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A NEW CLASS OF CONFORMABLE FRACTIONAL SOBOLEV SPACES AND p-ELLIPTIC PROBLEMS

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Abstract. In this article, we introduce two new mathematical notions: the conformable fractional Sobolev space ${}^{c}W^{\gamma,p}(\Omega)$ and the conformable fractional p-Laplacian operator, both defined via the conformable fractional derivative. We investigate several qualitative properties of these concepts, including embeddings and regularity results. As an application, we prove the existence of solutions to an elliptic boundary value problem involving the conformable fractional p-Laplacian operator.

1. Introduction

In recent years, elliptic operators and nonlocal fractional operators have attracted significant attention, motivated by both theoretical developments and applications in concrete models. In particular, nonlocal problems exhibiting p-growth structure have been extensively investigated, as they provide an effective framework for modeling anomalous diffusion phenomena. Such nonlocal behavior arises in various scientific contexts in which the underlying dynamics deviate from classical power-law growth, as documented in [12–14, 19]. A substantial body of work has been devoted to issues related to fractional diffusion.

$$(-\triangle)^s\mu(y) = C(N,s)\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B(y,\varepsilon)}\frac{\mu(y)-\mu(z)}{|y-z|^{N+2s}}dy,$$

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where $y \in \mathbb{R}^N$, $s \in (0,1)$, see, for example [8,17,31,32] and the references therein. To delve deeper into the study of nonlocal problems, one can refer to Di Nezza, Palatucci, and Valdinoci's work in [16], which provides a thorough introduction to the subject. Moreover, in recent references, the authors have explored the viability of results obtained when replacing the Laplacian with the fractional Laplacian. Moreover, the fractional p-Laplacian $(-\Delta)_p^s$ is the nonlinear nonlocal operator defined on smooth functions, for $p \in (1,+\infty)$, $s \in (0,1)$ and $N > sp$, by

$$(-\triangle)_p^s \mu(y) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B(y,\varepsilon)} \frac{|\mu(y)-\mu(z)|^{p-2}(\mu(y)-\mu(z))}{|y-z|^{N+2s}} dy, \quad y \in \mathbb{R}^N,$$

where $B(y,\varepsilon)$ is the ball with center y and radius ε.

The definition presented here is consistent with the standard definition of the linear fractional Laplacian operator $(-\Delta)^s$ for $p = 2$, up to a normalisation constant that depends on N and s. A substantial literature is currently emerging on problems involving these nonlocal operators (see [1,10,16,22] for more details).

In particular, the study of fractional Sobolev spaces and the associated nonlocal equations has gained considerable attention, owing to their wide range of physical applications, including phase transition models, thin obstacle problems, layered media, minimal surface theory, and materials science (see [6,7,9,29,31]). Moreover, a substantial amount of research has been devoted to partial differential equations involving the degenerate fractional p-Laplacian, often addressed through variational methods. For further developments in this direction, we refer the reader to [8,11,17].

It is natural to ask what results can be obtained by replacing the standard Laplacian operator with a new Laplacian operator based on the conformal fractional derivative given by

$$-\triangle_p^\gamma \mu(y) := -\sum_{i=1}^{N} \partial_i^\gamma \left(|\partial_i^\gamma \mu(y)|^{p-2} \partial_i^\gamma \mu \right), \quad y \in \Omega,$$

where $\partial_i^\gamma = \frac{\partial^\gamma}{\partial y_i}$ is a conformable fractional derivative which can be specified below.

On the other hand, for certain problems modelling inhomogeneous materials, such as the electro-rheology of fluids (sometimes called "smart fluids"), the standard approach based on Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{1,p}(\Omega)$ is not sufficient. This leads to the introduction of conformable fractional Sobolev spaces ${}^cW^{\gamma,p}(\Omega)$, where p and γ are real numbers such that $\gamma \in (0,1)$ and $p \in (1,+\infty)$.

The purpose of our paper is to introduce the Sobolev conformal fractional space ${}^cW^{\gamma,p}(\Omega)$ and the conformal fractional p-Laplacian operator, and to discuss their fundamental properties. In addition, we have investigated the existence of weak solutions in different cases for the following problem

$$\begin{cases} -\triangle_p^\gamma \mu = -\mathrm{div}_\gamma(|\nabla_\gamma \mu|^{p-2} \nabla_\gamma \mu) = |\mu|^{q-2}\mu + g(y,\mu), & y \in \Omega, \\ \mu = 0, & y \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $1 < p$, $q \in [1,p)$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory condition.

This article is organized as follows. In Section 2, we review definitions and findings related to the conformable fractional derivative. Additionally, we introduce and demonstrate fundamental properties of the conformable fractional Sobolev spaces ${}^{c}W^{\gamma,p}(\Omega)$. In Section 3, we present significant properties of the conformable fractional p-Laplacian operator. Finally, in Section 4, we discuss the existence of weak solutions for problem (1.1) in various cases.

2. Definitions and Preliminaries

In this section, we present several definitions and results related to the conformable fractional derivative. Various definitions of fractional derivatives exist in the literature; for example, in [23, 25], the authors introduce the following definition of the conformable fractional derivative.

Definition 2.1 ([25])**.** Given a function $g:\mathbb{R}^+\to\mathbb{R}$, $\gamma\in(0,1)$ and $t>0$, the conformable fractional derivative at order γ of g is defined by

$$(D^\gamma g)(t)=\lim_{\varepsilon\to 0}\frac{g\left(t+\varepsilon\, e^{1-\gamma}\right)-g(t)}{\varepsilon}.$$

If g is γ-differentiable in $(0,b)$, $b>0$ and $\lim_{t\to 0^+}(D^\gamma g)(t)$ exists, then

$$(D^\gamma g)(0)=\lim_{t\to 0^+}(D^\gamma g)(t).$$

Remark 2.1. Definition 2.1 remains valid for any function g defined on the set $\mathbb{R}$.

We will now give some properties of this derivative, for more details see [4,24,25,27].

Proposition 2.1 ([25])**.** *Let $0<\gamma\le 1$ and h,g be two γ-differentiable functions at a point $t>0$. Then,*

(a) $D^\gamma(ah+bg)=a\,D^\gamma h+b\,D^\gamma g$, *for all* $a,b\in\mathbb{R}$;
(b) $D^\gamma\left(t^p\right)=pe^{(\gamma-1)t}t^{p-1}$, *for any p real number;*
(c) *if λ is a constant function, then* $D^\gamma(\lambda)=0$;
(d) $D^\gamma\left(hg\right)=f\;D^\gamma g+g\;D^\gamma h$;
(e) $D^\gamma(\frac{h}{g})=\frac{hD^\gamma g+gD^\gamma h}{g^2}$;
(f) *in addition, if h is differentiable, then* $(D^\gamma h)(t)=e^{(\gamma-1)t}h'(t)$.

Definition 2.2 ([25])**.** Let $\gamma\in(n,n+1]$, $n\in\mathbb{N}$ and g be a function n-differentiable at $t>0$. Then, the γ-fractional derivative of g is given as follows

$$\left(D^\gamma g\right)(t)=\lim_{\varepsilon\to 0}\frac{g^{(n)}\left(t+\varepsilon\, e^{(\gamma-1-n)t}\right)-g^{(n)}(t)}{\varepsilon},$$

if the limit exists.

Definition 2.3 ([25])**.** Let $c\ge 0$, $\gamma\in(0,1)$ and g be a function defined on interval $(c,t]$. Then, the γ-fractional integral of g is given as follows

$$\mathcal{I}^c_\gamma(g)(t)=\int_c^t g(y)d_\gamma x=\int_c^t g(y)e^{(1-\gamma)y}dy.$$

Theorem 2.1 ([25])**.** *If $g : [c, +\infty) \to \mathbb{R}$ is a continuous function and $\gamma \in (0, 1]$, then, for $t > c$, we have*

$$D_c^\gamma \mathfrak{I}_\gamma^c g(t) = g(t).$$

Lemma 2.1 ([25])**.** *Let $g : (c, d) \to \mathbb{R}$ be a γ-differentiable function. Then, for $t > c$, we have*

$$\mathfrak{I}_\gamma^c D_c^\gamma g(t) = g(t) - g(c).$$

2.1. **The conformable fractional Sobolev space.** In this section, we present the definition and some basic properties of the conformable fractional Sobolev spaces.

Let Ω be a bounded open set of $\mathbb{R}^N$, $N \in \mathbb{N}^*$ and $p \in (1, +\infty)$. The conformable fractional Sobolev space, denoted by ${}^cW^{\gamma,p}(\Omega)$, is defined as the space of all real-valued functions $\mu \in L^p(\Omega)$ such that

$$\partial_i^\gamma \mu \in L^p(\Omega), \quad i = 1, \dots, N,$$

where $\partial_i^\gamma = \frac{\partial^\gamma}{\partial y_i}$ is a conformable fractional derivative in the sense of distributions. In addition, the following expression

$$\|\mu\|_{\gamma,p} = \left(\int_\Omega |\mu(y)|^p \, dy + \sum_{i=1}^N \int_\Omega |\partial_i^\gamma \mu(y)|^p \, dy \right)^{1/p} \tag{2.1}$$

is a norm in linear space ${}^cW^{\gamma,p}(\Omega)$.

Throughout this work, we set $\omega = \{\omega_i(y) = e^{(\gamma-1)py_i},\ 0 \le i \le N\}$. Then, ω is a vector of weight functions, i.e., each component $\omega_i(y)$ is a positive measurable function almost everywhere in Ω. Additionally, we have

$$w_i \in L^1_{\text{loc}}(\Omega) \quad \text{and} \quad w_i^{\frac{-1}{p-1}} \in L^1_{\text{loc}}(\Omega), \quad \text{for every } i \in \{1, 2, \dots, N\}. \tag{2.2}$$

Theorem 2.2. *${}^cW^{\gamma,p}(\Omega)$ is a Banach space.*

Proof. It suffices to show that ${}^cW^{\gamma,p}(\Omega)$ is complete. Let $\{\mu_n\}$ be a Cauchy sequence in $cW^p(\Omega)$. Then, the sequence $\{\mu_n\}$ is a Cauchy sequence in $L^p(\Omega)$, and for each $i = 1, \dots, N$, the sequence $\{\partial_i^\alpha \mu_n\}$ is also a Cauchy sequence in $L^p(\Omega)$.

By the completness of $L^p(\Omega)$, there exist two functions ϑ, $\mu \in L^p(\Omega)$ such that

$$\begin{cases} \lim\limits_{n\to+\infty} \partial_i^\gamma \mu_n = \vartheta, & \text{in } L^p(\Omega), \\ \lim\limits_{n\to+\infty} \mu_n = u, & \text{in } L^p(\Omega). \end{cases} \tag{2.3}$$

According to (2.3), a simple calculation shows that $\{\partial_i \mu_n\}$ is a convergent sequence in $L^p(\Omega, w)$, with with a usual derivative ∂_i. Then, there exists $\nu \in L^p(\Omega, w)$ such that

$$\begin{cases} \lim\limits_{n\to+\infty} \partial_i \mu_n = \nu, & \text{in } L^p(\Omega, w), \\ \lim\limits_{n\to+\infty} \mu_n = u, & \text{in } L^p(\Omega). \end{cases} \tag{2.4}$$

From (2.2) and (2.4), we get

$$\begin{cases} \lim_{n\to+\infty} \partial_i\mu_n = \nu, & \text{in } L^p(\Omega, w), \\ \lim_{n\to+\infty} \mu_n = u, & \text{in } L^p(\Omega, w). \end{cases}$$

By using the continuity of the derivation operator, we obtain

$$\mu \in L^p(\Omega), \quad \lim_{n\to+\infty} \partial_i\mu_n = \partial_i\mu, \quad \text{in } L^p(\Omega, w).$$

So, from the uniqueness of the limit we can get $\nu = \partial_i\mu$. Moreover, $\mu \in {}^cW^{\gamma,p}(\Omega)$ and

$$\begin{aligned} \|\mu_n - \mu\|\gamma, p &= \left(\int_\Omega |\mu_n(y) - \mu(y)|^p \, dy + \sum_{i=1}^N \int_\Omega |\partial_i^\gamma(\mu_n(y) - \mu(y))|^p \, dy \right)^{1/p} \\ &= \left(\int_\Omega |\mu_n(y) - \mu(y)|^p \, dy + \sum_{i=1}^N \int_\Omega |\partial_i \ (\mu_n(y) - \mu(y)) \, |^p w_i(y) \, dy \right)^{1/p} \\ &= \left(\int_\Omega |\mu_n(y) - \mu(y)|^p \, dy + \sum_{i=1}^N \int_\Omega |\partial_i\mu_n(y) - \partial_i\mu(y)|^p w_i(y) \, dy \right)^{1/p} \\ &= \left(\int_\Omega |\mu_n(y) - \mu(y)|^p \, dy + \sum_{i=1}^N \int_\Omega |\partial_i\mu_n(y) - \nu(y)|^p w_i(y) \, dy \right)^{1/p}. \end{aligned}$$

Taking $n \to +\infty$ and from (2.4), we get that Cauchy sequence $\{\mu_n\}$ converges to μ in ${}^cW^{\gamma,p}(\Omega)$. Hence, ${}^cW^{\gamma,p}(\Omega)$ is a complete space. ☐

From the first expression in the formula (2.2), we know that $\mathcal{C}_0^\infty(\Omega)$ is a subspace of ${}^cW^{\gamma,p}(\Omega)$ and therefore, we can define the subspace ${}^cW_0^{\gamma,p}(\Omega)$ of ${}^cW^{\gamma,p}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm (2.1). Moreover, the norm

$$\|\mu\|_\gamma = \left(\sum_{i=1}^N \int_\Omega |\partial_i^\gamma \mu(y)|^p \, dy \right)^{1/p}$$

is assigned to the Sobolev space ${}^cW_0^{\gamma,p}(\Omega)$ which is equivalent to the norm (2.1).

Theorem 2.3. *The spaces ${}^cW^{\gamma,p}(\Omega)$ and ${}^cW_0^{\gamma,p}(\Omega)$ are reflexive and separable Banach spaces.*

Proof. According to the second expression in the formula (2.2) and taking into account that $L^p(\Omega)$ is a reflexive and separable space, we can easily show that ${}^cW^{\gamma,p}(\Omega)$ and $W_0^{\gamma,p}(\Omega)$ are reflexive Banach spaces. ☐

Theorem 2.4 (Conformable fractional Sobolev space embedding)**.** *For $p \in (1, +\infty)$ we have the following.*

(a) *If $N > p$, then, for each r such that $p \le r < \frac{Np}{N-p}$,*

$${}^cW^{\gamma,p}(\Omega) \hookrightarrow L^r(\Omega).$$

More precisely, under the given conditions, there is $C > 0$ *such that*

$$\|\mu\|_{L^r} \leq C\|\mu\|_{\gamma,p}, \quad \textit{for every } \mu \in {}^cW^{\gamma,p}(\Omega).$$

(b) *If* $N = p$, *then for each* r *satisfying* $p \leq r \leq +\infty$, *then we have*

$${}^cW^{\gamma,p}(\Omega) \hookrightarrow L^r(\Omega).$$

Proof. Let $\mu \in {}^cW^{\gamma,p}(\Omega)$, we can easily established

$$\sum_{i=0}^{N} \int_\Omega |\partial_i \mu(y)|^p \leq C' \sum_{i=0}^{N} \int_\Omega |\partial_i^\gamma \mu(y)|^p, \quad C' > 0. \tag{2.5}$$

On the other hand, from $W^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$ and (2.5), we find the results. □

The embedding of conformable fractional Sobolev space remain true locally, i.e., in any open compactly included in Ω. They remain true globally if we replace ${}^cW^{\gamma,p}(\Omega)$ by ${}^cW_0^{\gamma,p}(\Omega)$. We notice that $({}^cW_0^{\gamma,p}(\Omega), \|\cdot\|\gamma)$ is a uniformly convex Banach space.

Remark 2.2. We know that $w_0(y) \equiv 1$ and for all $\lambda \in \left(\frac{N}{p}, +\infty\right) \cap \left(\frac{1}{p-1}, +\infty\right)$, we have

$$w_i^{-\lambda} \in L^1(\Omega), \quad \text{for every } i = 1, \dots, N.$$

We noticed that the expression (2.2) is stronger than the second expression in the formula (2.2). Then,

$$\|\mu\|_\gamma = \left(\sum_{i=1}^{N} \int_\Omega |\partial_i^\gamma \mu|^p \, dy \right)^{1/p}$$

is a norm defined on ${}^cW_0^{\gamma,p}(\Omega)$ and it is equivalent to the norm (2.1). Moreover, the imbedding

$${}^cW_0^{\gamma,p}(\Omega) \hookrightarrow L^r(\Omega)$$

is compact for every $r \geq 1$ if $p\lambda \geq N(\lambda+1)$ and for every $1 \leq r \leq p_1^*$ if $p\lambda < N(\lambda+1)$, where $p_1 = \frac{p\lambda}{\lambda+1}$ and p_1^* is the Sobolev conjugate of p_1 (see [18, p. 30, 31]).

3. Properties of Conformable Fractional p-Laplacian Operator

In this section, we will discuss the conformable fractional p-Laplacian operator defined by:

$$-\triangle_p^\gamma \mu = -\operatorname{div}_\gamma(|\nabla_\gamma \mu|^{p-2} \nabla_\gamma \mu) = -\sum_{i=1}^{N} \partial_i^\gamma \left(|\partial_i^\gamma \mu|^{p-2} \partial_i^\gamma \mu \right).$$

Let

$$\mathcal{K}(\mu) = \frac{1}{p} \sum_{i=1}^{N} \int_\Omega |\partial_i^\gamma \mu(y)|^p dy, \quad \mu \in \mathcal{Y} := {}^cW_0^{\gamma,p}(\Omega),$$

where $\mathcal{K} \in \mathcal{C}^1({}^cW_0^{\gamma,p}(\Omega), \mathbb{R})$ (for more details, see [15]), and the conformable fractional p-Laplacian operator is the derivative of operator $\mathcal{K}$ in the conformable weak sense. We define $\mathcal{Q} = D_\gamma \mathcal{K} : \mathcal{Y} \to \mathcal{Y}^*$, so

$$\langle \mathcal{Q}\mu, \nu \rangle = \sum_{i=1}^{N} \int_\Omega |\partial_i^\gamma \mu(y)|^{p-2} \partial_i^\gamma \mu(y) \partial_i^\gamma \nu(y) dy,$$

for all $\nu, \mu \in \mathcal{Y}$.

Lemma 3.1. *The operator* $\mathcal{Q}$ *is*

(i) *continuous, bounded and strictly monotone;*
(ii) *of type* $(\mathcal{S}^+)$*, i.e., if* $\limsup_{n\to+\infty} \langle \mathcal{Q}\mu_n - \mathcal{Q}\mu, \mu_n - \mu\rangle \leq 0$ *and* $\mu_n \rightharpoonup \mu$ *in* $\mathcal{Y}$*, then* $\mu_n \to \mu$ *in* $\mathcal{Y}$*;*
(iii) *is homeomorphism.*

Proof. (i) It is clear that $\mathcal{Q}$ is continuous and bounded. For all $\tau, \theta \in \mathbb{R}^N$, we have

$$\left[(|\tau|^{p-2}\tau - |\theta|^{p-2}\theta)(\tau - \theta)\right] (|\tau|^p + |\theta|^p)^{(2-p)/p} \geq (p-1)|\tau - \theta|^p, \quad 1 < p < 2, \tag{3.1}$$

and

$$\left(|\tau|^{p-2}\tau - |\theta|^{p-2}\theta\right)(\tau - \eta) \geq \left(\frac{1}{2}\right)^p |\tau - \theta|^p, \quad p \geq 2. \tag{3.2}$$

By using (3.1) and (3.2), we can obtain the strictly monotonicity of $\mathcal{Q}$ (see [26]).

(ii) According to (i), if $\mu_n \rightharpoonup \mu$ and $\limsup_{n\to+\infty} \langle \mathcal{Q}\mu_n - \mathcal{Q}\mu,\ \mu_n - \mu\rangle \leq 0$, then

$$\lim_{n\to+\infty} \langle \mathcal{Q}\mu_n - \mathcal{Q}\mu, \mu_n - \mu\rangle = 0.$$

From (3.1) and (3.2), $\nabla_\gamma u_n$ converges in measure to $\nabla_\gamma \mu$ in Ω, and thus

$$\nabla_\gamma \mu_n(y) \to \nabla_\gamma \mu(y), \quad \text{a.e. } y \in \Omega.$$

By Fatou Lemma, we infer

$$\liminf_{n\to+\infty} \frac{1}{p} \sum_{i=1}^{N} \int_\Omega |\partial_i^\gamma \mu_n(y)|^p dy \geq \frac{1}{p} \sum_{i=1}^{N} \int_\Omega |\partial_i^\gamma \mu(y)|^p dy. \tag{3.3}$$

Since $\mu_n \rightharpoonup \mu$, we obtain

$$\lim_{n\to+\infty} \langle \mathcal{Q}\mu_n, \mu_n - \mu\rangle = \lim_{n\to+\infty} \langle \mathcal{Q}\mu_n - \mathcal{Q}\mu, \mu_n - \mu\rangle = 0.$$

On the other hand, we have

$$\begin{aligned}
\langle \mathcal{Q}\mu_n, \mu_n - \mu\rangle &= \sum_{i=1}^{N} \int_\Omega |\partial_i^\gamma \mu_n(y)|^p dy - \sum_{i=1}^{N} \int_\Omega |\partial_i^\gamma \mu_n(y)|^{p-2} \partial_i^\gamma \mu_n(y) \partial_i^\gamma \mu(y) dy \\
&\geq \sum_{i=1}^{N} \int_\Omega |\partial_i^\gamma \mu_n(y)|^p dy - \sum_{i=1}^{N} \int_\Omega |\partial_i^\gamma \mu_n(y)|^{p-1} |\partial_i^\gamma \mu(y)| dy \\
&\geq \sum_{i=1}^{N} \int_\Omega |\partial_i^\gamma \mu_n(y)|^p dy - \sum_{i=1}^{N} \int_\Omega \left[\frac{p-1}{p} |\partial_i^\gamma \mu_n(y)|^p + \frac{1}{p} |\partial_i^\gamma \mu(y)|^p\right] dy
\end{aligned}$$

$$\geq \frac{1}{p}\sum_{i=1}^{N}\int_{\Omega}|\partial_i^{\gamma}\mu_n(y)|^p dy - \frac{1}{p}\sum_{i=1}^{N}\int_{\Omega}|\partial_i^{\gamma}\mu(y)|^p dy. \tag{3.4}$$

By using (3.3) and (3.4), we can get

$$\lim_{n\to+\infty}\frac{1}{p}\sum_{i=1}^{N}\int_{\Omega}|\partial_i^{\gamma}\mu_n(y)|^p dy = \frac{1}{p}\sum_{i=1}^{N}\int_{\Omega}|\partial_i^{\gamma}\mu(y)|^p dy. \tag{3.5}$$

According to (3.5), the integrals of the functions family $\left\{\frac{1}{p}\sum_{i=1}^{N}|\partial_i^{\gamma}\mu_n(y)|^p\right\}$ are absolutely equi-continuity on Ω (see [28, Chapiter 6, Section 3]. Moreover, since

$$\frac{1}{p}\sum_{i=1}^{N}|\partial_i^{\gamma}\mu_n(y)-\partial_i^{\gamma}\mu(y)|^p \leq C\Big[\frac{1}{p}\sum_{i=1}^{N}|\partial_i^{\gamma}\mu_n(y)|^p + \frac{1}{p}\sum_{i=1}^{N}|\partial_i^{\gamma}\mu(y)|^p\Big],$$

the integrals of the family $\left\{\frac{1}{p}\sum_{i=1}^{N}|\partial_i^{\gamma}\mu_n(y)-\partial_i^{\gamma}\mu(y)|^p\right\}$ are absolutely equi-continuous on Ω (cf. [28]), and thus

$$\lim_{n\to+\infty}\frac{1}{p}\sum_{i=1}^{N}\int_{\Omega}|\partial_i^{\gamma}\mu_n(y)-\partial_i^{\gamma}\mu(y)|^p dy = 0,$$

implies that

$$\lim_{n\to+\infty}\frac{1}{p}\sum_{i=1}^{N}\int_{\Omega}|\partial_i^{\gamma}\mu_n(y)-\partial_i^{\gamma}\mu(y)|^p dy = 0. \tag{3.6}$$

From (3.6), $\mu_n \to \mu$ (see [20, 21, 30]).

(iii) By using the strictly monotonicity, $\mathcal{Q}$ is an injection. As,

$$\lim_{\|\mu\|_{\gamma}\to+\infty}\frac{\langle \mathcal{Q}\mu,\mu\rangle}{\|\mu\|_{\gamma}} = \lim_{\|\mu\|_{\gamma}\to+\infty}\frac{\int_{\Omega}|\nabla_{\gamma}\mu|^p dy}{\|u\|_{\gamma}} = +\infty,$$

then $\mathcal{Q}$ is coercive, and from Minty-Browder Theorem (see [34]) $\mathcal{Q}$ is a surjection. So, $\mathcal{Q}$ admits an inverse mapping $\mathcal{Q}^{-1}:\mathcal{Y}^*\to\mathcal{Y}$.

On the other side, if $g_n, g\in\mathcal{Y}^*$ such that $\lim_{n\to+\infty} g_n = g$ and if

$$\mu_n = \mathcal{Q}^{-1}g_n, \quad \mu = \mathcal{Q}^{-1}g,$$

then

$$\mathcal{Q}\mu_n = g_n, \quad \mathcal{Q}\mu = g.$$

So, $\{\mu_n\}$ is bounded in $\mathcal{Y}$.

Without loss of generality, we suppose that $\mu_n \rightharpoonup \mu_0$. Since $g_n\to g$, then

$$\lim_{n\to+\infty}\langle \mathcal{Q}\mu_n-\mathcal{Q}\mu_0,\mu_n-\mu_0\rangle = \lim_{n\to+\infty}\langle g_n,\mu_n-\mu_0\rangle = 0.$$

As $\mathcal{Q}$ satisfies property (ii) of Lemma 3.1, we deduce that $\mu_n\to\mu$, so $\mathcal{Q}^{-1}$ is continuous. □

4. Existence of Solutions

In this section we will discuss the existence of weak solutions of the problem (1.1) in various cases.

Definition 4.1. Let $\mu \in {}^{c}W_0^{\gamma,p}(\Omega)$. We say that μ is a weak solution of the problem (1.1), if

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i^{\gamma} \mu(y)|^{p-2} \partial_i^{\gamma} \mu(y) \partial_i^{\gamma} \nu(y) dy = \int_{\Omega} \Big[|\mu|^{q-2}\mu + g(y,\mu) \Big] \nu dy,$$

for all $\nu \in \mathcal{Y} := {}^{c}W_0^{\gamma,p}(\Omega)$.

Case 1. In this case, we will investigate the existence of weak solutions of (1.1) when the right-hand side g depends only on x.

Theorem 4.1. *Assume that $g(y,\mu) = g(y)$ satisfies $g \in L^q(\Omega)$, where $q \in \mathbb{R}$, such that $q \in (1,+\infty)$ and $\frac{1}{q} + \frac{1}{p^*} < 1$. Then, the problem* (1.1) *admits a unique weak solution.*

Proof. By using [21, Proposition 2.5], for each $\nu \in \mathcal{Y}$, $\langle g, \nu \rangle := \int_{\Omega} g(y)\nu dy$ defines a continuous linear function on ${}^{c}W_0^{\gamma,p}(\Omega)$. As $\mathcal{Q}$ is a homeomorphism, (1.1) admits a unique weak solution.

From now on, it is assumed that $g(y,\mu)$ satisfies the following hypothesis:

$(\mathcal{A}_1)$ $|g(y,t)| \le c_1 + c_2|t|^{r-1}$, where $r \in [1,p^*)$, for all $(y,t) \in \Omega \times \mathbb{R}$.

Let

$$\phi(\mu) = \frac{1}{p} \sum_{i=1}^{N} \int_{\Omega} |\partial_i^{\gamma} \mu(y)|^p dy - \Psi(\mu), \quad \mu \in \mathcal{Y}, \tag{4.1}$$

where

$$\Psi(\mu) = \int_{\Omega} \mathcal{G}(y,\mu) dy + \frac{1}{q} \int_{\Omega} |\mu|^q dy. \tag{4.2}$$

In the formula (4.1), it is clear that $\phi \in \mathcal{C}^1(\mathcal{Y},\mathbb{R})$. Then, the weak solutions of the problem (1.1) are critical points of ϕ (see [15,35]). On the other hand from (4.2), $\Psi' : \mathcal{Y} \to \mathcal{Y}^*$ is completely continuous, and thus Ψ is weakly continuous. □

Case 2. It is assumed in this instance that g meets the following assumption:

$$|g(y,t)| \le C_1 + C_2|t|^{r-1}, \quad \text{where } r \in [1,p), \text{ for all } (y,t) \in \Omega \times \mathbb{R}. \tag{4.3}$$

Theorem 4.2. *Under assumption* (4.3)*, the problem* (1.1) *admits a weak solution.*

Proof. According to (4.3) we know that $|\mathcal{G}(y,t)| \le C(1 + |t|^r)$. Then,

$$\begin{aligned}\phi(\mu) &= \frac{1}{p} \sum_{i=1}^{N} \int_{\Omega} |\partial_i^{\gamma} \mu(y)|^p dy - \int_{\Omega} \mathcal{G}(y,\ \mu) dy - \frac{1}{q} \int_{\Omega} |\mu|^q dy \\ &\ge \frac{1}{p} \|\mu\|_{\gamma}^p - C \int_{\Omega} |\mu|^r dy - C_3 - \frac{1}{q} \int_{\Omega} |\mu|^q dy\end{aligned}$$

$$\geq \frac{1}{p}\|\mu\|_{\gamma}^{p} - C_4\|\mu\|_{\gamma}^{r} - C_5.$$

Therefore,

$$\lim_{\|\mu\|_{\gamma}\to+\infty} \phi(\mu) = +\infty.$$

As ϕ is weakly lower semi-continuous, then ϕ posses a minimum point μ in $\mathcal{Y}$, and therefore (1.1) admits a weak solution which is this minimum point of ϕ. □

Case 3. In this case, we will consider the problem (1.1), taking into account $(\mathcal{A}_1)$ and the following conditions

$(\mathcal{A}_2)$ exists $N > 0$, $\eta > p$ such that $0 < \eta\mathcal{G}(y,t) \leq tg(y,t)$, $|t| \geq N$, $y \in \Omega$, where $\mathcal{G}(y,t) = \int_0^t g(y,s)ds$;

$(\mathcal{A}_3)$ $g(y,t) = o_{t\to 0}(|t|^{r-1})$, $y \in \Omega$ uniformuly and $r > p$.

Definition 4.2. Let $\phi : \mathcal{Y} \to \mathbb{R}$ be a function. A functional ϕ is said to satisfy the Palais-Smale condition (denoted by (PS)) if any sequence $\{\mu_n\} \subset \mathcal{Y}$ with $\{\phi(\mu_n)\}$ is bounded and $\lim_{n\to+\infty} \|\phi'(\mu_n)\|_{\gamma} = 0$ admits a convergent subsequence.

In order to establish Theorem 4.3, we will first prove the following technical lemma.

Lemma 4.1. *Assume that* $(\mathcal{A}_2)$ *holds. Then,* ϕ *satisfies the* (PS) *condition.*

Proof. Assume that $\{\mu_n\} \subset \mathcal{Y}$, $\lim_{n\to+\infty} \|\phi'(\mu_n)\|_{\gamma} = 0$ and $\{\phi(\mu_n)\}$ is bounded. Thus,

$$\begin{aligned}
C \geq \phi(\mu_n) =& \frac{1}{p}\sum_{i=1}^{N}\int_{\Omega} |\partial_i^{\gamma}\mu_n(y)|^p dy - \int_{\Omega} \mathcal{G}(y,\mu)dy - \frac{1}{q}\int_{\Omega}|\mu|^q dy \\
\geq& \frac{1}{p}\sum_{i=1}^{N}\int_{\Omega} |\partial_i^{\gamma}\mu_n(y)|^p dy - \int_{\Omega}\frac{\mu_n}{\eta} f(y,\mu_n)dy - c - \frac{1}{q}\int_{\Omega}|\mu|^q dy \\
\geq& \left(\frac{1}{p} - \frac{1}{\eta}\right)\sum_{i=1}^{N}\int_{\Omega} |\partial_i^{\gamma}\mu_n(y)|^p dy \\
&+ \int_{\Omega}\frac{1}{\eta}\left(\sum_{i=1}^{N}|\partial_i^{\gamma}\mu_n(y)|^p - \mu_n f(y,\mu_n)\right)dy - c - \frac{1}{q}\int_{\Omega}|\mu|^q dy \\
\geq& \left(\frac{1}{p} - \frac{1}{\eta}\right)\|\nabla_{\gamma}\mu_n\|_{L^p}^p - \frac{1}{\eta}\|\phi'(\mu_n)\|_{\gamma}\|\mu_n\|_{\gamma} - C_6.
\end{aligned}$$

So, $\{\|\mu_n\|_{\gamma}\}$ is bounded.

Without loss of generality, we assume that $\mu_n \rightharpoonup \mu$, so

$$\lim_{n\to+\infty} \Psi'(\mu_n) = \Psi'(\mu).$$

Since $\lim_{n\to+\infty}(\mathcal{Q}\mu_n - \Psi'(\mu_n)) = \lim_{n\to+\infty}\phi'(\mu_n) = 0$, we obtain

$$\lim_{n\to+\infty} \mathcal{Q}\mu_n = \Psi'(\mu).$$

As $\mathcal{Q}$ is a homeomorphism, $\mu_n \to \mu$, which implies that ϕ satisfies the (PS) condition. □

Theorem 4.3. *Suppose that* $(\mathcal{A}_1)$-$(\mathcal{A}_3)$ *hold. Then, the problem* (1.1) *has a non-trivial solution.*

Proof. We will show that the assumptions of the Mountain Pass Lemma hold for ϕ.

By Lemma 4.1, we have ϕ satisfies the (PS) condition in $\mathcal{Y}$. As $p < r < p^*$ and thanks to Theorem 2.4, there exists $C_0 > 0$ such that

$$\|\mu\|_{L^p} \leq C_0\|\mu\|_\gamma, \quad \text{for all } \mu \in \mathcal{Y}.$$

Let $\theta > 0$ such that $\theta\, C_0^p \leq \frac{1}{2p}$, by variants of $(\mathcal{A}_1)$-$(\mathcal{A}_3)$, we have

$$\mathcal{G}(y,t) \leq \theta|t|^p + C_\theta|t|^r,$$

for all $(y,t) \in \Omega \times \mathbb{R}$. Since

$$\begin{aligned}\phi(\mu) &\geq \frac{1}{p}\sum_{i=1}^{N}\int_\Omega |\partial_i^\gamma \mu_n(y)|^p dy - \theta\int_\Omega |\mu|^p dy - C_\theta\int_\Omega |\mu|^r dy - \frac{1}{q}\int_\Omega |\mu|^q dy \\ &\geq \frac{1}{p}\|\mu\|_\gamma^p - \theta C_0^p\|\mu\|_\gamma^p - C(\theta)\|\mu\|_\gamma^r - C\|\mu\|_\gamma^q \\ &\geq \frac{1}{2p}\|\mu\|_\gamma^p - C'_\theta, \quad \text{when } \|\mu\|_\gamma \leq 1,\end{aligned}$$

there are two strictly positive numbers $\xi > 0$ and $\varrho > 0$ such that

$$\phi(\mu) \geq \varrho, \quad \text{for all } \mu \in \mathcal{Y} \text{ and } \|\mu\|_\gamma = \xi.$$

From $(\mathcal{A}_2)$, we can see that

$$\mathcal{G}(y,t) \geq C|t|^\eta, \quad \text{for all } y \in \overline{\Omega}, |t| \geq M.$$

For $\nu \in \mathcal{Y}\backslash\{0\}$ and $t > 1$, we can obtain

$$\begin{aligned}\phi(t\nu) &= \frac{1}{p}\sum_{i=1}^{N}\int_\Omega |t\, \partial_i^\gamma \nu(y)|^p dy - \int_\Omega \mathcal{G}(y,t\nu)dy - \frac{1}{q}\int_\Omega |t\nu|^q dy \\ &\leq \frac{t^p}{p}\sum_{i=1}^{N}\int_\Omega |\partial_i^\gamma \nu(y)|^p dy - ct^\eta\int_\Omega |\nu|^\eta dy - \frac{t^q}{q}\int_\Omega |\nu|^q dy.\end{aligned}$$

Therefore,

$$\lim_{t\to+\infty}\phi(t\nu) = -\infty.$$

As $\phi(0) = 0$, applying Mountain Pass Lemma (see [15]), ϕ has at least one non-trivial critical point. It follows that the problem (1.1) has at least one nontrivial solution. ☐

Case 4. Here, we make the same assumptions as in Theorem 4.3, except that we replace $(\mathcal{A}_3)$ with the following assumption

$(\mathcal{A}_4)$ $g(y,-t) = -g(y,t)$, for all $y \in \Omega,\ t \in \mathbb{R}$.

Let $\mathcal{Y}$ be a reflexive and separable Banach space. Then, there exist $\{e_i\} \subset \mathcal{Y}$ and $\{e_i^*\} \subset \mathcal{Y}^*$ such that

$$\mathcal{Y}^* = \overline{\text{span}\{e_i^*,\ i = 1,2,\ldots\}}, \quad \mathcal{Y} = \overline{\text{span}\{e_i,\ i = 1,2,\ldots\}},$$

and

$$\langle e_j^*, e_i \rangle = \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{otherwise.} \end{cases}$$

For convenience, we can write

$$\mathcal{X}_k = \oplus_{i=1}^{k} \mathcal{Y}_i, \quad \mathcal{Z}_k = \oplus_{i=k}^{+\infty} \mathcal{Y}_i,$$

where $\mathcal{Y}_i = \operatorname{span}\{e_i\}$.

For the sequel, we need the following lemma.

Lemma 4.2. *Let $r \in \mathbb{R}$ such that $1 < r < p^*$ and*

$$\lambda_k = \sup\Big\{ \|\mu\|_{L^r}, \|\mu\|_\gamma = 1, \mu \in \mathcal{Z}_k \Big\}. \tag{4.4}$$

Then, $\lim_{k\to+\infty} \lambda_k = 0$.

Proof. According to (4.4), it is clear that $0 < \lambda_{k+1} \leq \lambda_k$, so the sequence $\{\lambda_k\}$ is convergent, and thus

$$\lim_{k\to+\infty} \lambda_k = \lambda \geq 0.$$

Let $\mu_k \in \mathcal{Z}_k$ such that

$$\|\mu_k\|_\gamma = 1, \quad 0 \leq \lambda_k - \|\mu_k\|_{L^r} < \frac{1}{k}.$$

So there is a subsequence of $\{\mu_k\}$ (which we still denote by μ_k) such that

$$\mu_k \rightharpoonup \mu \quad \text{and} \quad \langle e_j^*, \mu \rangle = \lim_{k\to+\infty} \langle e_j^*, \mu_k \rangle = 0, \quad \text{where } j = 1, 2, \ldots$$

Then, $\mu = 0$, and thus $\mu_k \rightharpoonup 0$. Since ${}^cW_0^{\gamma,p}(\varOmega) \hookrightarrow L^r(\varOmega)$ is compact, so $\mu_k \to 0$, in $L^r(\varOmega)$. Hence, we obtain $\lim_{k\to+\infty} \lambda_k = 0$. ☐

Theorem 4.4. *Suppose that $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and $(\mathcal{A}_4)$ hold, then ϕ has a sequence of critical points $\{\mu_n\}$ such that*

$$\lim_{n\to+\infty} \phi(\mu_n) = +\infty$$

and (1.1) has infinite many pairs of solutions.

Proof. Combining $(\mathcal{A}_2)$ with $(\mathcal{A}_4)$, we conclude that ϕ is an even functional and verifies the condition (PS). We will show that if k is large enough there is $\delta_k > \rho_k > 0$ such that

$(\mathcal{B}_1)$ $b_k := \inf\{\phi(\mu),\ \mu \in \mathcal{Z}_k \text{ and } \|\mu\|_\gamma = \rho_k\}$;

$(\mathcal{B}_2)$ $a_k := \max\{\phi(\mu),\ \mu \in \mathcal{X}_k \text{ and } \|\mu\|_\gamma = \delta_k\}$.

It is clear that

$$\lim_{k\to+\infty} b_k = +\infty \quad \text{and} \quad a_k \leq 0.$$

By Fountain Theorem (see [33, Theorem 3.6]), we establish the assertion of Theorem 4.4.

($\mathcal{B}_1$) For each $\mu \in \mathcal{Z}_k,\ |\mu| = \rho_k = (cr\lambda_k^r)^{\frac{1}{p-r}}$, we can obtain

$$\begin{aligned}
\phi(\mu) =& \frac{1}{p}\sum_{i=1}^{N}\int_\Omega |\partial_i^\gamma \mu(y)|^p dy - \int_\Omega \mathcal{G}(y,\mu)dy - \frac{1}{q}\int_\Omega |\mu|^q dy \\
\geq& \frac{1}{p}\sum_{i=1}^{N}\int_\Omega |\partial_i^\gamma \mu(y)|^p dy - c\int_\Omega |\mu|^r dy - c_1 - \frac{1}{q}\int_\Omega |\mu|^q dy \\
\geq& \frac{1}{p}\|\mu\|_\gamma^p - c\|\mu\|_{L^r}^r - c_2 - \frac{1}{q}\|\mu\|_\gamma^q \\
\geq& \begin{cases} \frac{1}{p}\|\mu\|_\gamma^p - c - c_1 - \frac{1}{q}\|\mu\|_\gamma^q, & \text{if } \|\mu\|_{L^r} \leq 1, \\ \frac{1}{p}\|\mu\|_\gamma^p - c\ \lambda_k^r\|\mu\|_{L^r}^r - c_1 - \frac{1}{q}\|\mu\|_\gamma^q, & \text{if } \|\mu\|_{L^r} > 1, \end{cases} \\
\geq& \frac{1}{p}|\mu|^p - c\lambda_k^r|\mu|^r - c_3 - \frac{1}{q}\|\mu\|_\gamma^q \\
=& \frac{1}{p}(c\,r\lambda_k^r)^{\frac{p}{p-r}} - c\lambda_k^r(c\,r\lambda_k^r)^{\frac{r}{p-r}} - c_3 - \frac{1}{q}(c\,r\lambda_k^r)^{\frac{q}{p-r}}.
\end{aligned}$$

Since $p < r$ and $\lim_{k\to+\infty}\lambda_k = 0$. Then,

$$\lim_{k\to+\infty}\left[\left(\frac{1}{p} - \frac{2}{r}\right)(c\ r\ \lambda_k^r)^{\frac{p}{p-r}} - c_3\right] = +\infty.$$

($\mathcal{B}_2$) According ($\mathcal{A}_2$), we have $\mathcal{G}(y,t) \geq c_1|t|^\eta - c_2$. By using $\eta > p$ and $\dim\mathcal{X}_k = k$, we obtain

$$\lim_{\|\mu\|\to+\infty}\Psi(\mu) = -\infty, \quad \text{for } \mu \in \mathcal{X}_k.$$

□

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