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WEAKLY TRIPOTENT RINGS

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Abstract. We completely characterize those rings R, calling them *weakly tripotent*, whose elements satisfy the equations $x^3 = x$ or $x^3 = -x$. This enlarges a result due to Hirano-Tominaga in Bull. Austral. Math. Soc. (1988) concerning tripotent rings.

1. Introduction and Background

Everywhere in the text of the present paper, all our rings R are assumed to be associative, containing the identity element 1 which differs from the zero element 0. Our terminology and notations are mainly in agreement with [6], and the specific ones will be listed explicitly in the sequel. For instance, $U(R)$ denotes the unit group of such a ring R, $\mathrm{Inv}(R)$ is its subset consisting of all involutions (i.e., torsion units of order not exceeding 2) which is actually a subgroup when the former ring is commutative, $Id(R)$ stands for the set of all idempotents in R, and $J(R)$ designates the Jacobson radical of R. Recall that a ring R is *semiprimitive* (or, in other terms, Jacobson semi-simple), provided $J(R) = \{0\}$.

Imitating [2], we shall say that a ring is *invo-clean* if each its element is the sum of an involution and an idempotent, and mimicking [3] a ring is *weakly invo-clean* if each its element is the sum or the difference of an involution and an idempotent.

It is well known that a ring is said to be *tripotent* if any its element satisfies the equation $x^3 = x$. Such an element x is also called *tripotent*.

This motivates us to state the next more general concept.

Definition 1.1. A ring is called *weakly tripotent* if any its element satisfies the equations $x^3 = x$ or $x^3 = -x$.

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Obvious examples of weakly tripotent rings are $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_5$, whereas $\mathbb{Z}_4$ and $\mathbb{Z}_5 \times \mathbb{Z}_5$ need *not* be so.

The brief historical retrospection of the development of this and some other similar notions is as follows: A classical type of rings is the class of *boolean* rings that are rings whose elements are idempotents, i.e., having all elements satisfying the equation $x^2 = x$. Equivalently, any boolean ring is a subdirect product of isomorphic copies of the field $\mathbb{Z}_2$. Furthermore, rings with elements satisfying the equations $x^2 = x$ or $x^2 = -x$, called there *weakly nil-clean*, were explored in [4] proving that they are isomorphic to either a boolean ring, or to $\mathbb{Z}_3$, or to a direct product of two such rings.

On the other hand, concerning cubic equations, in [5] were studied those rings whose elements are solutions of the equation $x^3 = x$. It was shown there that these rings are necessarily commutative being a subdirect product of family of copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$. Even something more, each their element is a sum of two (commuting) idempotents.

However, for the cubic equation $x^3 = x$, which is implied by both the equations $x^2 = x$ and $x^2 = -x$ considered above, can be made the following reduction: $x^3 = x$ is equivalent to $x^2 = vx$, where $v = x^2 + x - 1$ is an involution, that is, $v^2 = 1$ (see, e.g., [1]).

Returning to our weakly tripotent rings, the equations $x^3 = x$ and $x^3 = -x$ can be written in a more general form as $x^3 = wx$ for some involution w with $w^2 = 1$ which certainly amounts to the equation $x^5 = x$. Etc., this could be successfully adapted for any natural number n-compare with the discussion at the end of the paper.

Our objective here is to extend significantly the aforementioned articles [5] and [4], by classifying up to an isomorphism weakly tripotent rings in a different aspect, doing that in Theorem 2.1 quoted below.

2. Main Results

We begin with the following very simple but useful technicality.

Lemma 2.1. *In a weakly tripotent ring R the equalities $6 = 0$ or $10 = 0$ are fulfilled, and hence $R \cong R_2 \times R_3$ or $R \cong R_2 \times R_5$, where $R_2 = \{0\}$ or R_2 is a weakly tripotent ring of characteristic 2, $R_3 = \{0\}$ or R_3 is a weakly tripotent ring of characteristic 3 and $R_5 = \{0\}$ or R_5 is a weakly tripotent ring of characteristic 5.*

Proof. Write $2^3 = 2$ or $2^3 = -2$, so that $6 = 0$ or $10 = 0$ are valid. Since $(2, 3) = 1$ and $(2, 5) = 1$, the second part-half is now an immediate consequence of the Chinese Remainder Theorem. □

We will be now focussed on weakly tripotent rings of characteristics 2, 3 and 5, respectively.

Proposition 2.1. *Let R be a weakly tripotent ring.*

(i) *If $2 = 0$, then R is boolean and so it is a subdirect product of $\prod_\lambda \mathbb{Z}_2$, where λ is an ordinal.*

(ii) *If* $3 = 0$*, then* R *is tripotent and thus it is a subdirect product of* $\prod_{\mu} \mathbb{Z}_3$*, where* μ *is an ordinal.*
(iii) *If* $5 = 0$*, then* R *is isomorphic to* $\mathbb{Z}_5$*, that is,* $R \cong \mathbb{Z}_5$*.*

Proof. (i) Each of the equations $x^3 = x$ and $x^3 = -x$ implies that $x^5 = x$ and thus, as it is well-known, R must be a subdirect product of isomorphic copies of the fields $\mathbb{Z}_2$, $\mathbb{Z}_3$ and/or $\mathbb{Z}_5$. But $2 = 0$ yields that the only members of this subdirect product are these isomorphic to $\mathbb{Z}_2$, as expected.

(ii) Each of the equations $x^3 = x$ and $x^3 = -x$ implies that $x^5 = x$ and so, as it is well-known, R has to be a subdirect product of isomorphic copies of the fields $\mathbb{Z}_2$, $\mathbb{Z}_3$ and/or $\mathbb{Z}_5$. But $3 = 0$ yields that the only members of this subdirect product are isomorphic to $\mathbb{Z}_3$, as expected.

(iii) As in the preceding two points, R is a subdirect product of $\prod_{\nu} \mathbb{Z}_5$ for some ordinal ν (see, e.g., [6]). We claim that $\nu = 1$, that is, R is embedded in $\mathbb{Z}_5$ which forces their isomorphism, as promised. In order to show that, we consider all elements of the kind (a, b), where a, b belong to the set $\{0, 1, 2, 3, 4\}$. However, all elements in R are $\overline{0} = (0, 0)$, $\overline{1} = (1, 1)$, $\overline{2} = (2, 2)$, $\overline{3} = (3, 3)$, $\overline{4} = (4, 4)$, which are exactly these $\overline{a} = (a, a)$ for $b = a$. Indeed, to verify this, one must to see that the elements $0, 1, 4$ are solutions of the equation $x^3 = x$ while the elements $2, 3$ are solutions of the equation $x^3 = -x$. Therefore, if some (a, b) lies in R for $a \neq b$, then $2(a, b) = (2a, 2b)$ or $(a, b)+(a, a) = (2a, a+b)$ have again to lie in R whence by a direct check, which we leave to the reader, we will obtain that $2a$, $2b$ and/or $a + b$ are solutions of either of the different equations $x^3 = x$ or $x^3 = -x$, which is a contradiction. That is why, R cannot be properly embedded in $\mathbb{Z}_5 \times \mathbb{Z}_5$, and so the claim is sustained after all.

We also have the following parallel confirmation of the validity of point (iii) like this: Let P be the subring of R generated by 1, and thus note that $P \cong \mathbb{Z}_5$. We claim that $P = R$, so we assume in a way of contradiction that there exists $b \in R \setminus P$. With no loss of generality, we shall also assume that $b^3 = b$ since $b^3 = -b$ obviously implies that $(2b)^3 = 2b$ as $5 = 0$ and $b \notin P \iff 2b \notin P$.

Let us now $(1+b)^3 = -(1+b)$. Hence $b = b^3$ along with $5 = 0$ enable us that $b^2 = 1$. This allows us to conclude that $(1+2b)^3 \neq \pm(1+2b)$, however. In fact, if $(1 + 2b)^3 = 1 + 2b$, then one deduces that $2b = 3$ and, by multiplying with 3, that $b = -1 \in P$ which is manifestly untrue. If now $(1 + 2b)^3 = -1 - 2b$, then one infers that $2b = 2 \in P$ which is false. That is why, $(1 + b)^3 = 1 + b$. This, in turn, guarantees that $b^2 = -b$. Moreover, $b^3 = b$ is equivalent to $(-b)^3 = -b$ as well as $b^3 = -b$ to $(-b)^3 = -(-b)$ and thus, by what we have proved so far applied to $-b \notin P$, it follows that $-b = b^2 = (-b)^2 = -(-b) = b$. Consequently, $2b = 0 = 6b = b \in P$ because $5 = 0$, which is the wanted contradiction. We thus conclude that $P = R$, as expected. □

We may discuss the last statement in the following way.

Remark 2.1. Actually, $x^3 = x$ for all x with $2 = 0$ ensures that $x^2 = x$, while $x^3 = x$ or $x^3 = -x$ for all x with $3 = 0$ assures that $x^3 = x$.

We now come to our chief tool here, which is the following one.

Theorem 2.1. *Suppose R is a ring. Then the following five items are tantamount:*

(0) *R is weakly tripotent;*
(1) *all elements of R satisfy the equations $x^3 = x$ or $x^3 = -x$;*
(2) *R is commutative such that every element is a sum of two idempotents with $6 = 0$, or R is commutative such that every element is a sum or a difference of an involution and an idempotent with $10 = 0$;*
(3) *R is a subdirect product of isomorphic copies of $\mathbb{Z}_2$ and/or $\mathbb{Z}_3$, or of $\mathbb{Z}_2$ and/or a single isomorphic copy of $\mathbb{Z}_5$;*
(4) *R is commutative semiprimitive weakly invo-clean with $6 = 0$ or $10 = 0$.*

Proof. First of all, notice that the equivalence (0)⇔(1) is just Definition 1.1 alluded to above.

(1)⇔(3) If (1) is valid, then we can combine Lemma 2.1 together with Proposition 2.1.

Conversely, if (3) holds, then all elements of R are solutions of the equations $x^3 = x$ or $x^3 = -x$, because the elements from $\mathbb{Z}_2$ and $\mathbb{Z}_3$ satisfy $x^3 = x$, whereas the elements of $\mathbb{Z}_2$ satisfy $x^2 = x = -x$ and hence $x^3 = -x$ as well as the elements of $\mathbb{Z}_5$ satisfy the same equation $x^3 = -x$ along with the equation $x^3 = x$, as needed.

(3)⇒(2) If R is in the first subdirect product, as noted before, it follows from [5] that any its element is the sum of two idempotents, so we may assume that R is embedded in the second subdirect product. Knowing that each element c of the subdirect product of copies of $\mathbb{Z}_2$ is an idempotent e and that each element d of $\mathbb{Z}_5$ is a sum or a difference of an involution and an idempotent, say $v + f$ or $v - f$, we detect that $(e, v + f) = (1, v) + (1 - e, f)$ which is again the sum of an involution and an idempotent as $2 = 0$ in $\mathbb{Z}_2$ and that $(e, v - f) = (1, v) - (1 - e, f)$ which is also the difference of an involution and an idempotent. With this at hand, point (2) is true.

(2)⇒(4) If $6 = 0$, what suffices to prove is that the sum $e + f$ for some $e, f \in Id(R)$ is a sum or a difference of an involution and an idempotent. In fact, when $2 = 0$, $e + f$ must be an idempotent, say h, and hence $h = 1 - (1 - h)$ satisfies our requirement. If now $3 = 0$, then $e + f = (1 + e) - (1 - f)$ where $(1 + e)^2 = 1 + 3e = 1$ and $(1 - f)^2 = 1 - f$, as required. Since with the aid of the Chinese Remainder Theorem R can be decomposed as the direct product of two such rings having characteristics 2 and 3, respectively, we are set.

Likewise, one observes that the sum $e + f$ satisfies $(e + f)^3 = e + f$ which is definitely right because $(e + f)^3 = e + 6ef + f = e + f$ taking into account that $6 = 0$.

(3)⇔(4) This equivalence follows by an application of [3] (cf. [2] too). □

Consulting with the proof of this theorem, one deduces that $\mathbb{Z}_2 \times \mathbb{Z}_5$ is weakly tripotent, but $\mathbb{Z}_3 \times \mathbb{Z}_5$ is not so. We close the work with the following challenging question.

Problem 2.1. What is the isomorphic structure of the so-called *weakly n-potent* rings, where $n \in \mathbb{N}$, i.e., rings whose elements satisfy the equations $x^n = x$ or $x^n = -x$?

What can be currently said is that $x^n = vx$, for $v^2 = 1$, is amounting to $x^{2n-1} = x$, where $v = x^{2n-2} + x^{n-1} - 1$, x^{2n-2} is an idempotent and x^{n-1} is a tripotent.

In particular, for $n = 3$, one may check that the equation $x^3 = vx$, that is, $x = vx^3$ is equivalent to $x^5 = x$, where $v = x^4 + x^2 - 1$ is an involution, i.e., $v^2 = 1$. Certainly, weakly n-potent rings must be commutative reduced and, therefore, they have to be a subdirect product of domains, each of which satisfies these two conditions.

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