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A BOUSSINESQ-TYPE EQUATION WITH ATANGANA-BALEANU TIME DERIVATIVE ON THE TORUS

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Abstract. We investigate a Boussinesq-type equation on the one-dimensional torus driven by an Atangana-Baleanu time derivative of order $\alpha \in (1, 2)$. After formulating a consistent extension of the AB-Caputo operator to this higher-order regime, we derive an equivalent integral representation suitable for analytical treatment. Local existence and uniqueness are established in Sobolev spaces $H^s(\mathbb{T})$ by means of a contraction argument that accommodates the quadratic term $(z^2)_{xx}$ and the fractional memory structure. In addition, a Laplace-based variational iteration scheme is constructed and compared with a Fourier pseudo-spectral discretization. Numerical experiments illustrate the short-time influence of the fractional order on the dispersive-diffusive dynamics.

1. Introduction

Boussinesq-type equations play a fundamental role in the mathematical description of long, small-amplitude waves in dispersive media, with classical motivations coming from water-wave theory and continuum mechanics. In the standard (integer-order) setting, the derivation and the linear and nonlinear theories have been extensively developed, we refer to [4, 5] for a systematic account. Depending on the sign structure of the highest-order term, one encounters the well-known distinction between the "bad" and the "good" Boussinesq models, which strongly affects stability and well-posedness,

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periodic dynamics on the torus provides a natural framework where dispersion, regularization, and nonlinear transfer can be investigated, see, e.g., [13, 16, 17]. On the numerical side, recent years have witnessed significant progress in Fourier-based methods and low-regularity integration strategies for dispersive equations, including Boussinesq-type models, see, for instance, [13, 21–23]. In parallel, time-fractional evolution equations have become a standard tool to incorporate memory and non-instantaneous relaxation in applied models. For classical fractional operators and their analytical foundations, we refer to [24], while numerical methodologies can be found in [25]. Beyond the traditional power-law kernels of Riemann-Liouville or Caputo type, several non-singular kernels have been proposed, such as the Caputo-Fabrizio derivative [7] and the Atangana-Baleanu derivative with Mittag-Leffler kernel [1]. The latter combines nonlocal memory with a regular kernel, which impacts both analysis (inversion identities, Laplace-domain representations) and numerical discretizations, we refer to [11] for representative results in this direction. Motivated by such developments, the work [3] proposed an AB-based framework for Burgers-type dynamics, combining a fixed-point well-posedness argument with a Laplace-based variational iteration methodology and numerical illustrations. The present paper addresses an Atangana-Baleanu time-fractional Boussinesq-type model on the one-dimensional torus $\mathbb{T}$:

$$ {}^{ABC}D_t^{\alpha} z + z_{xxxx} - z_{xx} - (z^2)_{xx} = 0, \quad x \in \mathbb{T}, t > 0, \alpha \in (1,2). \tag{1.1}$$

The regime $\alpha \in (1,2)$ is closer in spirit to second-order-in-time dispersive models, and it requires two initial conditions, namely $z(\cdot,0)$ and $z_t(\cdot,0)$. From an analytical viewpoint, this higher-order fractional setting calls for a consistent extension of the AB-Caputo operator and for an integral formulation compatible with the quadratic term $(z^2)_{xx}$ in Sobolev spaces. From a numerical viewpoint, it raises additional challenges related to the evaluation of the AB memory contribution and to the interplay between the linear dispersive operator and nonlinear spectral transfer. Our main contributions can be summarized as follows. First, we adopt a practical extension of the AB-Caputo derivative to orders $\alpha \in (1,2)$ and derive the associated Laplace-domain representation and integral formulation, in line with the AB calculus introduced in [1] and further exploited in [11]. Second, we prove local well-posedness in $H^s(\mathbb{T})$ for s sufficiently large, via a contraction argument that controls the mapping $z \mapsto (z^2)_{xx}$ together with the AB memory operator. Third, inspired by the strategy of [3], we construct a Laplace-based variational iteration method (LVIM) and compare it, over short times, with a Fourier pseudo-spectral discretization, following the spectral approach that is standard for periodic Boussinesq-type dynamics [13].

The paper is organized as follows: Section 2 recalls the Atangana-Baleanu calculus and introduces the order-$\alpha \in (1,2)$ framework used in this study. Section 3 fixes the functional setting on $\mathbb{T}$ and the Fourier tools needed for nonlinear estimates. Section 4 derives an equivalent integral formulation and introduces the fixed-point map. Local well-posedness is established in Section 5. Section 6 presents the LVIM scheme and

its local convergence. Finally, Section 7 reports numerical experiments illustrating the influence of the fractional order on the short-time dynamics.

2. Preliminaries

We briefly recall the main elements of the Atangana-Baleanu fractional calculus needed in this work. The original definition for orders in $(0,1)$ was introduced in [1], and further analytical properties can be found for instance in [11]. We state the definitions and fundamental identities without proof.

Definition 2.1 ([1])**.** Let $0<\beta<1$ and let $u\in C^1([0,T])$. The Atangana-Baleanu derivative in Caputo sense of order β is defined by

$$ {}^{ABC}D_t^{\beta}u(t)=\frac{M(\beta)}{1-\beta}\int_0^t E_\beta\!\left(-\frac{\beta}{1-\beta}(t-\tau)^{\beta}\right)u'(\tau)\,d\tau,$$

where E_β denotes the Mittag-Leffler function

$$E_\beta(z)=\sum_{n=0}^{+\infty}\frac{z^n}{\Gamma(\beta n+1)},$$

and $M(\beta)$ is a normalization factor satisfying $M(0)=M(1)=1$.

Definition 2.2 ([1])**.** For $0<\beta<1$ and $u\in L^1(0,T)$, the associated Atangana-Baleanu fractional integral of order β is defined by

$$ {}^{AB}I_t^{\beta}u(t)=\frac{1-\beta}{M(\beta)}\,u(t)+\frac{\beta}{M(\beta)\Gamma(\beta)}\int_0^t(t-\tau)^{\beta-1}u(\tau)\,d\tau.$$

These two operators are linked through an inversion property.

Proposition 2.1 ([1])**.** *Let $u\in C^1([0,T])$. Then,*

$$ {}^{AB}I_t^{\beta}\big({}^{ABC}D_t^{\beta}u\big)(t)=u(t)-u(0).$$

The Laplace transform representation is a key analytical tool in the study of AB operators.

Proposition 2.2 ([11])**.** *Let $u\in C^1([0,+\infty))$ with exponential growth and denote $U(s)=\mathcal{L}[u](s)$. Then,*

$$\mathcal{L}\Big[{}^{ABC}D_t^{\beta}u(t)\Big](s)=\frac{M(\beta)}{1-\beta}\cdot\frac{s^{\beta}U(s)-s^{\beta-1}u(0)}{s^{\beta}+\frac{\beta}{1-\beta}}.$$

In order to treat time derivatives of order between one and two, we adopt a natural extension consistent with the classical Caputo construction for higher orders.

Definition 2.3 ([1])**.** Let $\beta\in(0,1)$ and let $z\in C^2([0,T])$. The Atangana-Baleanu derivative in Caputo sense of order $\alpha=1+\beta\in(1,2)$ is defined by

$$ {}^{ABC}D_t^{1+\beta}z(t)={}^{ABC}D_t^{\beta}\big(z_t(t)\big).$$

This definition requires two initial data,

$$z(0) = z_0, \quad z_t(0) = z_1,$$

and coincides with the standard first-order derivative in the limit $\beta \to 0^+$.

Proposition 2.3. *Let $z \in C^2([0,+\infty))$ with exponential growth and set $Z(s) = \mathcal{L}[z](s)$. Then,*

$$\mathcal{L}\Big[{}^{ABC}D_t^{1+\beta} z(t)\Big](s) = \frac{M(\beta)}{1-\beta} \cdot \frac{s^\beta\Big(sZ(s) - z(0)\Big) - s^{\beta-1} z_t(0)}{s^\beta + \frac{\beta}{1-\beta}}.$$

Proof. The formula follows from the Laplace transform representation of the Atangana-Baleanu derivative of order $\beta \in (0,1)$ (see [1, 11]) applied to the function z_t, together with the identity $\mathcal{L}[z_t](s) = sZ(s) - z(0)$. □

In the PDE framework considered in this paper, the pointwise smoothness assumptions are replaced by time-Sobolev regularity with values in suitable spatial Sobolev spaces, ensuring that the AB derivative is well defined in a weak sense.

3. Functional Setting and Fourier Analysis on $\mathbb{T}$

We denote by $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ the one-dimensional torus endowed with the normalized Lebesgue measure. Every function $u \in L^2(\mathbb{T})$ admits a Fourier series representation

$$u(x) = \sum_{k\in\mathbb{Z}} \widehat{u}(k) e^{ikx}, \quad \widehat{u}(k) = \frac{1}{2\pi}\int_0^{2\pi} u(x) e^{-ikx}\, dx.$$

For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{T})$ is defined by

$$H^s(\mathbb{T}) = \Big\{ u \in \mathcal{D}'(\mathbb{T}) : \|u\|_{H^s}^2 = \sum_{k\in\mathbb{Z}} (1+k^2)^s |\widehat{u}(k)|^2 < +\infty \Big\}.$$

Equipped with the inner product

$$\langle u, v\rangle_{H^s} = \sum_{k\in\mathbb{Z}} (1+k^2)^s \widehat{u}(k)\overline{\widehat{v}(k)},$$

the space $H^s(\mathbb{T})$ is a Hilbert space. In particular $H^0(\mathbb{T}) = L^2(\mathbb{T})$ and, for any integer $m \geq 0$, the space $H^m(\mathbb{T})$ coincides with the set of functions in $L^2(\mathbb{T})$ whose weak derivatives up to order m belong to $L^2(\mathbb{T})$.

Lemma 3.1. *Let $s > \frac{1}{2}$. Then,*

$$H^s(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$$

continuously. In particular, there exists a constant $C_s > 0$ such that

$$\|u\|_{L^\infty(\mathbb{T})} \leq C_s \|u\|_{H^s(\mathbb{T})}, \quad \textit{for all } u \in H^s(\mathbb{T}).$$

Proof. Using the Fourier representation and the Cauchy-Schwarz inequality,

$$|u(x)| = \left|\sum_{k\in\mathbb{Z}} \widehat{u}(k)e^{ikx}\right| \leq \left(\sum_{k\in\mathbb{Z}}(1+k^2)^{-s}\right)^{1/2}\left(\sum_{k\in\mathbb{Z}}(1+k^2)^{s}|\widehat{u}(k)|^2\right)^{1/2}.$$

The series $\sum_k (1+k^2)^{-s}$ converges if and only if $s > \frac{1}{2}$. The desired estimate follows immediately. □

As a consequence of the Sobolev embedding, $H^s(\mathbb{T})$ is a Banach algebra when $s > \frac{1}{2}$.

Proposition 3.1. *Let $s > \frac{1}{2}$. Then, there exists $C_s > 0$ such that*

$$\|uv\|_{H^s} \leq C_s\|u\|_{H^s}\|v\|_{H^s}, \quad u, v \in H^s(\mathbb{T}).$$

Proof. By Lemma 3.1, $H^s(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$. Hence,

$$\|uv\|_{H^s} \leq C\Big(\|u\|_{L^\infty}\|v\|_{H^s} + \|v\|_{L^\infty}\|u\|_{H^s}\Big).$$

Using the embedding again yields $\|uv\|_{H^s} \leq C_s\|u\|_{H^s}\|v\|_{H^s}$. □

We now derive an estimate adapted to the quadratic nonlinearity appearing in the equation.

Lemma 3.2. *Let $s > \frac{5}{2}$ and $R > 0$. Then, there exists a constant $C_R > 0$ such that for all $z, w \in H^s(\mathbb{T})$ satisfying $\|z\|_{H^s}, \|w\|_{H^s} \leq R$ one has*

$$\Big\|(z^2)_{xx} - (w^2)_{xx}\Big\|_{H^{s-2}} \leq C_R\|z - w\|_{H^s}.$$

Proof. We first write $z^2 - w^2 = (z-w)(z+w)$. Applying two derivatives gives

$$(z^2)_{xx} - (w^2)_{xx} = \partial_x^2\Big[(z-w)(z+w)\Big].$$

Since $\partial_x^2 : H^s(\mathbb{T}) \to H^{s-2}(\mathbb{T})$ is bounded, we obtain

$$\Big\|(z^2)_{xx} - (w^2)_{xx}\Big\|_{H^{s-2}} \leq C\|(z-w)(z+w)\|_{H^s}.$$

Using the algebra property of $H^s(\mathbb{T})$ for $s > \frac{1}{2}$,

$$\|(z-w)(z+w)\|_{H^s} \leq C_s\|z-w\|_{H^s}\|z+w\|_{H^s}.$$

Since $\|z\|_{H^s}, \|w\|_{H^s} \leq R$, we obtain $\|z+w\|_{H^s} \leq 2R$. Combining the estimates yields

$$\Big\|(z^2)_{xx} - (w^2)_{xx}\Big\|_{H^{s-2}} \leq C_s(2R)\|z-w\|_{H^s}.$$

□

We now introduce the linear operator associated with the dispersive part of the equation

$$L = \partial_x^4 - \partial_x^2.$$

Proposition 3.2. *Let $u \in H^{s+4}(\mathbb{T})$ with Fourier series*

$$u(x) = \sum_{k\in\mathbb{Z}} \widehat{u}(k)e^{ikx}.$$

Then, $\widehat{Lu}(k) = (k^4 + k^2)\widehat{u}(k)$. Hence, the Fourier symbol of L is $\lambda_k = k^4 + k^2 \geq 0$.

Proof. Since $\partial_x e^{ikx} = ike^{ikx}$, we obtain

$$\partial_x^2 e^{ikx} = -k^2 e^{ikx}, \quad \partial_x^4 e^{ikx} = k^4 e^{ikx}.$$

Thus, $Le^{ikx} = (k^4 + k^2)e^{ikx}$. The formula follows by linearity. □

Corollary 3.1. *For every $s \in \mathbb{R}$, the operator*

$$L : H^{s+4}(\mathbb{T}) \to H^s(\mathbb{T})$$

is bounded and satisfies $\|Lu\|_{H^s} \leq C\|u\|_{H^{s+4}}$.

Proof. Since $\lambda_k = k^4 + k^2 \leq 2(1 + k^2)^2$, we obtain

$$(1 + k^2)^s |\lambda_k \widehat{u}(k)|^2 \leq C(1 + k^2)^{s+4} |\widehat{u}(k)|^2.$$

Summing over $k \in \mathbb{Z}$ yields the desired estimate. □

Finally, since $\lambda_k \geq 0$ for all k, the operator L is a nonnegative self-adjoint operator on $L^2(\mathbb{T})$ with domain $H^4(\mathbb{T})$. This spectral characterization allows us to treat the linear part mode by mode in Fourier variables and will be used later to derive the estimates required for the fixed point argument.

4. Equivalent Integral Formulation

We consider the time-fractional problem on $\mathbb{T}$

$${}^{ABC}D_t^\alpha z + Lz - (z^2)_{xx} = 0, \quad \alpha \in (1, 2),$$

where $L = \partial_x^4 - \partial_x^2$ and ${}^{ABC}D_t^\alpha$ denotes the Atangana-Baleanu derivative in the Caputo sense. Writing $\alpha = 1 + \beta$, $\beta \in (0, 1)$, and using the definition

$${}^{ABC}D_t^{1+\beta} z = {}^{ABC}D_t^\beta(z_t),$$

the equation takes the form

$${}^{ABC}D_t^\beta(z_t) = -Lz + (z^2)_{xx}.$$

We supplement this equation with the initial conditions $z(\cdot, 0) = z_0$, $z_t(\cdot, 0) = z_1$.

For convenience, we introduce the nonlinear operator

$$F(z) = -Lz + (z^2)_{xx}.$$

Proposition 4.1. *Let $s > \frac{5}{2}$ and let*

$$z \in C([0, T], H^s(\mathbb{T})) \cap C^1([0, T], H^{s-2}(\mathbb{T})).$$

Assume that ${}^{ABC}D_t^\beta(z_t)$ is well defined in $C([0, T], H^{s-2}(\mathbb{T}))$ and that

$${}^{ABC}D_t^\beta(z_t) = F(z), \quad \text{in } C([0, T], H^{s-2}(\mathbb{T})).$$

Then, the following assertions are equivalent:

(a) *z satisfies*

$$ {}^{ABC}D_t^{\beta}(z_t) = F(z), \quad \text{on } [0,T], $$

with $z(0) = z_0$, $z_t(0) = z_1$,

(b) *z satisfies the identity*

$$ z_t(t) - z_1 = {}^{AB}I_t^{\beta}\Big(F(z)\Big)(t), \quad t \in [0,T], $$

(c) *z satisfies the integral equation*

$$ z(t) = z_0 + tz_1 + \int_0^t {}^{AB}I_\tau^{\beta}\Big(F(z)\Big)(\tau)\, d\tau, \quad t \in [0,T]. $$

Proof. Assume first that

$$ {}^{ABC}D_t^{\beta}(z_t) = F(z). $$

Applying the Atangana-Baleanu fractional integral ${}^{AB}I_t^{\beta}$ to both sides and using the inversion identity

$$ {}^{AB}I_t^{\beta}\Big({}^{ABC}D_t^{\beta}u\Big) = u(t) - u(0), $$

with $u = z_t$, we obtain $z_t(t) - z_t(0) = {}^{AB}I_t^{\beta}\Big(F(z)\Big)(t)$. Since $z_t(0) = z_1$, this gives

$$ z_t(t) - z_1 = {}^{AB}I_t^{\beta}\Big(F(z)\Big)(t). $$

Integrating in time on $[0,t]$, we deduce

$$ z(t) - z(0) = tz_1 + \int_0^t {}^{AB}I_\tau^{\beta}\Big(F(z)\Big)(\tau)\, d\tau. $$

Since $z(0) = z_0$, we obtain

$$ z(t) = z_0 + tz_1 + \int_0^t {}^{AB}I_\tau^{\beta}\Big(F(z)\Big)(\tau)\, d\tau. $$

Conversely, assume that

$$ z(t) = z_0 + tz_1 + \int_0^t {}^{AB}I_\tau^{\beta}\Big(F(z)\Big)(\tau)\, d\tau. $$

Differentiating with respect to t, we get $z_t(t) = z_1 + {}^{AB}I_t^{\beta}\Big(F(z)\Big)(t)$. Applying ${}^{ABC}D_t^{\beta}$ to both sides and using the identity

$$ {}^{ABC}D_t^{\beta}\Big({}^{AB}I_t^{\beta}f\Big) = f $$

for sufficiently regular functions, we recover

$$ {}^{ABC}D_t^{\beta}(z_t) = F(z), $$

which is the desired equation. □

We now introduce the functional framework for the fixed-point argument. Fix $s > \frac{5}{2}$ and $T > 0$, and define $X_T = C([0,T], H^s(\mathbb{T}))$ endowed with the norm

$$\|z\|_{X_T} = \sup_{t\in[0,T]} \|z(t)\|_{H^s}.$$

For $R > 0$, we denote by $B_R = \{z \in X_T : \|z\|_{X_T} \le R\}$ the closed ball of radius R centered at the origin in X_T.

Definition 4.1. For $z \in X_T$, define

$$(\Phi z)(t) = z_0 + tz_1 + \int_0^t {}^{AB}I_\tau^\beta\Big(F(z)\Big)(\tau)\, d\tau, \quad t \in [0,T].$$

A mild solution on $[0,T]$ is a fixed point of Φ in X_T.

Lemma 4.1. *Let $r \in \mathbb{R}$ and let $f \in C([0,T], H^r(\mathbb{T}))$. Then,*

$${}^{AB}I_t^\beta f \in C([0,T], H^r(\mathbb{T})),$$

and there exists a constant $C_{\beta,T} > 0$ such that

$$\Big\|{}^{AB}I_t^\beta f\Big\|_{C([0,T],H^r)} \le C_{\beta,T}\|f\|_{C([0,T],H^r)}.$$

More precisely, one may take

$$C_{\beta,T} = \frac{1-\beta}{M(\beta)} + \frac{T^\beta}{M(\beta)\Gamma(\beta+1)}.$$

Proof. By definition,

$${}^{AB}I_t^\beta f(t) = \frac{1-\beta}{M(\beta)}\, f(t) + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau)\, d\tau.$$

Taking the H^r norm and using the triangle inequality, we obtain

$$\Big\|{}^{AB}I_t^\beta f(t)\Big\|_{H^r} \le \frac{1-\beta}{M(\beta)}\|f(t)\|_{H^r} + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1}\|f(\tau)\|_{H^r}\, d\tau.$$

Since $\|f(\tau)\|_{H^r} \le \|f\|_{C([0,T],H^r)}$, it follows that

$$\Big\|{}^{AB}I_t^\beta f(t)\Big\|_{H^r} \le \frac{1-\beta}{M(\beta)}\|f\|_{C([0,T],H^r)} + \frac{\beta}{M(\beta)\Gamma(\beta)} \left(\int_0^t (t-\tau)^{\beta-1}\, d\tau\right) \|f\|_{C([0,T],H^r)}.$$

Using

$$\int_0^t (t-\tau)^{\beta-1}\, d\tau = \frac{t^\beta}{\beta} \le \frac{T^\beta}{\beta},$$

we obtain

$$\Big\|{}^{AB}I_t^\beta f(t)\Big\|_{H^r} \le \left(\frac{1-\beta}{M(\beta)} + \frac{T^\beta}{M(\beta)\Gamma(\beta+1)}\right) \|f\|_{C([0,T],H^r)}.$$

Taking the supremum over $t \in [0,T]$ proves the desired estimate. □

Lemma 4.2. *Let $s > \frac{5}{2}$ and let $R > 0$. Then, there exists a constant $C_R > 0$ such that, for all $z, w \in B_R$,*

$$\|F(z) - F(w)\|_{C([0,T],H^{s-2})} \leq C_R \|z - w\|_{X_T}.$$

Moreover,

$$\|F(z)\|_{C([0,T],H^{s-2})} \leq C_R, \quad \text{for all } z \in B_R.$$

Proof. By definition, $F(z) - F(w) = -L(z-w) + \Big((z^2)_{xx} - (w^2)_{xx}\Big)$. By Corollary 3.1,

$$\|L(z-w)\|_{H^{s-2}} \leq C\|z-w\|_{H^s}.$$

On the other hand, by Lemma 3.2,

$$\|(z^2)_{xx} - (w^2)_{xx}\|_{H^{s-2}} \leq C_R \|z-w\|_{H^s}.$$

Hence, for every $t \in [0,T]$,

$$\|F(z(t)) - F(w(t))\|_{H^{s-2}} \leq C_R \|z(t) - w(t)\|_{H^s}.$$

Taking the supremum over $t \in [0,T]$ yields

$$\|F(z) - F(w)\|_{C([0,T],H^{s-2})} \leq C_R \|z - w\|_{X_T}.$$

Taking $w = 0$ and using again the boundedness of L together with the estimate

$$\Big\|(z^2)_{xx}\Big\|_{H^{s-2}} \leq C\|z\|_{H^s}^2,$$

we also obtain $\|F(z)\|_{C([0,T],H^{s-2})} \leq C_R$, for all $z \in B_R$. □

Proposition 4.2. *Let $s > \frac{5}{2}$ and $R > 0$. Then, Φ maps B_R into X_T. Moreover, there exists a constant $C_{\beta,T,R} > 0$ such that for all $z, w \in B_R$,*

$$\|\Phi z - \Phi w\|_{X_T} \leq T\, C_{\beta,T,R} \|z - w\|_{X_T}.$$

Proof. Let $z, w \in B_R$. By definition,

$$(\Phi z)(t) - (\Phi w)(t) = \int_0^t {}^{AB}I_\tau^\beta \Big(F(z) - F(w)\Big)(\tau)\, d\tau.$$

Therefore,

$$\|(\Phi z)(t) - (\Phi w)(t)\|_{H^s} \leq \int_0^t \Big\|{}^{AB}I_\tau^\beta \Big(F(z) - F(w)\Big)(\tau)\Big\|_{H^{s-2}}\, d\tau,$$

where we use that the integral term is controlled through the H^{s-2} regularity of $F(z) - F(w)$.

By Lemma 4.1 and Lemma 4.2,

$$\Big\|{}^{AB}I_t^\beta (F(z) - F(w))\Big\|_{C([0,T],H^{s-2})} \leq C_{\beta,T}\, C_R\, \|z - w\|_{X_T}.$$

Hence, $\|(\Phi z)(t) - (\Phi w)(t)\|_{H^s} \leq t\, C_{\beta,T} C_R\, \|z - w\|_{X_T} \leq T\, C_{\beta,T} C_R\, \|z - w\|_{X_T}$. Taking the supremum over $t \in [0,T]$ gives

$$\|\Phi z - \Phi w\|_{X_T} \leq T\, C_{\beta,T} C_R\, \|z - w\|_{X_T}.$$

This proves the Lipschitz estimate. □

The above results provide the basic estimates needed to apply a contraction argument on a sufficiently small time interval and thus to establish local well-posedness.

5. Local Well-posedness

In this section we establish a local well-posedness result for the Atangana-Baleanu time-fractional problem on $\mathbb{T}$ for orders $\alpha \in (1,2)$, that is,

$$\alpha = 1 + \beta, \quad \beta \in (0,1).$$

We work in the Sobolev scale $H^s(\mathbb{T})$ with $s > \frac{5}{2}$. In particular,

$$H^s(\mathbb{T}) \hookrightarrow C^2(\mathbb{T}),$$

and the quadratic mapping $z \mapsto (z^2)_{xx}$ is locally Lipschitz from $H^s(\mathbb{T})$ into $H^{s-2}(\mathbb{T})$.

In view of the integral formulation established in Section 4, it is natural to work in the space

$$Y_T = \Big\{ z \in C([0,T], H^s(\mathbb{T})) : z_t \in C([0,T], H^{s-2}(\mathbb{T})) \Big\}$$

endowed with the norm

$$\|z\|_{Y_T} = \sup_{t\in[0,T]} \|z(t)\|_{H^s} + \sup_{t\in[0,T]} \|z_t(t)\|_{H^{s-2}}.$$

For $R > 0$, we denote by $\mathbb{B}_R = \{z \in Y_T : \|z\|_{Y_T} \le R\}$ the closed ball of radius R centered at the origin.

We recall the nonlinear operator

$$F(z) = -Lz + (z^2)_{xx}, \quad L = \partial_x^4 - \partial_x^2.$$

For $z \in Y_T$, we define

$$(\Phi z)(t) = z_0 + t z_1 + \int_0^t {}^{AB}I_\tau^\beta \Big(F(z) \Big)(\tau)\, d\tau, \quad t \in [0,T].$$

A mild solution on $[0,T]$ is a fixed point of Φ in Y_T.

Lemma 5.1. *Let $s > \frac{5}{2}$ and let $R > 0$. Then, there exists a constant $C_R > 0$ such that for all $z, w \in H^s(\mathbb{T})$ with $\|z\|_{H^s}, \|w\|_{H^s} \le R$, one has*

$$\|F(z) - F(w)\|_{H^{s-2}} \le C_R \|z - w\|_{H^s}.$$

Moreover, $\|F(z)\|_{H^{s-2}} \le C(1 + \|z\|_{H^s})\|z\|_{H^s}$.

Proof. We write $F(z) - F(w) = -L(z-w) + \Big((z^2)_{xx} - (w^2)_{xx} \Big)$. Since $L : H^s(\mathbb{T}) \to H^{s-2}(\mathbb{T})$ is bounded, we have

$$\|L(z-w)\|_{H^{s-2}} \le C \|z - w\|_{H^s}.$$

On the other hand, by the nonlinear estimate established in Section 3,

$$\Big\| (z^2)_{xx} - (w^2)_{xx} \Big\|_{H^{s-2}} \le C_R \|z - w\|_{H^s}.$$

This proves the first estimate.

For the second one, we simply take $w = 0$ and use

$$\left\|(z^2)_{xx}\right\|_{H^{s-2}} \leq C\left\|z^2\right\|_{H^s} \leq C\|z\|_{H^s}^2,$$

together with $\|Lz\|_{H^{s-2}} \leq C\|z\|_{H^s}$. Hence,

$$\|F(z)\|_{H^{s-2}} \leq C\|z\|_{H^s} + C\|z\|_{H^s}^2 \leq C(1 + \|z\|_{H^s})\|z\|_{H^s}. \qquad \square$$

Lemma 5.2. *Let $r \in \mathbb{R}$, $\beta \in (0,1)$ and $T > 0$. Then, for every $f \in C([0,T], H^r(\mathbb{T}))$,*

$$^{AB}I_t^\beta f \in C([0,T], H^r(\mathbb{T})),$$

and there exists a constant $C_{\beta,T} > 0$ such that

$$\left\|^{AB}I_t^\beta f\right\|_{C([0,T],H^r)} \leq C_{\beta,T}\|f\|_{C([0,T],H^r)}.$$

More precisely, one may take

$$C_{\beta,T} = \frac{1-\beta}{M(\beta)} + \frac{T^\beta}{M(\beta)\Gamma(\beta+1)}.$$

Proof. The proof is identical to that of Lemma 4.1 in Section 4, and is included here for convenience. By definition,

$$^{AB}I_t^\beta f(t) = \frac{1-\beta}{M(\beta)}f(t) + \frac{\beta}{M(\beta)\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}f(\tau)\,d\tau.$$

Taking the H^r norm and using the triangle inequality, we obtain

$$\left\|^{AB}I_t^\beta f(t)\right\|_{H^r} \leq \frac{1-\beta}{M(\beta)}\|f(t)\|_{H^r} + \frac{\beta}{M(\beta)\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\|f(\tau)\|_{H^r}\,d\tau.$$

Since $\|f(\tau)\|_{H^r} \leq \|f\|_{C([0,T],H^r)}$, and

$$\int_0^t (t-\tau)^{\beta-1}\,d\tau = \frac{t^\beta}{\beta} \leq \frac{T^\beta}{\beta},$$

it follows that

$$\left\|^{AB}I_t^\beta f\right\|_{C([0,T],H^r)} \leq \left(\frac{1-\beta}{M(\beta)} + \frac{T^\beta}{M(\beta)\Gamma(\beta+1)}\right)\|f\|_{C([0,T],H^r)}.$$

$\square$

Lemma 5.3. *Let $s > \frac{5}{2}$, $(z_0, z_1) \in H^s(\mathbb{T}) \times H^{s-2}(\mathbb{T})$ and $R > 0$. Then, there exists $T_0 = T_0(\beta, s, R, \|z_0\|_{H^s}, \|z_1\|_{H^{s-2}}) > 0$ such that, for every $T \in (0, T_0]$, the map*

$$\Phi : \mathbb{B}_R \to Y_T$$

is well defined. Moreover, if R is chosen large enough and $T > 0$ is sufficiently small, then $\Phi(\mathbb{B}_R) \subset \mathbb{B}_R$.

Proof. Let $z \in \mathbb{B}_R$. By Lemma 5.1,

$$\|F(z)\|_{C([0,T],H^{s-2})} \leq C(1+R)R.$$

Applying Lemma 5.2, with $r = s-2$, we get

$$\left\| {}^{AB}I_t^{\beta}(F(z)) \right\|_{C([0,T],H^{s-2})} \leq C_{\beta,T} C(1+R)R.$$

We first estimate the time derivative of Φz. Differentiating the definition of Φz, we find

$$(\Phi z)_t(t) = z_1 + {}^{AB}I_t^{\beta}(F(z))(t).$$

Hence, $\|(\Phi z)_t\|_{C([0,T],H^{s-2})} \leq \|z_1\|_{H^{s-2}} + C_{\beta,T} C(1+R)R$.

We next estimate Φz itself in H^s. Since

$$(\Phi z)(t) = z_0 + \int_0^t (\Phi z)_t(\tau)\, d\tau,$$

we have $\|(\Phi z)(t)\|_{H^{s-2}} \leq \|z_0\|_{H^{s-2}} + T\|(\Phi z)_t\|_{C([0,T],H^{s-2})}$. However, because $z \in Y_T$ and $z_0 \in H^s$, the integral formulation is understood in the class $C([0,T],H^s) \cap C^1([0,T],H^{s-2})$, equivalently, the fixed point is sought among functions with prescribed H^s initial datum and H^{s-2} time derivative. Therefore the natural norm estimate is

$$\|\Phi z\|_{Y_T} \leq \|z_0\|_{H^s} + \|z_1\|_{H^{s-2}} + C_T\, C_{\beta,T} C(1+R)R,$$

for some constant $C_T > 0$ depending only on T.

Choosing

$$R \geq 2\Big(\|z_0\|_{H^s} + \|z_1\|_{H^{s-2}}\Big)$$

and then taking $T > 0$ sufficiently small so that

$$C_T\, C_{\beta,T} C(1+R) \leq \frac{1}{2},$$

we obtain $\|\Phi z\|_{Y_T} \leq R$. Hence, $\Phi(\mathbb{B}_R) \subset \mathbb{B}_R$. □

Lemma 5.4. *Let $s > \frac{5}{2}$ and $R > 0$. Then, there exists $T_1 > 0$ such that, for all $T \in (0, T_1]$, the map $\Phi : \mathbb{B}_R \to \mathbb{B}_R$ is a strict contraction with respect to the Y_T norm.*

Proof. Let $z, w \in \mathbb{B}_R$. By definition,

$$(\Phi z)(t) - (\Phi w)(t) = \int_0^t {}^{AB}I_\tau^{\beta}\Big(F(z) - F(w)\Big)(\tau)\, d\tau.$$

Differentiating in time gives

$$(\Phi z)_t(t) - (\Phi w)_t(t) = {}^{AB}I_t^{\beta}\Big(F(z) - F(w)\Big)(t).$$

Applying Lemma 5.2 with $r = s-2$ and Lemma 5.1, we obtain

$$\|(\Phi z)_t - (\Phi w)_t\|_{C([0,T],H^{s-2})} \leq C_{\beta,T} C_R \|z - w\|_{C([0,T],H^s)}.$$

Moreover,

$$\|(\Phi z)(t) - (\Phi w)(t)\|_{H^s} \leq \int_0^t \|(\Phi z)_t(\tau) - (\Phi w)_t(\tau)\|_{H^{s-2}}\, d\tau$$

and therefore

$$\|\Phi z - \Phi w\|_{C([0,T],H^s)} \le T\, C_{\beta,T} C_R \|z - w\|_{C([0,T],H^s)}.$$

Combining the two estimates, we find

$$\|\Phi z - \Phi w\|_{Y_T} \le (1+T)\, C_{\beta,T} C_R\, \|z - w\|_{Y_T}.$$

Hence, if $T > 0$ is chosen so small that $(1+T)\, C_{\beta,T} C_R < 1$, then Φ is a strict contraction on $\mathbb{B}_R$. □

Theorem 5.1 (Local well-posedness). *Let $\beta \in (0,1)$ and set $\alpha = 1+\beta \in (1,2)$. Fix $s > \frac{5}{2}$. Then, for every $(z_0, z_1) \in H^s(\mathbb{T}) \times H^{s-2}(\mathbb{T})$, there exists a time*

$$T = T\Big(\beta, s, \|z_0\|_{H^s}, \|z_1\|_{H^{s-2}}\Big) > 0$$

such that the problem

$${}^{ABC}D_t^{1+\beta} z + Lz - (z^2)_{xx} = 0, \quad x \in \mathbb{T}, t \in (0,T],$$

with initial conditions $z(\cdot,0) = z_0$, $z_t(\cdot,0) = z_1$, admits a unique mild solution

$$z \in C([0,T], H^s(\mathbb{T})) \cap C^1([0,T], H^{s-2}(\mathbb{T})).$$

Moreover, this solution is the unique fixed point of Φ in a suitable closed ball of Y_T.

Proof. The result follows immediately from Lemma 5.3, Lemma 5.4, and Banach's fixed point theorem. □

Theorem 5.2 (Continuous dependence on the initial data). *Under the assumptions of Theorem 5.1, let z and $\widetilde{z}$ be the solutions corresponding to the initial data (z_0, z_1), $(\widetilde{z}_0, \widetilde{z}_1)$ on a common time interval $[0,T]$ on which the contraction argument applies. Then, there exists a constant $C = C(\beta, s, T, R) > 0$ such that*

$$\|z - \widetilde{z}\|_{C([0,T],H^s)} + \|z_t - \widetilde{z}_t\|_{C([0,T],H^{s-2})} \le C\Big(\|z_0 - \widetilde{z}_0\|_{H^s} + \|z_1 - \widetilde{z}_1\|_{H^{s-2}}\Big).$$

Proof. We write $z = \Phi_{(z_0,z_1)}(z)$, $\widetilde{z} = \Phi_{(\widetilde{z}_0,\widetilde{z}_1)}(\widetilde{z})$, subtract the two identities, and argue exactly as in the contraction estimate. We obtain

$$\|z - \widetilde{z}\|_{Y_T} \le \|z_0 - \widetilde{z}_0\|_{H^s} + \|z_1 - \widetilde{z}_1\|_{H^{s-2}} + \kappa \|z - \widetilde{z}\|_{Y_T},$$

where $0 < \kappa < 1$ is the contraction constant. Absorbing the last term on the left-hand side gives the desired estimate. □

Proposition 5.1 (Continuation criterion). *Let z be the local solution given by Theorem 5.1, defined on a maximal interval $[0, T_{\max})$, $0 < T_{\max} \le +\infty$. If*

$$\sup_{t \in [0,T^*]} \|z(t)\|_{H^s} < +\infty,$$

for every $T^* < T_{\max}$, then the solution can be continued beyond $T_{\max}$. Consequently, either

$$T_{\max} = +\infty \quad \text{or} \quad \limsup_{t \uparrow T_{\max}} \|z(t)\|_{H^s} = +\infty.$$

Proof. The local existence time given by Theorem 5.1 depends only on β, s, $\|z(t_0)\|_{H^s}$, $\|z_t(t_0)\|_{H^{s-2}}$. Hence, as long as these quantities remain bounded, the same fixed point argument may be restarted at time t_0, yielding an extension of the solution on a further interval of length bounded from below by a positive constant independent of t_0. This proves the continuation criterion and the blow-up alternative. □

Remark 5.1. If one can complement the local theory with an a priori estimate controlling $\|z(t)\|_{H^s}$ uniformly on bounded time intervals, for instance through a suitable energy functional compatible with the Atangana-Baleanu memory term, then the continuation criterion yields global existence. Establishing such an estimate is a separate problem and depends on the interaction between the fractional time operator and the nonlinear dispersive structure of the equation.

6. Laplace-Fourier Realization of the Picard Iteration

In this section we derive a modewise representation of the fixed-point map associated with the Atangana-Baleanu fractional problem on $\mathbb{T}$. More precisely, we show that the integral iteration introduced in Section 5 can be written explicitly in Fourier variables and solved mode by mode in Laplace variables. This provides a practical Laplace-Fourier implementation of the Picard scheme.

Throughout this section we set

$$\alpha = 1 + \beta, \quad \beta \in (0,1), \quad L = \partial_x^4 - \partial_x^2, \quad \mathcal{N}(z) = -(z^2)_{xx}.$$

The equation reads

$$^{ABC}D_t^{1+\beta} z + Lz + \mathcal{N}(z) = 0, \quad x \in \mathbb{T}, t > 0,$$

with initial conditions $z(\cdot,0) = z_0$, $z_t(\cdot,0) = z_1$.

6.1. **Fourier representation.** Let

$$z(x,t) = \sum_{k\in\mathbb{Z}} \widehat{z}_k(t) e^{ikx}, \quad \widehat{z}_k(t) = \frac{1}{2\pi}\int_0^{2\pi} z(x,t)e^{-ikx}\,dx.$$

By Proposition 3.2, the operator L diagonalizes in the Fourier basis and satisfies

$$\widehat{(Lz)}_k(t) = \lambda_k \widehat{z}_k(t), \quad \lambda_k = k^4 + k^2.$$

Moreover, since

$$\widehat{z^2}_k(t) = \sum_{\ell\in\mathbb{Z}} \widehat{z}_\ell(t)\widehat{z}_{k-\ell}(t),$$

we obtain $\widehat{\mathcal{N}(z)}_k(t) = k^2\widehat{z^2}_k(t)$. Therefore, for each $k \in \mathbb{Z}$, the Fourier coefficient $\widehat{z}_k$ formally satisfies

$$^{ABC}D_t^{1+\beta}\widehat{z}_k(t) + \lambda_k \widehat{z}_k(t) + k^2\widehat{z^2}_k(t) = 0. \tag{6.1}$$

6.2. **Laplace representation of one mode.** Let

$$Z_k(s) = \mathcal{L}[\widehat{z}_k](s).$$

Using the Laplace transform formula for the Atangana-Baleanu derivative of order $1+\beta$, we obtain

$$\mathcal{L}\left[{}^{ABC}D_t^{1+\beta}\widehat{z}_k\right](s) = \frac{M(\beta)}{1-\beta}\cdot\frac{s^{\beta}\Big(sZ_k(s)-\widehat{z}_k(0)\Big)-s^{\beta-1}\widehat{z}_k'(0)}{s^{\beta}+\frac{\beta}{1-\beta}}.$$

Applying the Laplace transform to (6.1) yields

$$\frac{M(\beta)}{1-\beta}\cdot\frac{s^{\beta}\Big(sZ_k(s)-\widehat{z}_k(0)\Big)-s^{\beta-1}\widehat{z}_k'(0)}{s^{\beta}+\frac{\beta}{1-\beta}}+\lambda_k Z_k(s)+\mathcal{L}\left[\widehat{k^2z^2}_k\right](s)=0. \tag{6.2}$$

For later use, we define the linear Laplace operator

$$\mathscr{A}_k[Z](s) = \frac{M(\beta)}{1-\beta}\cdot\frac{s^{\beta}\Big(sZ(s)-\widehat{z}_k(0)\Big)-s^{\beta-1}\widehat{z}_k'(0)}{s^{\beta}+\frac{\beta}{1-\beta}}+\lambda_k Z(s).$$

Then, (6.2) can be rewritten as

$$\mathscr{A}_k[Z_k](s) = -\mathcal{L}\left[\widehat{k^2z^2}_k\right](s).$$

6.3. **Laplace-Fourier Picard iteration.** We now define an iterative sequence by freezing the nonlinear term at the previous step. Let $z^{(n)}$ be known and define

$$F^{(n)} = -(z^{(n)})_{xx}^2.$$

In Fourier variables,

$$\widehat{F^{(n)}}_k(t) = k^2\,\widehat{(z^{(n)})^2}_k(t), \quad \mathcal{F}_k^{(n)}(s) = \mathcal{L}[\widehat{F^{(n)}}_k](s).$$

The $(n+1)$-th iterate is defined as the solution of

$${}^{ABC}D_t^{1+\beta}\widehat{z}_k^{(n+1)}(t)+\lambda_k\widehat{z}_k^{(n+1)}(t) = -\widehat{F^{(n)}}_k(t), \tag{6.3}$$

with the prescribed initial traces

$$\widehat{z}_k^{(n+1)}(0)=\widehat{z}_{0,k}, \quad \partial_t\widehat{z}_k^{(n+1)}(0)=\widehat{z}_{1,k},$$

where $\widehat{z}_{0,k}=\widehat{z}_0(k)$, $\widehat{z}_{1,k}=\widehat{z}_1(k)$. Taking Laplace transforms in (6.3), we obtain

$$\mathscr{A}_k\left[Z_k^{(n+1)}\right](s) = -\mathcal{F}_k^{(n)}(s).$$

Solving for $Z_k^{(n+1)}$ gives

$$Z_k^{(n+1)}(s) = \frac{\frac{M(\beta)}{1-\beta}\cdot\frac{s^{\beta}\widehat{z}_{0,k}+s^{\beta-1}\widehat{z}_{1,k}}{s^{\beta}+\frac{\beta}{1-\beta}}-\mathcal{F}_k^{(n)}(s)}{\lambda_k+\frac{M(\beta)}{1-\beta}\cdot\frac{s^{\beta+1}}{s^{\beta}+\frac{\beta}{1-\beta}}}. \tag{6.4}$$

The next iterate is then recovered by inverse Laplace transform,

$$\widehat{z}_k^{(n+1)}(t) = \mathcal{L}^{-1}[Z_k^{(n+1)}](t),$$

and finally

$$z^{(n+1)}(x,t) = \sum_{k\in\mathbb{Z}} \widehat{z}_k^{(n+1)}(t) e^{ikx}.$$

Remark 6.1. A natural initialization is obtained by neglecting the quadratic term and solving the linear problem

$${}^{ABC}D_t^{1+\beta} z^{(0)} + Lz^{(0)} = 0, \quad z^{(0)}(\cdot,0) = z_0, \quad z_t^{(0)}(\cdot,0) = z_1.$$

Equivalently, for each $k \in \mathbb{Z}$, one solves

$${}^{ABC}D_t^{1+\beta} \widehat{z}_k^{(0)}(t) + \lambda_k \widehat{z}_k^{(0)}(t) = 0,$$

with initial data $\widehat{z}_k^{(0)}(0) = \widehat{z}_{0,k}$, $\partial_t \widehat{z}_k^{(0)}(0) = \widehat{z}_{1,k}$.

6.4. **Relation with the abstract fixed-point map.** The previous modewise construction is simply the Fourier-Laplace realization of the abstract Picard iteration introduced in Section 5. Indeed, if Φ denotes the mild solution map

$$(\Phi z)(t) = z_0 + tz_1 + \int_0^t {}^{AB}I_\tau^\beta \Big(-Lz - (z^2)_{xx}\Big)(\tau)\, d\tau,$$

then the iterative procedure above is exactly $z^{(n+1)} = \Phi(z^{(n)})$. Consequently, the convergence of the Laplace-Fourier iteration follows directly from the contraction argument already established for the local well-posedness theory.

Theorem 6.1. *Let $s > \frac{5}{2}$ and let $(z_0, z_1) \in H^s(\mathbb{T}) \times H^{s-2}(\mathbb{T})$. Assume that, for some $T > 0$ and some radius $R > 0$, the map Φ is a strict contraction on the closed ball $\mathbb{B}_R \subset Y_T$, $Y_T = C([0,T], H^s(\mathbb{T})) \cap C^1([0,T], H^{s-2}(\mathbb{T}))$, with contraction constant $0 < \kappa < 1$. Let $\{z^{(n)}\}_{n\geq 0}$ be defined by*

$$z^{(n+1)} = \Phi(z^{(n)}), \quad n \geq 0,$$

with an initial guess $z^{(0)} \in \mathbb{B}_R$. Then, $\{z^{(n)}\}_{n\geq 0}$ converges in Y_T to the unique mild solution $z \in \mathbb{B}_R$, and

$$\Big\|z^{(n)} - z\Big\|_{Y_T} \leq \frac{\kappa^n}{1-\kappa} \Big\|z^{(1)} - z^{(0)}\Big\|_{Y_T}, \quad n \geq 0.$$

Proof. Since Φ is a strict contraction on $\mathbb{B}_R$, Banach's fixed point theorem ensures the existence of a unique fixed point $z \in \mathbb{B}_R$. Moreover, for every initial guess $z^{(0)} \in \mathbb{B}_R$, the Picard sequence $z^{(n+1)} = \Phi(z^{(n)})$ converges in Y_T to z.

Furthermore,

$$\Big\|z^{(n+1)} - z^{(n)}\Big\|_{Y_T} = \Big\|\Phi(z^{(n)}) - \Phi(z^{(n-1)})\Big\|_{Y_T} \leq \kappa \Big\|z^{(n)} - z^{(n-1)}\Big\|_{Y_T},$$

and therefore $\|z^{(n+1)} - z^{(n)}\|_{Y_T} \leq \kappa^n \Big\|z^{(1)} - z^{(0)}\Big\|_{Y_T}$. Using the telescopic representation

$$z - z^{(n)} = \sum_{m=n}^{+\infty} \Big(z^{(m+1)} - z^{(m)}\Big),$$

we infer

$$\left\|z - z^{(n)}\right\|_{Y_T} \leq \sum_{m=n}^{+\infty} \left\|z^{(m+1)} - z^{(m)}\right\|_{Y_T} \leq \sum_{m=n}^{+\infty} \kappa^m \left\|z^{(1)} - z^{(0)}\right\|_{Y_T}.$$

Since the geometric series converges, we obtain

$$\left\|z - z^{(n)}\right\|_{Y_T} \leq \frac{\kappa^n}{1-\kappa} \left\|z^{(1)} - z^{(0)}\right\|_{Y_T}.$$

This proves the claim. □

Remark 6.2. The previous result should be understood as a convergence statement for the Picard iteration written in Laplace-Fourier variables. It is therefore a local-in-time analytical construction, not a standalone numerical convergence theorem. A genuine numerical analysis of the fully discretized scheme would require additional estimates on Laplace inversion, truncation in Fourier modes, and time discretization errors.

7. Numerical Illustration

In this section we present a Fourier pseudo-spectral approximation of the Atangana-Baleanu fractional Boussinesq-type equation on the torus and report representative short-time numerical experiments. The purpose of this section is illustrative: we aim to visualize the influence of the fractional order and to compare the fully discrete spectral solver with the Laplace-Fourier Picard iteration introduced in Section 6. No claim of full numerical convergence is made here.

We consider the equation

$${}^{ABC}D_t^{1+\beta} z + Lz - (z^2)_{xx} = 0, \quad L = \partial_x^4 - \partial_x^2, \quad \beta \in (0,1),$$

or equivalently,

$${}^{ABC}D_t^{\beta}(z_t) = -Lz + (z^2)_{xx} =: f(z),$$

supplemented with the initial conditions $z(\cdot,0) = z_0$, $z_t(\cdot,0) = z_1$.

7.1. **Spatial discretization.**

All computations are performed on the uniform grid

$$x_j = \frac{2\pi j}{N}, \quad j = 0, 1, \ldots, N-1,$$

and the numerical solution is approximated by the truncated Fourier series

$$z_N(x,t) = \sum_{|k| \leq K} \widehat{z}_k(t) e^{ikx}, \quad K = \frac{N}{2}.$$

Spatial derivatives are computed spectrally according to

$$\widehat{(\partial_x^m z_N)}_k(t) = (ik)^m \widehat{z}_k(t), \quad m \in \{1, 2, 4\}.$$

In particular, the linear part is evaluated exactly in Fourier variables:

$$\widehat{(Lz_N)}_k(t) = (k^4 + k^2)\widehat{z}_k(t).$$

The quadratic term is treated pseudo-spectrally. More precisely, at each time step we reconstruct z_N in physical space, compute the pointwise square z_N^2, transform back to Fourier variables, and then apply the standard 2/3-dealiasing rule: all modes satisfying $|k| > \frac{2K}{3}$ are set to zero. The second derivative of the filtered nonlinear term is then computed spectrally.

7.2. **Time discretization of the AB integral term.** We use the integral identity

$$z_t(t) - z_1 = {}^{AB}I_t^{\beta} f(z)(t),$$

where

$${}^{AB}I_t^{\beta} g(t) = \frac{1-\beta}{M(\beta)} g(t) + \frac{\beta}{M(\beta)\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} g(\tau)\, d\tau.$$

This suggests a time-marching strategy based on a discrete approximation of the weakly singular convolution.

Let $t^n = n\Delta t$, $n = 0, 1, \ldots, N_t$, and denote by $z^n \approx z(\cdot, t^n)$, $v^n \approx z_t(\cdot, t^n)$ the numerical approximations of the solution and its time derivative. At time level t^n, we define $f^n = f(z^n) = -Lz^n + \left((z^n)^2\right)_{xx}$. To approximate the Volterra term, we use a piecewise-constant quadrature:

$$\int_0^{t^n} (t^n - \tau)^{\beta-1} f(\tau)\, d\tau \approx \sum_{j=0}^{n-1} \left(\int_{t^j}^{t^{j+1}} (t^n - \tau)^{\beta-1}\, d\tau \right) f^j.$$

The convolution weights are therefore given by

$$\omega_{n,j} = \int_{t^j}^{t^{j+1}} (t^n - \tau)^{\beta-1}\, d\tau = \frac{(\Delta t)^{\beta}}{\beta} \Big((n-j)^{\beta} - (n-j-1)^{\beta} \Big), \quad 0 \le j \le n-1.$$

Hence the AB integral at time t^n is approximated by

$${}^{AB}I_{t^n}^{\beta}(f(z)) \approx \frac{1-\beta}{M(\beta)}\, f^n + \frac{\beta}{M(\beta)\Gamma(\beta)} \sum_{j=0}^{n-1} \omega_{n,j}\, f^j.$$

We then define the approximate velocity by

$$v^n = z_1 + \frac{1-\beta}{M(\beta)}\, f^n + \frac{\beta}{M(\beta)\Gamma(\beta)} \sum_{j=0}^{n-1} \omega_{n,j}\, f^j,$$

and update the state through the explicit Euler step $z^{n+1} = z^n + \Delta t\, v^n$.

Comments on the scheme. The above method is intended as a simple and robust procedure for generating short-time numerical profiles. Because the linear operator L contains fourth-order derivatives, an explicit time update may require small values of Δt in order to remain stable. For more demanding simulations, one could replace the explicit Euler step by a semi-implicit treatment of the linear part or by a predictor-corrector strategy adapted to Volterra equations with weakly singular kernels.

In the present work, numerical consistency is assessed empirically by refining both the spatial resolution N and the time step Δt. In particular, we monitor the decay of

the high-frequency Fourier tail and verify that the diagnostics $\|z(t)\|_{L^2}$ and $\|z(t)\|_{H^1}$ stabilize under mesh refinement.

7.3. **Test configurations and diagnostics.** For the simulations reported below, we choose smooth initial data, for instance

$$z_0(x) = A\cos x + B\cos(2x), \quad z_1(x) = 0,$$

and compare several values of $\alpha = 1 + \beta \in (1, 2)$ under identical numerical conditions.

The discrete L^2 norm is computed by

$$\|z^n\|_{L_h^2} = \left(\frac{1}{N} \sum_{j=0}^{N-1} |z^n(x_j)|^2 \right)^{1/2},$$

and the discrete H^1 norm by

$$\|z^n\|_{H_h^1} = \left(\|z^n\|_{L_h^2}^2 + \|(\partial_x z^n)\|_{L_h^2}^2 \right)^{1/2}.$$

When needed, we also examine energy-type quantities involving $\|(\partial_x z^n)\|_{L_h^2}$ and $\|(\partial_x^2 z^n)\|_{L_h^2}$.

Finally, in order to connect the computations with the analytical construction of Sections 5–6, we compare the fully discrete spectral approximation with one Picard iterate obtained from the Laplace-Fourier formulation. As expected, the agreement is best on short time intervals contained in the local contraction regime and for parameter choices for which the solution remains bounded in the Sobolev ball used in the fixed-point argument.

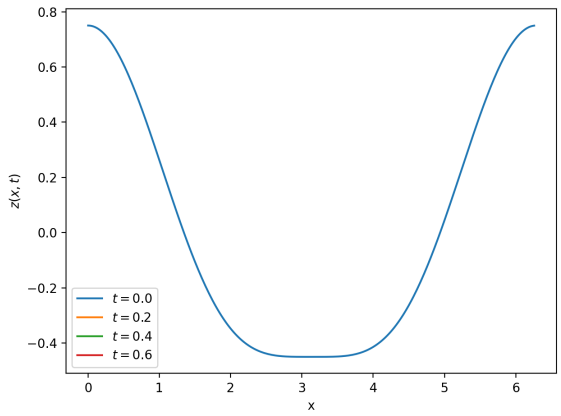


FIGURE 1. Profiles $x \mapsto z(x, t)$ at selected times for a representative choice $\alpha = 1.5$, computed by the Fourier pseudo-spectral scheme with 2/3-dealiasing.

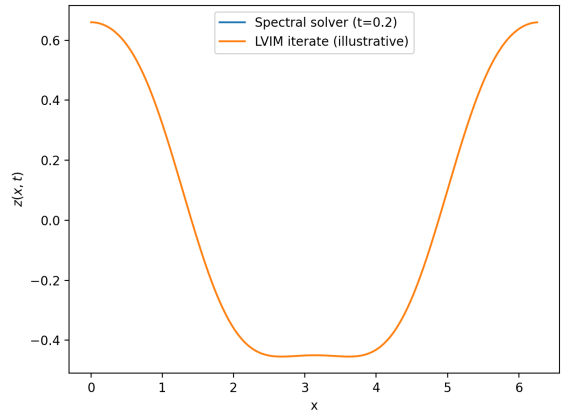


FIGURE 2. Short-time comparison between the Fourier pseudo-spectral approximation and one Picard iterate generated from the Laplace-Fourier integral formulation.

8. Conclusion

We have developed a local well-posedness theory for a Boussinesq-type equation on $\mathbb{T}$ with an Atangana-Baleanu time derivative of order $\alpha \in (1, 2)$, based on a precise integral formulation and a contraction argument in Sobolev spaces. The analysis identifies the functional framework required to control the quadratic nonlinearity together with the fractional memory term. On the numerical side, a Fourier pseudo-spectral scheme combined with a discrete AB convolution illustrates the influence of α on the short-time dynamics. A Laplace-based variational iteration procedure was shown to be consistent with the fixed-point construction. Future work includes global-in-time analysis, refined energy methods adapted to the AB structure, and investigations of stability, long-time behavior, and possible instability or chaotic regimes.

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