)$, the following holds for the points x_n, x_{n+1}, x_{n+1}:

$$\begin{aligned} \tau + F(\sigma_{n+1}) &= \tau + F(\omega(Tx_n, Tx_{n+1}, Tx_{n+1})) \\ &\leq F(\theta(x_n, x_{n+1}, x_{n+1})), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} &\theta(x_n, x_{n+1}, x_{n+1}) \\ &= \max \left\{ \begin{array}{c} \omega(x_n, x_{n+1}, x_{n+1}), \omega(x_n, x_{n+1}, Tx_n), \\ \omega(x_n, Tx_n, x_{n+1}), \omega(x_n, Tx_n, Tx_n), \\ \omega(x_{n+1}, Tx_{n+1}, Tx_{n+1}), \omega(x_{n+1}, Tx_{n+1}, Tx_{n+1}), \\ \frac{1}{2}[\omega(x_n, Tx_{n+1}, Tx_{n+1}) + \omega(Tx_n, x_{n+1}, x_{n+1})], \end{array} \right\} \\ &= \max \left\{ \omega(x_n, x_{n+1}, x_{n+1}), \omega(x_{n+1}, x_{n+2}, x_{n+2}), \frac{1}{2}\omega(x_n, x_{n+2}, x_{n+2}) \right\} \\ &= \max \left\{ \sigma_n, \sigma_{n+1}, \frac{1}{2}\omega(x_n, x_{n+2}, x_{n+2}) \right\}. \end{aligned}$$

By (ω1) we have

$$\omega(x_n, x_{n+2}, x_{n+2}) \leq \omega(x_n, x_{n+1}, x_{n+1}) + \omega(x_{n+1}, x_{n+2}, x_{n+2}).$$

Thus

$$\begin{aligned} \theta(x_n, x_{n+1}, x_{n+1}) &= \max\{\omega(x_n, x_{n+1}, x_{n+1}), \omega(x_{n+1}, x_{n+2}, x_{n+2})\} \\ &= \max\{\sigma_n, \sigma_{n+1}\}. \end{aligned}$$

From (3.3) we have

$$\tau + F(\sigma_{n+1}) \leq F(\max\{\sigma_n, \sigma_{n+1}\}). \tag{3.4}$$

If $\sigma_n \leq \sigma_{n+1}$ for some $n \in \mathbb{N}$, then from (3.4) we have $\tau + F(\sigma_{n+1}) \leq F(\sigma_{n+1})$, which is a contradiction since $\tau > 0$. Thus $\sigma_n > \sigma_{n+1}$, for all $n \in \mathbb{N}$ and so from (3.4) we have

$$F(\sigma_{n+1}) \leq F(\sigma_n) - \tau.$$

Therefore, we derive

$$F(\sigma_{n+1}) \leq F(\sigma_n) - \tau \leq F(\sigma_{n-1}) - 2\tau \leq \cdots \leq F(\sigma_0) - (n+1)\tau,$$

that is,

$$F(\sigma_{n+1}) \leq F(\sigma_0) - (n+1)\tau, \text{ for all } n \in \mathbb{N}. \tag{3.5}$$

From (3.5), we get $F(\sigma_{n+1}) \to -\infty$ as $n \to \infty$. Thus, from (F2), we have

$$\lim_{n\to\infty} \sigma_n = 0. \tag{3.6}$$

Now by the property (F3), there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} (\sigma_n)^k F(\sigma_n) = 0. \tag{3.7}$$

By (3.7), the following holds for all $n \in \mathbb{N}$:

$$(\sigma_n)^k F(\sigma_n) - (\sigma_n)^k F(\sigma_0) \leq (\sigma_n)^k(-n\tau) \leq 0. \tag{3.8}$$

Passing to the limit as $n \to \infty$ in (3.8), and using (3.6)-(3.7) we obtain

$$\lim_{n\to\infty} n(\sigma_n)^k = 0. \tag{3.9}$$

From (3.9), there exits $n_1 \in \mathbb{N}$ such that $n(\sigma_n)^k \leq 1$, for all $n \geq n_1$. So, we have, for all $n \geq n_1$

$$\sigma_n \leq \frac{1}{n^{1/k}}. \tag{3.10}$$

In order to show that $\{x_n\}$ is a $\mathcal{G}$-Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the property $(\omega 1)$ and (3.10), we have

$$\begin{aligned}
\omega(x_n, x_m, x_m) \leq & \omega(x_n, x_{n+1}, x_{n+1}) + \omega(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + \omega(x_{m-1}, x_m, x_m) \\
= & \sigma_n + \sigma_{n+1} + \cdots + \sigma_{m-1} \\
= & \sum_{i=n}^{m-1} \sigma_i \leq \sum_{i=n}^{\infty} \sigma_i \leq \sum_{i=n}^{\infty} \frac{1}{n^{1/k}}.
\end{aligned}$$

By the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^{1/k}}$, passing to the limit as $n \to \infty$, we get $\omega(x_n, x_m, x_m) \to 0$ and hence $\{x_n\}$ is a $\mathcal{G}$-Cauchy sequence in $(\mathcal{X}, \mathcal{G})$. Since $\mathcal{X}$ is a complete $\mathcal{G}$-metric space, there exists an $x^* \in \mathcal{X}$ such that

$$\lim_{n\to\infty} x_n = x^*. \tag{3.11}$$

Suppose that (i) holds. Using $(\omega 2)$ and (3.6), since T is $\mathcal{G}$-continuous, we have

$$\omega(x^*, Tx^*, Tx^*) = \lim_{n\to\infty} \omega(x_n, Tx_n, Tx_n) = \lim_{n\to\infty} \omega(x_n, x_{n+1}, x_{n+1}) = 0.$$

Therefore, x^* is a fixed point of T.

Suppose that (ii) holds. Since $\{x_n\}$ is a non-decreasing sequence such that $x_n \to x^*$ and $\mathcal{X}$ is regular, it follows that $x_n \preceq x^*$, for all $n \in \mathbb{N}^*$. Therefore, we have

$$\begin{aligned}
\tau + F(\omega(x_{n+1}, Tx^*, Tx^*)) &= \tau + F(\omega(Tx_n, Tx^*, Tx^*)) \\
&\leq F(\theta(x_n, x^*, x^*)),
\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
&\theta(x_n, x^*, x^*) \\
&= \max\left\{ \begin{array}{c} \omega(x_n, x^*, x^*), \omega(x_n, x^*, Tx_n), \omega(x_n, Tx_n, x^*), \\ \omega(x_n, Tx_n, Tx_n), \omega(x^*, Tx^*, Tx^*), \omega(x^*, Tx^*, Tx^*), \\ \frac{1}{2}[\omega(x_n, Tx^*, Tx^*) + \omega(Tx_n, x^*, x^*)] \end{array} \right\} \\
&= \max\left\{ \begin{array}{c} \omega(x_n, x^*, x^*), \omega(x_n, x_{n+1}, x^*), \\ \omega(x_n, x_{n+1}, x_{n+1}), \omega(x^*, Tx^*, Tx^*), \\ \frac{1}{2}[\omega(x_n, Tx^*, Tx^*) + \omega(x_{n+1}, x^*, x^*)] \end{array} \right\}.
\end{aligned} \tag{3.13}$$

Taking $n \to \infty$ in inequality (3.12) and using (3.11), (3.13) with Lemma 2.2, we obtain

$$\tau + F(\omega(x^*, Tx^*, Tx^*)) \leq F(\omega(x^*, Tx^*, Tx^*)),$$

a contradiction and hence $Tx^* = x^*$. Thus x^* is a fixed point of T.

Suppose that (iii) holds. As $\lim_{n\to\infty} \omega(T^n x_0, T^m x_0, T^p x_0) = 0$, therefore, for each $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $m \geq n > N_\epsilon$ implies $\omega(T^m x_0, T^m x_0, T^n x_0) < \epsilon$. Since $\lim_{n\to\infty} T^n x_0 = x^*$ and by lower semi-continuity of ω, we have

$$\omega(T^m x_0, T^p x_0, x^*) \leq \liminf_{n\to\infty} w(T^m x_0, T^p x_0, T^n x_0) < \epsilon, \;\; m \geq p$$
$$\omega(T^m x_0, x^*, T^l x_0) \leq \liminf_{n\to\infty} w(T^m x_0, T^n x_0, T^l x_0) < \epsilon, \;\; m \geq l.$$

Therefore, $\omega(T^m x_0, T^p x_0, x^*) \leq \epsilon$ and $\omega(T^m x_0, x^*, T^l x_0) < \epsilon$. Set $\epsilon = \frac{1}{k}$. Assume that $Tx^* \neq x^*$. Then by hypothesis (iii) with $T^{n_k} x_0 \preceq T^{n_k+1} x_0$, we have

$$\begin{aligned}
0 &< \inf\{\omega(x, y, x) + \omega(x, y, Tx) + \omega(x, Tx, y) : x \preceq Tx\} \\
&\leq \inf \left\{ \begin{array}{c} \omega(T^{n_k} x_0, x^*, T^{n_k} x_0) + \omega(T^{n_k} x_0, x^*, T^{n_k+1} x_0) \\ +\omega(T^{n_k} x_0, T^{n_k+1} x_0, x^*) : n \in \mathbb{N} \end{array} \right\} \\
&< \frac{3}{k} \to 0 \text{ as } k \to \infty,
\end{aligned}$$

a contradiction. Therefore, $Tx^* = x^*$. □

To ensure the uniqueness of the fixed point, we will consider the following hypothesis.

$$(H0) : \begin{cases} \text{for all } u^*, v^* \in \mathcal{X} \text{ there exists } w^* \in \mathcal{X}, w^* \preceq Tw^* \\ \text{satisfying both } u^* \preceq w^* \text{ and } v^* \preceq w^*. \end{cases}$$

Theorem 3.2. *Adding condition* $(H0)$ *to the hypotheses of Theorem* 3.1 *we have that* $Tu^* = u^*$, $Tv^* = v^*$ *imply that* $u^* = v^*$.

Proof. Suppose that $Tu^* = u^*$, $Tv^* = v^*$, and, to the contrary, $u^* \neq v^*$, hence $\omega(u^*, v^*, v^*) > 0$. Consider the following two possible cases.

1° Suppose u^* and v^* are comparable. Without loss of generality, assume that $\omega(v^*, u^*, u^*) < \omega(u^*, v^*, v^*)$. By the assumption, we can replace x by u^*, and y and z by v^* in the condition (3.2), and we get

$$\tau + F(\omega(Tu^*, Tv^*, Tv^*)) \leq F(\theta(u^*, v^*, v^*)), \tag{3.14}$$

where

$$\begin{aligned}
\theta(u^*, v^*, v^*) &= \max \left\{ \begin{array}{c} \omega(u^*, v^*, v^*), \omega(u^*, v^*, Tu^*), \\ \omega(u^*, Tu^*, v^*), \omega(u^*, Tu^*, Tu^*), \\ \omega(v^*, Tv^*, Tv^*), \omega(v^*, Tv^*, Tv^*), \\ \frac{1}{2}[\omega(u^*, Tv^*, Tv^*) + \omega(Tu^*, v^*, v^*)], \end{array} \right\} \\
&= \max\{\omega(u^*, v^*, v^*), \omega(v^*, u^*, u^*)\} \\
&= \omega(u^*, v^*, v^*).
\end{aligned}$$

Therefore, from (3.14), we have

$$\tau + F(\omega(u^*, v^*, v^*)) \leq F(\omega(u^*, v^*, v^*)),$$

a contradiction, which implies that $u^* = v^*$.

2° Suppose now that u^* and v^* are not comparable. Choose an element $w^* \in \mathcal{X}$, $w^* \preceq Tw^*$ comparable with both of them. Then also $u^* = T^n u^*$ is comparable with $T^n w^*$ for each n (since T is non-decreasing). Applying (3.2) one obtains that

$$\begin{aligned}\tau + F(\omega(u^*, T^n w^*, T^n w^*)) &= \tau + F(\omega(TT^{n-1}u^*, TT^{n-1}w^*, TT^{n-1}w^*)) \\ &\leq F(\theta(T^{n-1}u^*, T^{n-1}w^*, T^{n-1}w^*)),\end{aligned}$$

where

$$\begin{aligned}&\theta(T^{n-1}u^*, T^{n-1}w^*, T^{n-1}w^*) \\ &= \max\left\{\begin{array}{c} \omega(T^{n-1}u^*, T^{n-1}w^*, T^{n-1}w^*), \omega(T^{n-1}u^*, T^{n-1}w^*, TT^{n-1}u^*), \\ \omega(T^{n-1}u^*, TT^{n-1}u^*, T^{n-1}u^*), \omega(T^{n-1}u^*, TT^{n-1}u^*, TT^{n-1}u^*), \\ \omega(T^{n-1}w^*, TT^{n-1}w^*, TT^{n-1}w^*), \omega(T^{n-1}w^*, TT^{n-1}w^*, TT^{n-1}w^*), \\ \frac{1}{2}[\omega(T^{n-1}u^*, TT^{n-1}w^*, TT^{n-1}w^*) + \omega(TT^{n-1}u^*, T^{n-1}w^*, T^{n-1}w^*)] \end{array}\right\} \\ &= \max\left\{\begin{array}{c} \omega(u^*, T^{n-1}w^*, T^{n-1}w^*), \omega(u^*, T^{n-1}w^*, u^*), \\ \omega(T^{n-1}w^*, T^n w^*, T^n w^*), \end{array}\right\},\end{aligned}$$

for n sufficiently large, because $\omega(T^{n-1}w^*, T^n w^*, T^n w^*) \to 0$ when $n \to \infty$ (the last assertion can be proved, starting from the assumption $w^* \preceq Tw^*$, in the same way as the similar conclusion in the proof of Theorem 3.1).

Similarly as in the proof of Theorem 3.1, it can be shown that

$$\omega(u^*, T^n w^*, T^n w^*) \leq \theta(u^*, T^{n-1}w^*, T^{n-1}w^*) \leq \omega(u^*, T^{n-1}w^*, T^{n-1}w^*).$$

It follows that the sequence $\omega(u^*, T^n w^*, T^n w^*)$ is nonincreasing and it has a limit $\ell \geq 0$. Assuming that $\ell > 0$ and passing to the limit in the relation

$$\tau + F(\omega(u^*, T^n w^*, T^n w^*)) \leq F(\theta(u^*, T^{n-1}w^*, T^{n-1}w^*),$$

one obtains that $\ell = 0$, a contradiction with (F1). In the same way it can be deduced that $\omega(v^*, T^n w^*, T^n w^*) \to 0$ as $n \to \infty$. Now, passing to the limit in $\omega(u^*, v^*, v^*) \leq \omega(u^*, T^n w^*, T^n w^*) + \omega(T^n w^*, v^*, v^*)$, it follows that $\omega(u^*, v^*, v^*) = 0$. Hence, $u^* = v^*$ and the uniqueness of the fixed point is proved. □

Remark 3.1. Theorem 3.1 is also true if $\theta(x, y, z)$ is replaced by

$$\theta_1(x, y, z) = \max\left\{\begin{array}{c} \omega(x, y, z), \omega(x, Tx, y), \omega(x, Tx, z), \\ \omega(x, Tx, Tx), \omega(y, Ty, Ty), \omega(z, Tz, Tz), \\ \omega(x, Ty, Tz), \omega(Tx, y, z) \end{array}\right\}$$

and condition (3.2) by

$$\tau + F(\omega(Tx, Ty, Tz)) \leq F(\theta_1(x, y, z)).$$

Remark 3.2. Taking various concrete functions $F \in \mathfrak{F}$ in the condition (3.2) of Theorems 3.1-3.2, we can get various (F, ω)-contractive conditions. We state just a few examples (recall that $\theta(x, y, z)$ is defined in (3.2)).

(I) Taking $F(t) = \ln t$, $t > 0$, and $\tau = \ln\left(\frac{1}{\lambda}\right)$, where $\lambda \in (0, 1)$, we have the condition

$$x, y, z \in \mathcal{X} \text{ with } z \preceq y \preceq x \text{ and } \omega(Tx, Ty, Tz) > 0 \text{ implies}$$
$$\omega(Tx, Ty, Tz) \le \lambda\, \theta(x, y, z). \tag{3.15}$$

(II) Taking $F(t) = \ln t + t$, $t > 0$, and $\tau = \ln\left(\frac{1}{\lambda}\right)$, where $\lambda \in (0, 1)$, we have the condition

$$x, y, z \in \mathcal{X} \text{ with } z \preceq y \preceq x \text{ and } \omega(Tx, Ty, Tz) > 0 \text{ implies}$$
$$\omega(Tx, Ty, Tz) e^{\omega(Tx,Ty,Tz)-\theta(x,y,z)} \le \lambda\, \theta(x, y, z). \tag{3.16}$$

(III) Taking $F(t) = -\frac{1}{\sqrt{t}}$, $t > 0$, the condition is

$$x, y, z \in \mathcal{X} \text{ with } z \preceq y \preceq x \text{ and } \omega(Tx, Ty, Tz) > 0 \text{ implies}$$
$$\omega(Tx, Ty, Tz) \le \frac{1}{(1 + \tau\sqrt{\theta(x, y, z)})^2}\theta(x, y, z).$$

(IV) Taking $F(t) = \ln(t^2 + t)$, $t > 0$, and $\tau = \ln\left(\frac{1}{\lambda}\right)$, where $\lambda > 0$, we have

$$x, y, z \in \mathcal{X} \text{ with } z \preceq y \preceq x \text{ and } \omega(Tx, Ty, Tz) > 0 \text{ implies}$$
$$\omega(Tx, Ty, Tz)[\omega(Tx, Ty, Tz) + 1] \le \lambda\, \theta(x, y, z)[\theta(x, y, z) + 1].$$

We illustrate the previous result by the following example which is modified according to [9, Example 3.1].

Example 3.1. Let $\mathcal{X} = \{0, 1, 2, 3, \dots\}$ and let $\mathcal{G} : \mathcal{X}^3 \to \mathbb{R}^+$ and $\omega : \mathcal{X}^3 \to \mathbb{R}^+$ be defined by

$$\mathcal{G}(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ x + y + z, & \text{otherwise,} \end{cases} \qquad \omega(x, y, z) = x + 2\max\{y, z\}.$$

Then it is easy to see that $\mathcal{G}$ is a complete $\mathcal{G}$-metric and ω is an ω-distance on $(\mathcal{X}, \mathcal{G})$. Define an order $\preceq$ on $\mathcal{X}$ by $x \preceq y$ iff $x \ge y$ and a $\mathcal{G}$-continuous mapping $T : \mathcal{X} \to \mathcal{X}$ by

$$Tx = \begin{cases} 0, & \text{if } x = 0, \\ x - 1, & \text{otherwise.} \end{cases}$$

Then T is a nondecreasing mapping w.r.t. $\preceq$ and $1 \preceq T1$ holds. We will show that $T \in \Lambda(F, \omega)$ with $F(t) = t + \ln t$ and $\tau = 2$, i.e., we will show that condition (3.16) holds with $\lambda = e^{-2}$. The following two cases are non-trivial.

1. $x = 0$, $\max\{2, y\} \le z$. Then it is easy to show that $\omega(Tx, Ty, Tz) = 2z - 2$ and $\theta(x, y, z) = 3z - 2$, and hence

$$\omega(Tx, Ty, Tz)e^{\omega(Tx,Ty,Tz)-\theta(x,y,z)} = (2z - 2)e^{-z} \le e^{-2}(3z - 2) = \lambda\theta(x, y, z).$$

2. $0 < x \le y \le z$ and $x < z$ (hence, $z \ge 2$). Then $\omega(Tx, Ty, Tz) = x + 2z - 3$, $\theta(x, y, z) = \max\{x + 2z, 3z - 2\}$.

2.1. If $\theta(x, y, z) = x + 2z$ then

$$\begin{aligned}\omega(Tx, Ty, Tz)e^{\omega(Tx,Ty,Tz)-\theta(x,y,z)} &= (x + 2z - 3)e^{-3}\\ &\le e^{-2}(x + 2z) = \lambda\theta(x, y, z).\end{aligned}$$

2.2. If $\theta(x, y, z) = 3z - 2$ then

$$\begin{aligned}\omega(Tx, Ty, Tz)e^{\omega(Tx,Ty,Tz)-\theta(x,y,z)} &= (x + 2z - 3)e^{x-z-1}\\ &\le e^{-2}(3z - 2) = \lambda\theta(x, y, z).\end{aligned}$$

Thus, all the conditions of Theorem 3.2 are fulfilled and the mapping T has a unique fixed point (which is $x^* = 0$). Note that the same conclusion cannot be obtained using the simplest function $F(t) = \ln t$, i.e., the Banach-type condition (3.15). Indeed, in the case 2.1,

$$\lim_{z\to\infty} \frac{\omega(Tx, Ty, Tz)}{\theta(x, y, z)} = 1.$$

3.2. **Result-2.**

Definition 3.3. Let $(\mathcal{X}, \mathcal{G}, \preceq)$ be an ordered $\mathcal{G}$-metric space and ω be an ω-distance on $\mathcal{X}$. A self-mapping T on $\mathcal{X}$ is called an (F, ω)-contraction of type-II, if there exists $F \in \mathfrak{F}$ and $\tau \in \mathbb{R}^+$ such that

$$\begin{gathered}x, y \in \mathcal{X} \text{ with } y \preceq x \text{ and } \omega(Tx, Ty, T^2y) > 0 \text{ implies}\\ \tau + F(\omega(Tx, Ty, T^2y)) \le F(\mathcal{M}(x, y)),\end{gathered}$$

where

$$\mathcal{M}(x, y) = \max\left\{\begin{array}{c}\omega(x, y, Ty), \omega(x, Tx, Ty), \omega(x, Tx, T^2x),\\ \frac{1}{2}[\omega(y, Ty, T^2y) + \omega(y, T^2x, T^2y)]\end{array}\right\}. \tag{3.17}$$

We denote by $\Phi(F, \omega)$ the collection of all (F, ω)-contractive type-II mappings on $(\mathcal{X}, \mathcal{G}, \preceq)$.

If we take $F(t) = \ln t$ and $\tau = \ln\left(\frac{1}{\lambda}\right)$, where $\lambda = (0, 1)$, we see that every ω-contraction of type-II is also an (F, ω)-contraction of type-II in an ordered $\mathcal{G}$-metric space. However, for other functions $F \in \mathfrak{F}$, new conditions can be obtained.

Our next main result is the following.

Theorem 3.3. *Let $(\mathcal{X}, \mathcal{G}, \preceq)$ be an ordered complete $\mathcal{G}$-metric space and ω be an ω-distance on $\mathcal{X}$. Let $T : \mathcal{X} \to \mathcal{X}$ be a given mapping. Suppose that the following conditions hold:*

(I) *there exists $x_0 \in \mathcal{X}$ such that $x_0 \preceq Tx_0$;*
(II) *T is a non-decreasing mapping;*
(III) *$T \in \Phi(F, \omega)$;*
(IV) (i) *T is a continuous mapping, or*

(ii) $(\mathcal{X}, \mathcal{G}, \preceq)$ *is regular, or*

(iii) *for every* $x \in \mathcal{X}$*and for every* $y \in \mathcal{X}$ *with* $y \neq Ty$,

$$\inf\{\omega(x, y, x) + \omega(x, y, Tx) + \omega(x, T^2x, y) : x \preceq Tx\} > 0.$$

Then T *has a fixed point* $x^* \in \mathcal{X}$.

Proof. As in the proof of Theorem 3.1, starting with $x_0 \in \mathcal{X}$ with $x_0 \preceq Tx_0$, construct the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$, with $x_n \preceq x_{n+1}$ for $n \in \mathbb{N}^* = \mathbb{N} \cup \{0\}$. We will prove that $\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \omega(x_n, x_{n+1}, x_{n+2}) = 0$.

Since $x_n \preceq x_{n+1}$ and $T \in \Phi(F, \omega)$, we have the following for the points x_n, x_{n+1}:

$$\begin{aligned}\tau + F(\sigma_{n+1}) &= \tau + F(\omega(Tx_n, Tx_{n+1}, T^2x_{n+1})) \\ &\leq F(\mathcal{M}(x_n, x_{n+1})),\end{aligned}$$

where

$$\begin{aligned}\mathcal{M}(x_n, x_{n+1}) &= \max\left\{ \begin{array}{c} \omega(x_n, x_{n+1}, Tx_{n+1}), \omega(x_n, Tx_n, Tx_{n+1}), \omega(x_n, Tx_n, T^2x_n), \\ \frac{1}{2}[\omega(x_{n+1}, Tx_{n+1}, T^2x_{n+1}) + \omega(x_{n+1}, T^2x_n, T^2x_{n+1})] \end{array} \right\} \\ &= \max\left\{ \omega(x_n, x_{n+1}, x_{n+2}), \omega(x_{n+1}, x_{n+2}, x_{n+3}) \right\} \\ &= \max\{\sigma_n, \sigma_{n+1}\}.\end{aligned}$$

Thus from (3.3) we have

$$\tau + F(\sigma_{n+1}) \leq F(\max\{\sigma_n, \sigma_{n+1}\}).$$

Similarly as in the proof of Theorem 3.1, one gets that $\lim_{n\to\infty} \sigma_n = 0$, then that $\{x_n\}$ is a $\mathcal{G}$-Cauchy sequence converging to some $x^* \in \mathcal{X}$. We have to prove that x^* is a fixed point of T.

If (i) holds, the proof is similar as in Theorem 3.1.

Suppose that (ii) holds. Since $\{x_n\}$ is a non-decreasing sequence such that $x_n \to x^*$ and $\mathcal{X}$ is regular, it follows that $x_n \preceq x^*$, for all $n \in \mathbb{N}^*$. Therefore, we have

$$\begin{aligned}\tau + F(\omega(x_{n+1}, Tx^*, T^2x^*)) &= \tau + F(\omega(Tx_n, Tx^*, T^2x^*)) \\ &\leq F(\mathcal{M}(x_n, Tx^*)),\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}\mathcal{M}(x_n, x^*) &= \max\left\{ \begin{array}{c} \omega(x_n, x^*, Tx^*), \omega(x_n, Tx_n, Tx^*), \omega(x_n, Tx_n, T^2x_n), \\ \frac{1}{2}[\omega(x^*, Tx^*, T^2x^*) + \omega(x^*, T^2x_n, T^2x^*)] \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} \omega(x_n, x^*, Tx^*), \omega(x_n, x_{n+1}, Tx^*), \omega(x_n, x_{n+1}, x_{n+1}), \\ \omega(x^*, Tx^*, T^2x^*), \omega(x^*, x_{n+2}, T^2x^*) \end{array} \right\}.\end{aligned} \tag{3.19}$$

Taking $n \to \infty$ in inequality (3.19) and using (3.18) with Lemma 2.2, we obtain

$$\tau + F(\omega(x^*, Tx^*, T^2x^*)) \leq F(\max\{\omega(x^*, x^*, Tx^*), \omega(x^*, Tx^*, T^2x^*), \omega(x^*, x^*, T^2x^*)\}),$$

which implies that

$$\tau + F(\omega(x^*, Tx^*, T^2x^*)) \leq F(\omega(x^*, Tx^*, T^2x^*)),$$

a contradiction and hence $Tx^* = x^*$. Thus x^* is a fixed point of T.

Suppose that (iii) holds. As $\lim_{n\to\infty} \omega(T^n x_0, T^m x_0, T^p x_0) = 0$, therefore for each $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $m \geq n > N_\epsilon$ implies $\omega(T^m x_0, T^m x_0, T^n x_0) < \epsilon$. Since $\lim_{n\to\infty} T^n x_0 = x^*$ and by lower semi-continuity of ω, we have

$$\omega(T^m x_0, T^p x_0, x^*) \leq \liminf_n w(T^m x_0, T^p x_0, T^n x_0) < \epsilon, \ \ m \geq p,$$

$$\omega(T^m x_0, x^*, T^l x_0) \leq \liminf_n w(T^m x_0, T^n x_0, T^l x_0) < \epsilon, \ \ m \geq l.$$

Therefore, $\omega(T^m x_0, T^p x_0, x^*) \leq \epsilon$ and $\omega(T^m x_0, x^*, T^l x_0) < \epsilon$. Set $\epsilon = \frac{1}{k}$. Assuming that $Tx^* \neq x^*$, by hypothesis (iii) with $T^{n_k} x_0 \preceq T^{n_k+1} x_0$, we have

$$\begin{aligned} 0 &< \inf\{\omega(x, y, x) + \omega(x, y, Tx) + \omega(x, T^2 x, y) : x \in \mathcal{X}\} \\ &\leq \inf \left\{ \begin{array}{c} \omega(T^{n_k} x_0, x^*, T^{n_k} x_0) + \omega(T^{n_k} x_0, x^*, T^{n_k+2} x_0) \\ +\omega(T^{n_k} x_0, T^{n_k+1} x_0, x^*) : n \in \mathbb{N} \end{array} \right\} \\ &< \frac{3}{k} \to 0 \text{ as } k \to \infty, \end{aligned}$$

a contradiction. Therefore $Tx^* = x^*$. □

Theorem 3.4. *Adding condition* $(H0)$ *to the hypotheses of Theorem* 3.3 *we have that* $Tu^* = u^*$, $Tv^* = v^*$ *imply that* $u^* = v^*$.

Proof. Suppose that $Tu^* = u^*$, $Tv^* = v^*$, and, to the contrary, $u^* \neq v^*$, hence $\omega(u^*, v^*, v^*) > 0$. Consider the following two possible cases.

1° Suppose u^* and v^* are comparable. Without loss of generality, assume that $\omega(v^*, u^*, u^*) < \omega(u^*, v^*, v^*)$. By the assumption, we can replace x by u^*, and y and z by v^* in the condition (3.17), and we get

$$\tau + F(\omega(Tu^*, Tv^*, T^2 v^*)) \leq F(\mathcal{M}(u^*, v^*)),$$

where

$$\begin{aligned} \mathcal{M}(u^*, v^*) &= \max \left\{ \begin{array}{c} \omega(u^*, v^*, Tv^*), \omega(u^*, Tu^*, Tv^*), \omega(u^*, Tu^*, T^2 u^*), \\ \frac{1}{2}[\omega(v^*, Tv^*, T^2 v^*) + \omega(v^*, T^2 u^*, T^2 v^*)] \end{array} \right\} \\ &= \max \left\{ \ \omega(u^*, v^*, v^*), \omega(u^*, u^*, v^*) \ \right\} \\ &= \omega(u^*, u^*, v^*). \end{aligned}$$

Therefore, we have

$$\tau + F(\omega(u^*, v^*, v^*)) \leq F(\omega(u^*, v^*, v^*)),$$

a contradiction, which implies that $u^* = v^*$.

2° Suppose now that u^* and v^* are not comparable. Choose an element $w^* \in \mathcal{X}$, $w^* \preceq Tw^*$ comparable with both of them. Then also $u^* = T^n u^*$ is comparable with $T^n w^*$ for each n (since T is non-decreasing). Applying (3.17) one obtains that

$$\begin{aligned} \tau + F(\omega(u^*, T^n w^*, T^{n+1} w^*)) &= \tau + F(\omega(TT^{n-1} u^*, TT^{n-1} w^*, T^2 T^{n-1} u^*)) \\ &\leq F(\mathcal{M}(T^{n-1} u^*, T^{n-1} w^*)), \end{aligned}$$

where

$$\mathcal{M}(T^{n-1}u^*, T^{n-1}w^*) = \max\left\{\begin{array}{c} \omega(T^{n-1}u^*, T^{n-1}w^*, TT^{n-1}w^*), \\ \omega(T^{n-1}u^*, TT^{n-1}u^*, TT^{n-1}w^*), \\ \omega(T^{n-1}u^*, TT^{n-1}u^*, T^2T^{n-1}u^*), \\ \frac{1}{2}[\omega(T^{n-1}w^*, TT^{n-1}w^*, T^2T^{n-1}w^*) \\ +\omega(T^{n-1}w^*, T^2T^{n-1}u^*, T^2T^{n-1}w^*)] \end{array}\right\}$$
$$= \max\left\{\begin{array}{c} \omega(u^*, T^{n-1}w^*, T^n w^*), \omega(u^*, u^*, T^n w^*), \\ \omega(T^{n-1}w^*, T^n w^*, T^{n+1}w^*), \omega(T^n w^*, u^*, T^n w^*) \end{array}\right\}.$$

The rest of proof follows from Theorem 3.2 and the proof is concluded. □

Remark 3.3. Similarly to Remark 3.2 (I)-(IV), we can define some new contractive conditions corresponding to $\mathcal{M}(x, y)$ given in (3.17).

4. Ulam-Hyers Stability in $\mathcal{G}$-Metric Spaces

We introduce the notion of generalized Ulam-Hyers stability of fixed point problems in $\mathcal{G}$-metric spaces under ω-distance.

Definition 4.1. Let $(\mathcal{X}, \mathcal{G})$ be a $\mathcal{G}$-metric space and ω be an ω-distance on $\mathcal{X}$. Let $T : \mathcal{X} \to \mathcal{X}$ be a given mapping. The fixed point equation

$$u = Tu, \ u \in \mathcal{X}, \tag{4.1}$$

is said to be generalized Ulam-Hyers stable in the framework of $\mathcal{G}$-metric spaces under ω-distance if there exists an increasing function $\psi : [0, \infty) \to [0, \infty)$, continuous at 0, with $\psi(0) = 0$, such that for each $\epsilon > 0$ and an ϵ-solution $v \in \mathcal{X}$, that is,

$$\omega(v, Tv, Tv) \leq \epsilon,$$

there exists a solution $w \in \mathcal{X}$ of the fixed point equation (4.1) such that

$$\omega(v, w, w) \leq \psi(\epsilon). \tag{4.2}$$

If $\psi(t) = ct$ for $t \in [0, \infty)$, where $c > 0$, then (4.1) is said to be Ulam-Hyers stable in the framework $\mathcal{G}$-metric spaces under ω-distance.

Remark 4.1. If $\omega = \mathcal{G}$, then Definition 4.1 reduces to the definition of generalized Ulam-Hyers stability in $\mathcal{G}$-metric spaces. If, moreover, $\psi(t) = ct$ for all $t \in [0, \infty)$, where $c > 0$, then it reduces to the definition of Ulam-Hyers stability in $\mathcal{G}$-metric spaces.

Theorem 4.1. *Let $(\mathcal{X}, \mathcal{G}, \preceq)$ be a complete ordered $\mathcal{G}$-metric space and ω be an ω-distance on $\mathcal{X}$. Suppose that all the hypotheses of Theorem* 3.2 *hold, using the contraction condition in the form* (3.15). *Then this equation is Ulam-Hyers stable in the framework of $\mathcal{G}$-metric spaces under ω-distance.*

Proof. Following Theorem 3.2, we have $Tu^* = u^*$, that is, $u^* \in \mathcal{X}$ is a solution of the fixed point equation (4.1). Let $\epsilon > 0$ and $v^* \in \mathcal{X}$ be an ϵ-solution of (4.1), that is,

$$\omega(v^*, Tv^*, Tv^*) \leq \epsilon.$$

Since $\omega(u^*, Tu^*, Tu^*) = \omega(u^*, u^*, u^*) = 0 \leq \epsilon$, u^* and v^* are ϵ-solutions. By hypothesis, we get

$$\begin{aligned}\omega(u^*, v^*, v^*) = \omega(Tu^*, v^*, v^*) &\leq \omega(Tu^*, Tv^*, Tv^*) + \omega(Tv^*, v^*, v^*)\\ &\leq \lambda\theta(u^*, v^*, v^*) + \epsilon,\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}\theta(u^*, v^*, v^*) &= \max\left\{\begin{array}{c}\omega(u^*, v^*, v^*), \omega(u^*, v^*, Tu^*), \omega(u^*, Tu^*, v^*),\\ \omega(u^*, Tu^*, Tu^*), \omega(v^*, Tv^*, Tv^*), \omega(v^*, Tv^*, Tv^*),\\ \frac{1}{2}[\omega(u^*, Tv^*, Tv^*) + \omega(Tu^*, v^*, v^*)]\end{array}\right\}\\ &= \max\left\{\omega(u^*, v^*, v^*), \epsilon\right\}.\end{aligned}$$

Consider the two possible cases.

1° If $\theta(u^*, v^*, v^*) = \omega(u^*, v^*, v^*)$, then we get

$$\omega(u^*, v^*, v^*) \leq \lambda\omega(u^*, v^*, v^*) + \epsilon,$$

which implies that $\omega(u^*, v^*, v^*)(1-\lambda) \leq \epsilon$. Taking $c = (1-\lambda)^{-1}$, where $c > 1$, and $\psi(t) = ct$, we have

$$\omega(u^*, v^*, v^*) \leq \psi(\epsilon).$$

2° If $\theta(u^*, v^*, v^*) = \epsilon$, then (4.3) gives that

$$\omega(u^*, v^*, v^*) \leq (1+\lambda)\epsilon \leq \psi(\epsilon).$$

Thus, the inequality (4.2) holds in all cases and, therefore, the fixed point equation (4.1) is Ulam-Hyers stable in the framework of $\mathcal{G}$-metric spaces under ω-distance. ☐

5. Well Posed and Limit Shadowing Fixed Point Problems in $\mathcal{G}$-Metric Space

The notion of well-posedness of a fixed point problem has evoked much interest of several mathematicians, see, for example, Popa [5] and others. We discuss this notion under ω-distance on $\mathcal{G}$-metric spaces.

Definition 5.1. Let $(\mathcal{X}, \mathcal{G}, \preceq)$ be a complete ordered $\mathcal{G}$-metric space and ω be an ω-distance on $\mathcal{X}$. Let $T : \mathcal{X} \to \mathcal{X}$ be a given mapping, having a unique fixed point u in $\mathcal{X}$ such that $u \preceq Tu$.

(a) The fixed point problem (4.1) is said to be well posed with respect to ω if for any sequence $\{u_n\} \subseteq \mathcal{X}$ such that $\lim_{n\to\infty} \omega(Tu_n, u_n, u_n) = 0$, we have $\lim_{n\to\infty} \omega(u_n, u, u) = 0$.

(b) The fixed point problem (4.1) is said to have limit shadowing property in $\mathcal{X}$ with respect to ω if for any sequence $\{u_n\} \subseteq \mathcal{X}$ such that $\omega(u_n, Tu_n, Tu_n) \to 0$ as $n \to \infty$, there exists $u \in \mathcal{X}$ such that $\omega(u_n, T^n u, T^n u) \to 0$ as $n \to \infty$.

Theorem 5.1. *Let* $(\mathcal{X}, \mathcal{G}, \preceq)$ *be a complete ordered* $\mathcal{G}$*-metric space and* ω *be an* ω*-distance on* $\mathcal{X}$*. Suppose that all the hypotheses of Theorem* 3.2 *hold, using the contraction condition in the form* (3.15)*. Then*

(a) *the fixed point problem for* T *is well posed with respect to* ω*;*
(b) T *has the limit shadowing property with respect to* ω*.*

Proof. Following Theorem 3.2, T has a unique fixed point $u^* = Tu^* \in \mathcal{X}$, such that $u^* \preceq Tu^*$.

1. Let $\{u_n\} \subset \mathcal{X}$ be such that $\lim_{n\to\infty} \omega(u_n, Tu_n, Tu_n) = 0$. Then

$$\begin{aligned}\omega(u_n, u^*, u^*) &\le \omega(u_n, Tu_n, Tu_n) + \omega(Tu_n, Tu^*, Tu^*)\\ &\le \omega(u_n, Tu_n, Tu_n) + \lambda\theta(u_n, u^*, u^*),\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}\lim_{n\to\infty} \theta(u_n, u^*, u^*) &= \lim_{n\to\infty} \max\left\{\begin{array}{c}\omega(u_n, u^*, u^*), \omega(u_n, u^*, Tu_n), \omega(u_n, Tu_n, u^*),\\ \omega(u_n, Tu_n, Tu_n), \omega(u^*, Tu^*, Tu^*), \omega(u^*, Tu^*, Tu^*),\\ \frac{1}{2}[\omega(u_n, Tu^*, Tu^*) + \omega(Tu_n, u^*, u^*)]\end{array}\right\}\\ &= 0 \quad (\text{using property } (\omega 2)).\end{aligned}$$

Passing to the limit as $n \to \infty$ in (5.1) and using $(\omega 2)$ property, we get that $\omega(u_n, u^*, u^*) \to 0$ as $n \to \infty$ which is equivalent to saying that $u_n \to u^*$ as $n \to \infty$.

2. Owing to Theorem 3.2, we know that the fixed point $u^* = Tu^* \in \mathcal{X}$ satisfies $\omega(u^*, Tu^*, Tu^*) = 0$. Let $\{u_n\} \subset \mathcal{X}$ be such that $\lim_{n\to\infty} \omega(u_n, Tu_n, Tu_n) = 0$. Then, as in the previous proof,

$$\omega(u_n, u^*, u^*) \le \omega(u_n, Tu_n, Tu_n) + \lambda\theta(u_n, u^*, u^*).$$

Passing to the limit as $n \to \infty$ in the above inequality and using $(\omega 2)$ property, it follows that $\omega(u_n, T^n u^*, T^n u^*) = \omega(u_n, u^*, u^*) \to 0$ as $n \to \infty$. □

6. Applications

6.1. **Application to integral equations.** This subsection is devoted to the existence of solutions of an integral equation as an application of Theorem 3.3.

Let $\mathcal{X} = C(I, \mathbb{R})$, where $I = [0, 1]$, be endowed with the standard maximum norm $\|u\| = \max_{t\in I} |u(t)|$, and let

$$\mathcal{G}(u, v, w) = \frac{1}{3}[\,\|u - v\| + \|v - w\| + \|w - u\|\,]$$

be the corresponding $\mathcal{G}$-metric on $\mathcal{X}$. Define $\omega : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ by

$$\omega(u, v, w) = \frac{1}{2}[\,\|u - v\| + \|u - w\|\,].$$

Then clearly ω is an ω-distance function (see Example 2.1).

Define an order relation $\preceq$ on $\mathcal{X}$ by

$$x \preceq y \text{ iff } x(t) \le y(t), \text{ for all } t \in I.$$

Then $(\mathcal{X}, \preceq)$ is a partially ordered set.

We will consider the integral equation

$$u(t) = \int_0^t G(t,s) f(s,u(s))\, ds, \tag{6.1}$$

where $f : I \times \mathbb{R} \to \mathbb{R}$ and $G : I \times I \to [0,\infty)$ are given functions. Let

$$Tu(t) = \int_0^t G(t,s) f(s,u(s))\, ds \tag{6.2}$$

define the respective operator $T\colon \mathcal{X} \to \mathcal{X}$.

Theorem 6.1. *Suppose the following assertions hold:*

(C1) *f is a continuous function, non-decreasing in the second variable;*
(C2) *there exists $u_0 \in \mathcal{X}$ such that $u_0 \preceq Tu_0$;*
(C3) *if $\{u_n\}$ is a sequence in $\mathcal{X}$ such that $u_n \to u$ in $(\mathcal{X}, \mathcal{G}, \preceq)$ and $u_n \preceq u_{n+1}$, for all $n \in \mathbb{N}$, then $u_n \preceq u$, for all $n \in \mathbb{N}$;*
(C4) *there exists $\lambda_1 \in (0, \frac{1}{2})$ such that for $u, v \in \mathcal{X}$ with $u \preceq v$ and $t \in I$ we have*

$$\max\{|f(t,u(t)) - f(t,v(t))|, |f(t,Tv(t)) - f(t,u(t))|\} \le A^{-1}\lambda_1 \Delta(u,v)(t),$$

where $A = \sup_{t\in I} \int_0^t G(t,r)\, dr$ and

$$\begin{aligned}&\Delta(u,v)(t)\\ &= \max\left\{\begin{array}{c} |u(t)-v(t)| + |Tv(t)-u(t)|, |u(t)-Tu(t)| + |Tv(t)-u(t)|,\\ |u(t)-Tu(t)| + |T^2u(t)-u(t)|,\\ \frac{1}{2}[|v(t)-Tv(t)| + |T^2v(t)-v(t)| + |v(t)-T^2u(t)| + |T^2v(t)-v(t)|] \end{array}\right\}.\end{aligned} \tag{6.3}$$

Then the equation (6.1) *has at least one solution $u^* \in \mathcal{X}$.*

Proof. We are going to check that $T \in \Phi(F, \omega)$. For this, let $u, v \in \mathcal{X}$ be such that $u \preceq v$, i.e., $u(t) \le v(t)$, for all $t \in I$. For each $t \in I$, by the definition (6.2) of operator T, we have

$$\begin{aligned}|Tu(t) - Tv(t)| &= \left|\int_0^t G(t,r) f(r,u(r))\, dr - \int_0^t G(t,r) f(r,v(r))\, dr\right|\\ &\le \int_0^t G(t,r) |f(r,u(r)) - f(r,v(r))|\, dr.\end{aligned}$$

Now, using the assumption (C4), after routine calculations, we obtain

$$\|Tu - Tv\| \le 2\lambda_1 \mathcal{M}(u,v).$$

Similarly, we can prove that

$$\|T^2v - Tu\| \le 2\lambda_1 \mathcal{M}(u,v).$$

Therefore we have

$$\omega(Tu, Tv, T^2v) = \frac{1}{2}[\|Tu - Tv\| + \|T^2v - Tu\|] \le 2\lambda_1 \mathcal{M}(u,v) = \lambda \mathcal{M}(u,v),$$

where $\lambda \in (0,1)$ and $\mathcal{M}(u,v)$ is given in (3.17).

Now, by considering $F \in \mathfrak{F}$ given by $F(t) = \ln t$, $t > 0$, and $\tau = \ln\left(\frac{1}{\lambda}\right)$ (similarly as in the condition (3.15) of Remark 3.2), we have the condition

$$u, v \in \mathcal{X} \text{ with } v \preceq u \Rightarrow \tau + F(\omega(Tu, Tv, T^2v)) \leq F(\mathcal{M}(u, v)),$$

for all $u, v \in \mathcal{X}$ with $\omega(Tu, Tv, T^2v) > 0$. Thus $T \in \Phi(F, \omega)$. Therefore, all the hypotheses of Theorem 3.3 are satisfied and we conclude that there is a fixed point $u^* \in \mathcal{X}$ of the operator T. It is obvious that in this case u^* is a solution of the integral equation (6.1). □

Example 6.1. Consider the following second-order initial value problem which is an example of spring-mass problem when damped-forced function is distributed, that is, the so-called “Underdamped Spring-Mass System”:

$$\begin{cases} mu''(t) + cu'(t) + ku(t) = f(t, u(t)), & t \in I = [0, 1], \\ u(0) = u'(0) = 0, \end{cases}$$

where f is an external force, k is the spring constant, m is the mass of spring and c is the damping constant with restriction $-\omega^2 = c^2 - 4mk < 0$.

This problem is known to be equivalent to the integral equation (6.1), where $G : I \times I \to [0, \infty)$ is the Green function given by

$$G(t, s) = \begin{cases} A \exp\left[\left(\frac{-c}{2m}\right)(t - s)\right] \sinh\left[\left(\frac{\omega}{2m}\right)(t - s)\right], & 0 \leq s \leq t \leq 1, \\ 0, & 0 \leq t \leq s \leq 1. \end{cases}$$

Here $A = \frac{c^2 - 2mk}{m^2\omega}$. Then all the conditions of Theorem 6.1 are satisfied if the corresponding integral equation or operator equation of the form (6.1) fulfils the conditions (C1)-(C4). The situation can be better understood by an illustration.

For example, consider the equation

$$\begin{cases} u''(t) + 8u'(t) + 20u(t) = \frac{1}{14} \cos u(t), & 0 < t < 1, \\ u(0) = u'(0) = 0. \end{cases} \tag{6.4}$$

In this case, $m = 1$, $c = 8$, $k = 20$ and so $c^2 - 4km < 0$ which shows that the considered Spring Mass System is “underdamped”. Here $A = 6$ and $f(t, u(t)) = \frac{1}{14} \cos u(t)$.

The function f is continuous and non-decreasing in the second variable, i.e., the condition (C1) holds. It is obvious that the constant function $u_0(t)$ satisfies (C2). Condition (C3) follows as in the paper [4] from the definition of ω. Finally, we have

$$\begin{aligned} |f(t, u(t)) - f(t, v(t))| &= \frac{1}{14} |\cos u(t) - \cos v(t)| \leq \frac{1}{14} |u(t) - v(t)|, \\ |f(t, Tv(t)) - f(t, u(t))| &\leq \frac{1}{14} |Tv(t) - u(t)|, \end{aligned}$$

wherefrom, after routine calculations,

$$\begin{aligned}&\max\Big\{|f(t,u(t))-f(t,v(t))|,|f(t,Tv(t))-f(t,u(t))|\Big\}\\ \le&\frac{1}{14}\Big(|u(t)-v(t)|+|Tv(t)-u(t)|\Big)\\ \le&\frac{1}{6}\cdot\frac{3}{20}\Delta(u,v)(t)=A^{-1}\lambda_1\Delta(u,v)(t).\end{aligned}$$

Hence, the condition (C4) also holds. Thus there exists a solution $u^*\in\mathcal{X}$ of the problem (6.4) via Theorem 6.1.

Remark 6.1. We remark that a similar problem

$$\begin{cases}\dfrac{d^2u}{dt^2}+\dfrac{c}{m}\dfrac{du}{dt}=f(t,u(t)),\\ u(0)=0,u'(0)=a,\end{cases} \tag{6.5}$$

was treated in the paper [10]. The authors claimed that the equivalent integral equation had the form (6.1), and that the Green's function $G(t,s)$ was given by

$$G(t,s)=\begin{cases}(t-s)e^{\tau(t-s)}, & 0\le s\le t\le 1,\\ 0, & 0\le t\le s\le 1.\end{cases}$$

where $\tau>0$ is a constant, calculated in terms of c and m.

However, this claim is not true. The correct form of the equivalent integral equation is

$$u(t)=g(t)+\int_0^t G(t,s)f(s,u(s))\,ds,\ t\in I, \tag{6.6}$$

where $g(t)=\frac{a}{\tau}(1-\exp(-\tau t))$ with $\tau=\frac{c}{m}$ and the respective Green's function is

$$G(t,s)=\begin{cases}\frac{1}{\tau}\Big(1-e^{\tau(t-s)}\Big), & 0\le s\le t\le 1,\\ 0, & 0\le t\le s\le 1.\end{cases} \tag{6.7}$$

If in the problem (6.5), $a=0$, then $g(t)=0$ in integral equation (6.6) but $G(t,s)$ will still be given by (6.7). In any case, this problem can be treated using ω-distance similarly as the one in Example 6.1.

6.2. **Application to fractional differential equations.** Recall the following definitions.

Definition 6.1. For a continuous function $f:[0,\infty)\to\mathbb{R}$, the Caputo derivative of fractional order β is defined as

$${}^cD^\beta(g(t))=\frac{1}{\Gamma(n-\beta)}\int_0^t(t-s)^{n-\beta-1}g^{(n)}(s)\,ds,\ n-1<\beta<n,\ n=[\beta]+1,$$

where $g\in C^n([0,\infty))$, $[\beta]$ denotes the integer part of the positive real number β and Γ is the gamma function.

Definition 6.2. The Riemann-Liouville fractional integral of order β for a continuous function $f(t)$ is defined as

$$I^{\beta}f(t)=\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\beta-1}f(s)\,ds,\ \beta>0,$$

provided that such integral exists.

We consider the three-point fractional integral boundary value problem of the form

$$\begin{cases} {}^{c}D^{\beta}u(t)=f(t,u(t)), & 2<\beta\le 3,\ t\in[0,1],\\ u(\eta)=0,\ u'(0)=0,\ I^{\gamma}u(1)=0,\ 0<\eta<1, \end{cases} \tag{6.8}$$

where $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function and $\eta^2\neq\frac{2}{\gamma^2+3\gamma+2}$. Here we discuss the existence of solutions of (6.8) as an application of Theorem 3.1.

We endow $\mathcal{X}$ with the norm, $\mathcal{G}$-metric and ω-distance in the same way as in Subsection 6.1. Let the corresponding operator $T\colon\mathcal{X}\to\mathcal{X}$ be defined by

$$\begin{aligned} Tu(t)=&\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}f(s,u(s))\,ds-\frac{1}{\Gamma(\beta)}\int_0^{\eta}(\eta-s)^{\beta-1}f(s,u(s))\,ds\\ &+\frac{(\eta^2-t^2)\Gamma(\gamma+3)}{\Gamma(\beta+\gamma)[2-\eta^2(\gamma^2+3\gamma+2)]}\int_0^1(1-s)^{\gamma+\beta-1}f(s,u(s))\,ds\\ &-\frac{(\gamma+2)(p+1)(\eta^2-t^2)}{\Gamma(\beta)[2-\eta^2(\gamma^2+3\gamma+2)]}\int_0^{\eta}(\eta-s)^{\beta-1}f(s,u(s))\,ds, \end{aligned} \tag{6.9}$$

for $u\in\mathcal{X}$, $t\in I$.

Theorem 6.2. *Under the assumptions* $(C1)$-$(C3)$ *of Theorem* 6.1 *and*

(C4') *there exists* $\lambda_1\in(0,\frac{1}{2})$ *such that for* $u,v,w\in\mathcal{X}$ *with* $w\preceq v\preceq u$ *we have*

$$\max\{|f(t,u(t))-f(t,v(t))|,|f(t,v(t))-f(t,w(t))|\}\le B^{-1}\lambda_1\Delta_1(u,v,w)(t) \tag{6.10}$$

where

$$\begin{aligned} B=&\frac{2}{\Gamma(\beta+1)}+\frac{(\gamma+2)(\gamma+1)}{\Gamma(\beta+1)[2-\eta^2(\gamma^2+3\gamma+2)]}\\ &+\frac{\Gamma(\gamma+3)}{\Gamma(\gamma+\beta+1)[2-\eta^2(\gamma^2+3\gamma+2)]} \end{aligned}$$

and

$$\begin{aligned} &\Delta_1(u,v,w)(t)\\ &=\max\left\{\begin{array}{c} |u(t)-v(t)|+|w(t)-u(t)|,|u(t)-v(t)|+|Tu(t)-u(t)|,\\ |u(t)-Tu(t)|+|w(t)-u(t)|,2|u(t)-Tu(t)|,\\ 2|v(t)-Tv(t)|,2|w(t)-Tw(t)|,\\ \frac{1}{2}[|u(t)-Tv(t)|+|Tw(t)-u(t)|\\ +|Tu(t)-v(t)|+|w(t)-Tu(t)|] \end{array}\right\}, \end{aligned}$$

the problem (6.8) *has at least one solution* $u^*\in\mathcal{X}$.

Proof. Here we have to check the contraction condition (3.15) for $u, v, w \in \mathcal{X}$ (see Remark 3.2.(I)).

For each $t \in I$, by the definition (6.9) of operator T, we have

$$\begin{aligned}
&|Tu(t) - Tv(t)| \\
=&\Bigg| \Bigg[\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,u(s))\,ds - \frac{1}{\Gamma(\beta)} \int_0^\eta (\eta-s)^{\beta-1} f(s,u(s))\,ds \\
&+ \frac{(\eta^2 - t^2)\Gamma(\gamma+3)}{\Gamma(\beta+\gamma)[2-\eta^2(\gamma^2+3\gamma+2)]} \int_0^1 (1-s)^{\gamma+\beta-1} f(s,u(s))\,ds \\
&- \frac{(\gamma+2)(p+1)(\eta^2-t^2)}{\Gamma(\beta)[2-\eta^2(\gamma^2+3\gamma+2)]} \int_0^\eta (\eta-s)^{\beta-1} f(s,u(s))\,ds\Bigg] \\
&- \Bigg[\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,v(s))\,ds - \frac{1}{\Gamma(\beta)} \int_0^\eta (\eta-s)^{\beta-1} f(s,v(s))\,ds \\
&+ \frac{(\eta^2 - t^2)\Gamma(\gamma+3)}{\Gamma(\beta+\gamma)[2-\eta^2(\gamma^2+3\gamma+2)]} \int_0^1 (1-s)^{\gamma+\beta-1} f(s,v(s))\,ds \\
&- \frac{(\gamma+2)(p+1)(\eta^2-t^2)}{\Gamma(\beta)[2-\eta^2(\gamma^2+3\gamma+2)]} \int_0^\eta (\eta-s)^{\beta-1} f(s,v(s))\,ds\Bigg]\Bigg| \\
\le&\frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |f(s,u(s)) - f(s,v(s))|\,ds \\
&+ \frac{1}{\Gamma(\beta)} \int_0^\eta (\eta-s)^{\beta-1} |f(s,u(s)) - f(s,v(s))|\,ds \\
&+ \frac{\Gamma(p+3)}{\Gamma(\gamma+\beta)[2-\eta^2(\gamma^2+3\gamma+2)]} \int_0^1 (1-s)^{\gamma+\beta-1} |f(s,u(s)) - f(s,v(s))|\,ds \\
&+ \frac{(\gamma+2)(\gamma+1)}{\Gamma(\beta)[2-\eta^2(\gamma^2+3\gamma+2)]} \int_0^\eta (\eta-s)^{\beta-1} |f(s,u(s)) - f(s,v(s))|\,ds.
\end{aligned}$$

Now, using the assumption (C4'), after routine calculations, we obtain

$$\|Tu - Tv\| \le 2\lambda_1 \theta(u,v,w).$$

Similarly, we can prove that

$$\|Tw - Tu\| \le 2\lambda_1 \theta(u,v,w).$$

This implies that

$$\begin{aligned}
\omega(Tu, Tv, Tw) &= \frac{1}{2}[\|Tu - Tv\| + \|Tw - Tu\|] \\
&\le 2\lambda_1 \theta(u,v,w) = \lambda\theta(u,v,w),
\end{aligned}$$

where $2\lambda_1 = \lambda \in (0,1)$ and $\theta(u,v,w)$ is given in (3.2).

Now, by considering $F \in \mathfrak{F}$ given by $F(t) = \ln t$, $t > 0$, and $\tau = \ln\left(\frac{1}{\lambda}\right)$, we have the condition

$$u, v, w \in \mathcal{X} \text{ with } w \preceq v \preceq u \Rightarrow \tau + F(\omega(Tu, Tv, Tw)) \leq F(\theta(u, v, w)),$$

for all $u, v, w \in \mathcal{X}$ with $\omega(Tu, Tv, Tw) > 0$. Thus $T \in \Lambda(F, \omega)$. Therefore, all the hypotheses of Theorem 3.1 are satisfied and we conclude that there is a fixed point $u^* \in \mathcal{X}$ of the operator T. It is clear that in this case u^* is also a solution of the integral equation (6.9), as well as the fractional differential equation (6.8). $\square$

Example 6.2. Consider the following nonlinear fractional differential equation

$$(6.11) \qquad {}^cD^{\frac{5}{2}}u(t) = \frac{1}{(t+8)^2}\frac{|u(t)|}{1+|u(t)|}, \ t \in [0, 1]$$

with the three-point integral boundary value condition

$$(6.12) \qquad u\left(\frac{1}{2}\right) = 0, \ u'(0) = 0, \ I^{3/2}u(1) = 0.$$

Here $\beta = \frac{5}{2}$, $\eta = \frac{1}{2}$, $\gamma = \frac{3}{2}$, $\eta^2 = \frac{1}{4} \neq \frac{2}{\gamma^2+3\gamma+2} = \frac{8}{35}$ and $f(t, u(t)) = \frac{1}{(t+8)^2}\frac{|u(t)|}{1+|u(t)|}$. Further, $|f(t, u)| \leq \frac{1}{64}$ and $B \approx 28.62$. Therefore, the considered system (6.11)-(6.12) is an example of the system (6.8). With the similar argument of Example 6.1, f and T satisfy the conditions (C1)-(C4'). In particular, routine calculations show that (6.10) holds with $\lambda_1 = \frac{12}{25} \in \left(0, \frac{1}{2}\right)$. Hence, we can apply Theorem 6.2 and conclude that there exists a solution $u^* \in \mathcal{X}$ of the equation (6.11) with the conditions (6.12).

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