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PICTURE FUZZY SUBGROUP

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Abstract. Picture fuzzy subgroup of a crisp group is established here and some properties connected to it are investigated. Also, normalized restricted picture fuzzy set, conjugate picture fuzzy subgroup, picture fuzzy coset, picture fuzzy normal subgroup and the order of picture fuzzy subgroup are defined. The order of picture fuzzy subgroup is defined using the cardinality of a special type of crisp subgroup. Some corresponding properties are established in this regard.

Significant Statement. Subgroup is an important algebraic structure in the field of Pure Mathematics. Study of different properties of subgroup in fuzzy sense is an interesting fact to the readers because fuzzy sense is the extension of classical sense. Readers can easily observe how the properties of subgroup hold in fuzzy sense like classical sense. Picture fuzzy sense is the generalization of fuzzy sense. In other words, picture fuzzy sense can be treated as advanced fuzzy sense. Readers will be interested to study how the properties of subgroup hold when the number of components increases in fuzzy environment. Our study is actually the study of an important type of advanced fuzzy algebraic structure.

1. Introduction

Generalizing the concept of classical set theory, Zadeh [12] initiated fuzzy set theory which leads a vital role for handling uncertainty in practical field. Considering the limitation of fuzzy set and generalizing fuzzy set, Atanassov [1] introduced intuitionistic fuzzy set. After the invention of fuzzy set, Rosenfeld [9] introduced fuzzy group. Intuitionistic fuzzy subgroup came in the light of study by Zhan and Tan [13]. Sharma

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[10] investigated t-intuitionistic fuzzy subgroup. As the time goes, different researchers have done a lot of research works in the context of fuzzy set and intuitionistic fuzzy set. Intuitionistic fuzzy set deals with the measure of membership and the measure of non-membership such that their sum does not exceed unity. It was observed that the measure of neutrality was not taken into account in intuitionistic fuzzy set. Cuong and Kreinovich [4] initiated the notion of picture fuzzy set including the measure of neutral membership with the intuitionistic fuzzy set. So, picture fuzzy set can be treated as an immediate generalization of intuitionistic fuzzy set by togethering three components namely positive, neutral and negative. With the advancement of time, different kinds of research works under picture fuzzy environment were performed by several researchers [2, 3, 5–8, 11].

Here an attempt has been made to define picture fuzzy subgroup, normalized restricted picture fuzzy set, conjugate picture fuzzy subgroup, picture fuzzy coset, picture fuzzy normal subgroup and the order of picture fuzzy subgroup. Different corresponding properties have also been studied.

2. Preliminaries

Here, some primary concepts of fuzzy set (FS), fuzzy subgroup (FSG), intuitionistic fuzzy set (IFS), intuitionistic fuzzy subgroup (IFSG), picture fuzzy set (PFS) and some basic operations on picture fuzzy sets (PFSs) are recapitulated.

Definition 2.1 ([12]). Let A be the set of universe. Then a FS P over A is defined as $P = \{(a, \mu_P(a)) : a \in A\}$, where $\mu_P(a) \in [0, 1]$ is the measure of membership of a in P.

Realizing the absence of non-membership, Atanassov [1] included it in IFS.

Definition 2.2 ([1]). Let A be the set of universe. An IFS P over A is defined by $P = \{(a, \mu_P(a), v_P(a)) : a \in A\}$, where $\mu_P(a) \in [0, 1]$ is the measure of membership of a in P and $v_P(a) \in [0, 1]$ is the measure of non-membership of a in P with the condition $0 \leqslant \mu_P(a) + v_P(a) \leqslant 1$ for all $a \in A$.

Here, $S_P(a)= 1 - (\mu_P(a) + v_P(a))$ is the measure of suspicion of a in P, which excludes the measure of membership and non-membership.

Based on the notion of FS given by Zadeh, Rosenfeld [9] defined FSG.

Definition 2.3 ([9]). Let $(G, *)$ be a group and $P = \{(a, \mu_P(a)) : a \in G\}$ be a FS in G. Then P is said to be FSG of G if $\mu_P(a * b) \geqslant \mu_P(a) \wedge \mu_P(b)$ and $\mu_P(a^{-1}) \geqslant \mu_P(a)$ for all $a, b \in G$. Here a^{-1} is the inverse of a in G.

Definition 2.4 ([13]). Let $(G, *)$ be a crisp group and $P = \{(a, \mu_P(a), v_P(a)) : a \in G\}$ be an IFS in G. Then P is said to be IFSG of G if

(i) $\mu_P(a * b) \geqslant \mu_P(a) \wedge \mu_P(b)$, $v_P(a * b) \leqslant v_P(a) \vee v_P(b)$;

(ii) $\mu_P(a^{-1}) \geqslant \mu_P(a)$, $v_P(a^{-1}) \leqslant v_P(a)$ for all $a, b \in G$. Here a^{-1} is the inverse of a in G.

Cuong and Kreinovich [4] included more possible types of uncertainty upon IFS and initiated a new set namely PFS.

Definition 2.5 ([4])**.** Let A be the set of universe. Then a PFS P over the universe A is defined as $P = \{(a, \mu_P(a), \eta_P(a), v_P(a)) : a \in A\}$, where $\mu_P(a) \in [0, 1]$ is the measure of positive membership of a in P, $\eta_P(a) \in [0, 1]$ is the measure of neutral membership of a in P and $v_P(a) \in [0, 1]$ is the measure of negative membership of a in P with the condition $0 \leqslant \mu_P(a) + \eta_P(a) + v_P(a) \leqslant 1$ for all $a \in A$. For all $a \in A$ $1 - (\mu_P(a) + \eta_P(a) + v_P(a))$ is the measure of denial membership a in P.

The basic operations on PFSs consisting equality, union and intersection are defined below.

Definition 2.6 ([4])**.** Let $P = \{(a, \mu_P(a), \eta_P(a), v_P(a)) : a \in A\}$ and $Q = \{(a, \mu_Q(a), \eta_Q(a), v_Q(a)) : a \in A\}$ be two PFSs over the universe A. Then

(i) $P \subseteq Q$ if and only if $\mu_P(a) \leqslant \mu_Q(a)$, $\eta_P(a) \leqslant \eta_Q(a)$, $v_P(a) \geqslant v_Q(a)$ for all $a \in A$;

(ii) $P = Q$ if and only if $\mu_P(a) = \mu_Q(a)$, $\eta_P(a) = \eta_P(a)$, $v_P(a) = v_Q(a)$ for all $a \in A$;

(iii) $P \cup Q = \{(a, \max(\mu_P(a), \mu_Q(a)), \min(\eta_P(a), \eta_Q(a)), \min(v_P(a), v_Q(a))) : a \in A\}$;

(iv) $P \cap Q = \{(a, \min(\mu_P(a), \mu_Q(a)), \min(\eta_P(a), \eta_Q(a)), \max(v_P(a), v_Q(a))) : a \in A\}$.

Definition 2.7. Let $P = \{(a, \mu_P, \eta_P, v_P) : a \in A\}$ be a PFS over the universe A. Then (θ, ϕ, ψ)-cut of P is the crisp set in A denoted by $C_{\theta,\phi,\psi}(P)$ and is defined by $C_{\theta,\phi,\psi}(P) = \{a \in A : \mu_P(a) \geqslant \theta, \eta_P(a) \geqslant \phi, v_P(a) \leqslant \psi\}$, where $\theta, \phi, \psi \in [0, 1]$ with the condition $0 \leqslant \theta + \phi + \psi \leqslant 1$.

Throughout the paper, we write PFS $P = \{(a, \mu_P(a), \eta_P(a), v_P(a)) : a \in A\}$ as $P = (\mu_P, \eta_P, v_P)$.

3. Picture Fuzzy Subgroup

Now, we are going to define PFSG of a crisp group as the extension of FSG and IFSG.

Definition 3.1. Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a PFS in G. Then P is said to be a PFSG of G if

(i) $\mu_P(a * b) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\eta_P(a * b) \geqslant \eta_P(a) \wedge \eta_P(b)$, $v_P(a * b) \leqslant v_P(a) \vee v_P(b)$ for all $a, b \in G$;

(ii) $\mu_P(a^{-1}) \geqslant \mu_P(a)$, $\eta_P(a^{-1}) \geqslant \eta_P(a)$, $v_P(a^{-1}) \leqslant v_P(a)$ for all $a \in G$, where a^{-1} is the inverse of a in G.

Example 3.1. A PFS $P = (\mu_P, \eta_P, v_P)$ in a group $G = (\mathbb{Z}, +)$ is considered here in the following way:

$$\mu_P(a) = \begin{cases} 0.35, & \text{when } a \in 2\mathbb{Z}, \\ 0.2, & \text{when } a \in 2\mathbb{Z} + 1, \end{cases}$$

$$\eta_P(a) = \begin{cases} 0.45, & \text{when } a \in 2\mathbb{Z}, \\ 0.2, & \text{when } a \in 2\mathbb{Z}+1, \end{cases}$$

$$v_P(a) = \begin{cases} 0.2, & \text{when } a \in 2\mathbb{Z}, \\ 0.4, & \text{when } a \in 2\mathbb{Z}+1. \end{cases}$$

It is not very tough to show that P is a PFSG of G.

Now, we will develop a proposition in two parts. First part gives the relationship between the identity element and any other element of the universal group in case of a PFSG while the second part gives the relationship between the inverse of an element and the element itself of the universal group in case of a PFSG.

Proposition 3.1. *Let $(G, *)$ be a group and $P = (\mu_P, \eta_P, v_P)$ be a PFSG of G. Then*
(i) *$\mu_P(e) \geqslant \mu_P(a)$, $\eta_P(e) \geqslant \eta_P(a)$, $v_P(e) \leqslant v_P(a)$ for all $a \in G$, where e is the identity in G;*
(ii) *$\mu_P(a^{-1}) = \mu_P(a)$, $\eta_P(a^{-1}) = \eta_P(a)$, $v_P(a^{-1}) = v_P(a)$ for all $a \in G$. Here, a^{-1} is the inverse of a in G.*

Proof. (i) It is observed that

$$\begin{aligned} \mu_P(e) &= \mu_P(a * a^{-1}) \\ &\geqslant \mu_P(a) \wedge \mu_P(a^{-1}) \quad [\text{because } P \text{ is a PFSG of } G] \\ &= \mu_P(a) \quad [\text{because } \mu_P(a^{-1}) \geqslant \mu_P(a) \text{ as } P \text{ is a PFSG of } G], \\ \eta_P(e) &= \eta_P(a * a^{-1}) \\ &\geqslant \eta_P(a) \wedge \eta_P(a^{-1}) \quad [\text{because } P \text{ is a PFSG of } G] \\ &= \eta_P(a) \quad [\text{because } \eta_P(a^{-1}) \geqslant \eta_P(a) \text{ as } P \text{ is a PFSG of } G], \\ \text{and } v_P(e) &= v_P(a * a^{-1}) \\ &\leqslant v_P(a) \vee v_P(a^{-1}) \quad [\text{because } P \text{ is a PFSG of G}] \\ &= v_P(a) \quad [\text{because } v_P(a^{-1}) \leqslant v_P(a) \text{ as } P \text{ is a PFSG of G}], \end{aligned}$$

for all $a \in G$.
(ii) Since P is a PFSG of G, therefore $\mu_P(a^{-1}) \geqslant \mu_P(a)$, $\eta_P(a^{-1}) \geqslant \eta_P(a)$ and $v_P(a^{-1}) \leqslant v_P(a)$ for all $a \in G$. Replacing a by a^{-1}, it is obtained that $\mu_P(a) \geqslant \mu_P(a^{-1})$, $\eta_P(a) \geqslant \eta_P(a^{-1})$ and $v_P(a) \leqslant v_P(a^{-1})$ for all $a \in G$. Thus, $\mu_P(a^{-1}) = \mu_P(a)$, $\eta_P(a^{-1}) = \eta_P(a)$ and $v_P(a^{-1}) = v_P(a)$ for all $a \in G$. □

The following proposition suggests the necessary and sufficient condition under which a PFS will be a PFSG.

Proposition 3.2. *Let $(G, *)$ be a group and $P = (\mu_P, \eta_P, v_P)$ be a PFS in G. Then P is a PFSG of G if and only if $\mu_P(a * b^{-1}) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\eta_P(a * b^{-1}) \geqslant \eta_P(a) \wedge \eta_P(b)$ and $v_P(a * b^{-1}) \leqslant v_P(a) \vee v_P(b)$ for all $a, b \in G$.*

Proof. Since P is a PFSG of G, therefore $\mu_P(a*b^{-1}) \geqslant \mu_P(a) \wedge \mu_P(b^{-1}) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\eta_P(a*b^{-1}) \geqslant \eta_P(a) \wedge \eta_P(b^{-1}) \geqslant \eta_P(a) \wedge \eta_P(b)$ and $v_P(a*b^{-1}) \leqslant v_P(a) \vee v_P(b^{-1}) \leqslant v_P(a) \vee v_P(b)$ for all $a, b \in G$.

Conversely, let the condition be hold. Then

$$\begin{aligned}
\mu_P(e) &= \mu_P(a*a^{-1}) \geqslant \mu_P(a) \wedge \mu_P(a) = \mu_P(a),\\
\eta_P(e) &= \eta_P(a*a^{-1}) \geqslant \eta_P(a) \wedge \eta_P(a) = \eta_P(a),\\
v_P(e) &= v_P(a*a^{-1}) \leqslant v_P(a) \vee v_P(a) = v_P(a),
\end{aligned}$$

for all $a \in G$, e is the identity in G. Thus, $\mu_P(e) \geqslant \mu_P(a)$, $\eta_P(e) \geqslant \eta_P(a)$ and $v_P(e) \leqslant v_P(a)$ for all $a \in G$.

Now,

$$\begin{aligned}
\mu_P(b^{-1}) &= \mu_P(e*b^{-1}) \geqslant \mu_P(e) \wedge \mu_P(b) = \mu_P(b),\\
\eta_P(b^{-1}) &= \eta_P(e*b^{-1}) \geqslant \eta_P(e) \wedge \eta_P(b) = \eta_P(b),\\
v_P(b^{-1}) &= v_P(e*b^{-1}) \leqslant v_P(e) \vee v_P(b) = v_P(b), \quad \text{for all } b \in G.
\end{aligned}$$

Thus, $\mu_P(b^{-1}) \geqslant \mu_P(b)$, $\eta_P(b^{-1}) \geqslant \eta_P(b)$, $v_P(b^{-1}) \leqslant v_P(b)$ for all $b \in G$.

It is observed that

$$\begin{aligned}
\mu_P(a*b) &= \mu_P(a*(b^{-1})^{-1}) \geqslant \mu_P(a) \wedge \mu_P(b^{-1}) \geqslant \mu_P(a) \wedge \mu_P(b),\\
\eta_P(a*b) &= \eta_P(a*(b^{-1})^{-1}) \geqslant \eta_P(a) \wedge \eta_P(b^{-1}) \geqslant \eta_P(a) \wedge \eta_P(b),\\
v_P(a*b) &= v_P(a*(b^{-1})^{-1}) \leqslant v_P(a) \vee v_P(b^{-1}) \leqslant v_P(a) \vee v_P(b), \quad \text{for all } a, b \in G.
\end{aligned}$$

Consequently, P is a PFSG of G. □

Proposition 3.3. *Let $(G, *)$ be a group and $P = (\mu_P, \eta_P, v_P)$, $Q = (\mu_Q, \eta_Q, v_Q)$ be two PFSGs in G. Then $P \cap Q$ is a PFSG of G.*

Proof. Let $P \cap Q = R = (\mu_R, \eta_R, v_R)$. Then $\mu_R(a) = \mu_P(a) \wedge \mu_Q(a)$, $\mu_R(a) = \eta_P(a) \wedge \eta_Q(a)$ and $v_R(a) = v_P(a) \vee v_Q(a)$ for all $a \in G$. Since P, Q are PFSGs of G, therefore

$$\begin{aligned}
\mu_R(a*b^{-1}) &= \mu_P(a*b^{-1}) \wedge \mu_Q(a*b^{-1})\\
&\geqslant (\mu_P(a) \wedge \mu_P(b)) \wedge (\mu_Q(a) \wedge \mu_Q(b))\\
&= (\mu_P(a) \wedge \mu_Q(a)) \wedge (\mu_P(b) \wedge \mu_Q(b)) = \mu_R(a) \wedge \mu_R(b),\\
\eta_R(a*b^{-1}) &= \eta_P(a*b^{-1}) \wedge \eta_Q(a*b^{-1})\\
&\geqslant (\eta_P(a) \wedge \eta_P(b)) \wedge (\eta_Q(a) \wedge \eta_Q(b))\\
&= (\eta_P(a) \wedge \eta_Q(a)) \wedge (\eta_P(b) \wedge \eta_Q(b)) = \eta_R(a) \wedge \eta_R(b),\\
v_R(a*b^{-1}) &= v_P(a*b^{-1}) \vee v_Q(a*b^{-1})\\
&\leqslant (v_P(a) \vee v_P(b)) \vee (v_Q(a) \vee v_Q(b))\\
&= (v_P(a) \vee v_Q(a)) \vee (v_P(b) \vee v_Q(b)) = v_R(a) \vee v_R(b) \text{ for all } a, b \in G.
\end{aligned}$$

Consequently, $R = P \cap Q$ is a PFSG of G. □

We have proved that the intersection of two PFSGs is also a PFSG. But, this is not true for union. If P and Q are two PFSGs then $P \cup Q$ may or may not be PFSG. This observation is proved by examples. Below we consider two examples. Example 3.2 shows that $P \cup Q$ is not a PFSG and Example 3.3 shows that $P \cup Q$ is a PFSG.

Example 3.2. Two PFSGs $P = (\mu_P, \eta_P, \upsilon_P)$, $Q = (\mu_Q, \eta_Q, \upsilon_Q)$ in a group $G = (\mathbb{Z}, +)$ considered here in the following way:

$$\mu_P(a) = \begin{cases} 0.25, & \text{when } a \in 7\mathbb{Z}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\eta_P(a) = \begin{cases} 0.35, & \text{when } a \in 7\mathbb{Z}, \\ 0.2, & \text{otherwise}, \end{cases}$$

$$\upsilon_P(a) = \begin{cases} 0, & \text{when } a \in 7\mathbb{Z}, \\ 0.5, & \text{otherwise}, \end{cases}$$

and

$$\mu_Q(a) = \begin{cases} 0.15, & \text{when } a \in 5\mathbb{Z}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\eta_Q(a) = \begin{cases} 0.25, & \text{when } a \in 5\mathbb{Z}, \\ 0.15, & \text{otherwise}, \end{cases}$$

$$\upsilon_Q(a) = \begin{cases} 0.2, & \text{when } a \in 5\mathbb{Z}, \\ 0.3, & \text{otherwise}. \end{cases}$$

Then

$$\mu_{P\cup Q}(a) = \begin{cases} 0.25, & \text{when } a \in 7\mathbb{Z}, \\ 0.15, & \text{when } a \in 5\mathbb{Z}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\eta_{P\cup Q}(a) = \begin{cases} 0.15, & \text{when } a \in 7\mathbb{Z}, \\ 0.2, & \text{when } a \in 5\mathbb{Z}, \\ 0.15, & \text{otherwise}, \end{cases}$$

$$\upsilon_{P\cup Q}(a) = \begin{cases} 0, & \text{when } a \in 7\mathbb{Z}, \\ 0.2, & \text{when } a \in 5\mathbb{Z}, \\ 0.3, & \text{otherwise}. \end{cases}$$

Here, $\mu_{P\cup Q}(7 + (-5)) = \mu_{P\cup Q}(2) = 0 \not\geq \mu_{P\cup Q}(7) \wedge \mu_{P\cup Q}(5) = 0.25 \wedge 0.15 = 0.15$ and $\upsilon_{P\cup Q}(7 + (-5)) = \upsilon_{P\cup Q}(2) = 0.3 \not\leq \upsilon_{P\cup Q}(7) \vee \upsilon_{P\cup Q}(5) = 0 \vee 0.2 = 0.2$. But, $\eta_{P\cup Q}(7 + (-5)) = \eta_{P\cup Q}(2) = 0.15 \geq \eta_{P\cup Q}(7) \wedge \eta_{P\cup Q}(5) = 0.15 \wedge 0.2 = 0.15$. Thus, $P \cup Q$ is not a PFSG.

Example 3.3. A PFS P=(μ_P, η_P, υ_P) in a group G is considered in the following way:

$$\mu_P(a) = \begin{cases} 0.45, & \text{when } a = 0, \\ 0.3, & \text{when } a \neq 0, \end{cases}$$

$$\eta_P(a) = \begin{cases} 0.4, & \text{when } a = 0, \\ 0.2, & \text{when } a \neq 0, \end{cases}$$

$$v_P(a) = \begin{cases} 0.1, & \text{when } a = 0, \\ 0.15, & \text{when } a \neq 0, \end{cases}$$

and

$$\mu_Q(a) = \begin{cases} 0.35, & \text{when } a = 0, \\ 0.25, & \text{when } a \neq 0, \end{cases}$$

$$\mu_Q(a) = \begin{cases} 0.25, & \text{when } a = 0, \\ 0.2, & \text{when } a \neq 0, \end{cases}$$

$$v_Q(a) = \begin{cases} 0.15, & \text{when } a = 0, \\ 0.2, & \text{when } a \neq 0. \end{cases}$$

Therefore,

$$\mu_{P\cup Q}(a) = \begin{cases} 0.45, & \text{when } a = 0, \\ 0.3, & \text{when } a \neq 0, \end{cases}$$

$$\eta_{P\cup Q}(a) = \begin{cases} 0.25, & \text{when } a = 0, \\ 0.2, & \text{when } a \neq 0, \end{cases}$$

$$\eta_{P\cup Q}(a) = \begin{cases} 0.1, & \text{when } a = 0, \\ 0.15, & \text{when } a \neq 0. \end{cases}$$

Clearly, $P \cup Q$ is a PFSG of G.

Proposition 3.4. *Let $(G, *)$ be a group and $P = (\mu_P, \eta_P, v_P)$, $Q = (\mu_Q, \eta_Q, v_Q)$ be PFSGs in G. Then $P \cup Q$ is a PFSG of G if $P \subseteq Q$ or $Q \subseteq P$.*

Proof. Let $P \cup Q = R = (\mu_R, \eta_R, v_R)$. Then $\mu_R(a) = \mu_P(a) \vee \mu_Q(a)$, $\eta_R(a) = \eta_P(a) \wedge \eta_Q(a)$ and $v_R(a) = v_P(a) \wedge v_Q(a)$ for $a \in G$.

Case 1. Let $P \subseteq Q$. Then $\mu_P(a) \leqslant \mu_Q(a)$, $\eta_P(a) \leqslant \eta_Q(a)$ and $v_P(a) \geqslant v_Q(a)$ for all $a \in G$. Now,

$$\begin{aligned}
\mu_R(a * b^{-1}) &= \mu_P(a * b^{-1}) \vee \mu_Q(a * b^{-1}) \\
&= \mu_Q(a * b^{-1}) \\
&\geqslant \mu_Q(a) \wedge \mu_Q(b) \quad [\text{because } Q \text{ is a PFSG of } G] \\
&= (\mu_P(a) \vee \mu_Q(a)) \wedge (\mu_P(b) \vee \mu_Q(b)) \\
&= \mu_R(a) \wedge \mu_R(b), \\
\eta_R(a * b^{-1}) &= \eta_P(a * b^{-1}) \wedge \eta_Q(a * b^{-1}) \\
&= \eta_P(a * b^{-1}) \\
&\geqslant \eta_P(a) \wedge \eta_P(b) \quad [\text{because } Q \text{ is a PFSG of } G] \\
&= (\eta_P(a) \wedge \eta_Q(a)) \wedge (\eta_P(b) \wedge \eta_Q(b)) \\
&= \eta_R(a) \wedge \eta_R(b),
\end{aligned}$$

$$\begin{aligned} v_R(a*b^{-1}) &= v_P(a*b^{-1}) \wedge v_Q(a*b^{-1}) \\ &= v_Q(a*b^{-1}) \\ &\leqslant v_Q(a) \vee v_Q(b) \quad \text{[because } Q \text{ is a PFSG of } G] \\ &= (v_P(a) \wedge v_Q(a)) \vee (v_P(b) \wedge v_Q(b)) \\ &= v_R(a) \vee v_R(b), \quad \text{for all } a,b \in G. \end{aligned}$$

Consequently, R is a PFSG of G.

Case 2. When $Q \subseteq P$ then it can be proceeded in the similar way to get $\mu_R(a*b) \geqslant \mu_R(a) \wedge \mu_R(b)$, $\eta_R(a*b) \geqslant \eta_R(a) \wedge \eta_R(b)$ and $v_R(a*b) \leqslant v_R(a) \vee v_R(b)$ for all $a,b \in G$. □

Definition 3.2. Let $P = (\mu_P, \eta_P, v_P)$ and $Q = (\mu_Q, \eta_Q, v_Q)$ be two PFSs over the universe A. Then the Cartesian product of P and Q is the PFS $P \times Q = (\mu_{P\times Q}, \eta_{P\times Q}, v_{P\times Q})$, where $\mu_{P\times Q}((a,b)) = \mu_P(a) \wedge \mu_Q(b)$, $\eta_{P\times Q}((a,b)) = \eta_P(a) \wedge \eta_Q(b)$ and $v_{P\times Q}((a,b)) = v_P(a) \vee v_Q(b)$ for all $(a,b) \in A \times A$.

Proposition 3.5. *Let $(G,*)$ be a group and $P = (\mu_P, \eta_P, v_P)$, $Q = (\mu_Q, \eta_Q, v_Q)$ be two PFSGs in G. Then $P \times Q$ is a PFSG of $G \times G$.*

Proof. Let $P \times Q = R = (\mu_R, \eta_R, v_R)$. Then $\mu_R((a,b)) = \mu_P(a) \wedge \mu_Q(b)$, $\eta_R((a,b)) = \eta_P(a) \wedge \eta_Q(b)$ and $v_R((a,b)) = v_P(a) \vee v_Q(b)$ for all $(a,b) \in G \times G$.

Now,

$$\begin{aligned} \mu_R((a,b)*(c,d)^{-1}) &= \mu_R((a,b)*(c^{-1},d^{-1})) = \mu_P(a*c^{-1}) \wedge \mu_Q(b*d^{-1}) \\ &\geqslant (\mu_P(a) \wedge \mu_P(c)) \wedge (\mu_Q(b) \wedge \mu_Q(d)) \quad \text{[as } P,Q \text{ are PFSGs of } G] \\ &= (\mu_P(a) \wedge \mu_Q(b)) \wedge (\mu_P(c) \wedge \mu_Q(d)) \\ &= \mu_R((a,b)) \wedge \mu_R((c,d)), \\ \eta_R((a,b)*(c,d)^{-1}) &= \eta_R((a,b)*(c^{-1},d^{-1})) = \eta_P(a*c^{-1}) \wedge \eta_Q(b*d^{-1}) \\ &\geqslant (\eta_P(a) \wedge \eta_P(c)) \wedge (\eta_Q(b) \wedge \eta_Q(d)) \quad \text{[as } P,Q \text{ are PFSGs of } G] \\ &= (\eta_P(a) \wedge \eta_Q(b)) \wedge (\eta_P(c) \wedge \eta_Q(d)) \\ &= \eta_R((a,b)) \wedge \eta_R((c,d)), \\ v_R((a,b)*(c,d)^{-1}) &= v_R((a,b)*(c^{-1},d^{-1})) = v_P(a*c^{-1}) \vee v_Q(b*d^{-1}) \\ &\leqslant (v_P(a) \vee v_P(c)) \vee (v_Q(b) \vee v_Q(d)) \quad \text{[as } P,Q \text{ are PFSGs of } G] \\ &= (v_P(a) \vee v_Q(b)) \vee (v_P(c) \vee v_Q(d)) \\ &= v_R((a,b)) \vee v_Q((c,d)), \quad \text{for all } (a,b),(c,d) \in G \times G. \end{aligned}$$

Consequently, $P \times Q$ is a PFSG of $G \times G$. □

The following proposition gives the relationship between the identity element and any other element in case of Cartesian product of two PFSGs.

Proposition 3.6. *Let $(G_1,*)$ and $(G_2,*)$ be two crisp groups and $P = (\mu_P, \eta_P, v_P)$, $Q = (\mu_Q, \eta_Q, v_Q)$ be two PFSGs of G_1 and G_2, respectively. Then $\mu_{P\times Q}((e_1,e_2)) \geqslant$*

$\mu_{P\times Q}((a_1,a_2))$, $\eta_{P\times Q}((e_1,e_2)) \geqslant \eta_{P\times Q}((a_1,a_2))$ and $v_{P\times Q}((e_1,e_2)) \leqslant v_{P\times Q}((a_1,a_2))$ for all $(a_1,a_2) \in G_1 \times G_2$, where (e_1,e_2) is the identity in $G_1 \times G_2$.

Proof. Here, we have

$$\begin{aligned}
\mu_{P\times Q}((e_1,e_2)) =& \mu_P(e_1) \wedge \mu_Q(e_2) \\
&\geqslant \mu_P(a_1) \wedge \mu_Q(a_2) \quad \text{[by Proposition 3.1]} \\
&= \mu_{P\times Q}((a_1,a_2)), \\
\eta_{P\times Q}((e_1,e_2)) &= \eta_P(e_1) \wedge \eta_Q(e_2) \\
&\geqslant \eta_P(a_1) \wedge \eta_Q(a_2) \quad \text{[by Proposition 3.1]} \\
&= \eta_{P\times Q}((a_1,a_2)), \\
\text{and } v_{P\times Q}((e_1,e_2)) &= v_P(e_1) \vee v_Q(e_2) \\
&\leqslant v_P(a_1) \vee v_Q(a_2) \quad \text{[by Proposition 3.1]} \\
&= v_{P\times Q}((a_1,a_2)),
\end{aligned}$$

for all $a_1 \in G_1$ and for all $a_2 \in G_2$. Thus, it is obtained that $\mu_{P\times Q}((e_1,e_2)) \geqslant \mu_{P\times Q}((a_1,a_2))$, $\eta_{P\times Q}((e_1,e_2)) \geqslant \eta_{P\times Q}((a_1,a_2))$ and $v_{P\times Q}((e_1,e_2)) \leqslant v_{P\times Q}((a_1,a_2))$ for all $(a_1,a_2) \in G_1 \times G_2$. □

Proposition 3.7. *Let $(G_1,*)$ and $(G_2,*)$ be two crisp groups and $P = (\mu_P,\eta_P,v_P)$, $Q = (\mu_Q,\eta_Q,v_Q)$ be two PFSs of G_1 and G_2 respectively such that $P \times Q$ is PFSG of $G_1 \times G_2$. Then one of the following conditions must hold:*

(i) *$\mu_Q(e_2) \geqslant \mu_P(a)$, $\eta_Q(e_2) \geqslant \eta_P(a)$, $v_Q(e_2) \leqslant v_P(a)$ for all $a \in G_1$, where e_2 is the identity in G_2;*

(ii) *$\mu_P(e_1) \geqslant \mu_Q(b)$, $\eta_P(e_1) \geqslant \eta_Q(b)$, $v_P(e_1) \leqslant v_Q(b)$ for all $b \in G_2$, where e_1 is the identity in G_1.*

Proof. Let none of the conditions be hold. Then there exists some $a \in G_1$ and some $b \in G_2$ such that $\mu_Q(e_2) < \mu_P(a)$, $\mu_P(e_1) < \mu_Q(b)$, $\eta_Q(e_2) < \eta_P(a)$, $\eta_P(e_1) < \eta_Q(b)$, $v_Q(e_2) > v_P(a)$, $v_P(e_1) > v_Q(b)$. Then we have

$$\begin{aligned}
\mu_{P\times Q}((a,b)) &= \mu_P(a) \wedge \mu_Q(b) > \mu_Q(e_2) \wedge \mu_P(e_1) = \mu_{P\times Q}((e_1,e_2)), \\
\eta_{P\times Q}((a,b)) &= \eta_P(a) \wedge \eta_Q(b) > \eta_Q(e_2) \wedge \eta_P(e_1) = \eta_{P\times Q}(e_1,e_2), \\
v_{P\times Q}((a,b)) &= v_P(a) \vee v_Q(b) < v_Q(e_2) \vee v_P(e_1) = v_{P\times Q}((e_1,e_2)).
\end{aligned}$$

Thus, it is obtained that $\mu_{P\times Q}((a,b)) > \mu_{P\times Q}((e_1,e_2))$, $\eta_{P\times Q}((a,b)) > \eta_{P\times Q}((e_1,e_2))$ and $v_{P\times Q}((a,b)) < v_{P\times Q}((e_1,e_2))$. This is a contradiction because (e_1,e_2) is the identity in $G_1 \times G_2$ and by Proposition 3.6, it is known that $\mu_{P\times Q}((e_1,e_2)) \geqslant \mu_{P\times Q}((a_1,a_2))$, $\eta_{P\times Q}((e_1,e_2)) \geqslant \eta_{P\times Q}((a_1,a_2))$ and $v_{P\times Q}((e_1,e_2)) \leqslant v_{P\times Q}((a_1,a_2))$ for all $(a_1,a_2) \in G_1 \times G_2$. Hence, one of the conditions must hold.

The power of a PFS P can be defined by taking the power of measure of three types of membership of each element. It is easy to verify that k-th power P^k of P is also a PFS. Now, it is the time to define power of a PFS below. □

Definition 3.3. Let A be the set of universe and $P=(\mu_P,\eta_P,\upsilon_P)$ be a PFS in A. Then for a positive integer k, k-th power of the PFS P is the PFS $P^k=(\mu_P^k,\eta_P^k,\upsilon_P^k)$, where $\mu_P^k(a)=(\mu_P(a))^k$, $\eta_P^k(a)=(\eta_P(a))^k$ and $\upsilon_P^k(a)=(\upsilon_P(a))^k$ for all $a\in A$.

Obviously, $(\mu_P(a))^k\leqslant\mu_P(a)$, $(\eta_P(a))^k\leqslant\eta_P(a)$ and $(\upsilon_P(a))^k\leqslant\upsilon_P(a)$ and $0\leqslant\mu_P(a)+\eta_P(a)+\upsilon_P(a)\leqslant 1$ for all $a\in A$. So, clearly, $0\leqslant(\mu_P(a))^k+(\eta_P(a))^k+(\upsilon_P(a))^k\leqslant 1$ for all $a\in A$.

Proposition 3.8. *Let $(G,*)$ be a group and $P=(\mu_P,\eta_P,\upsilon_P)$ be a PFSG of G. Then $P^k=(\mu_P^k,\eta_P^k,\upsilon_P^k)=((\mu_P(a))^k,(\eta_P(a))^k,(\upsilon_P(a))^k)$ is a PFSG of G for a positive integer k.*

Proof. Since P is a PFSG, therefore

$$\begin{aligned}
\mu_P^k(a*b^{-1}) &= (\mu_P(a*b^{-1}))^k\\
&\geqslant (\mu_P(a)\wedge\mu_P(b))^k\\
&= (\mu_P(a))^k\wedge(\mu_P(b))^k=\mu_P^k(a)\wedge\mu_P^k(b),\\
\eta_P^k(a*b^{-1}) &= (\eta_P(a*b^{-1}))^k\\
&\geqslant (\eta_P(a)\wedge\eta_P(b))^k\\
&= (\eta_P(a))^k\wedge(\eta_P(b))^k=\eta_P^k(a)\wedge\eta_P^k(b),\\
\upsilon_P^k(a*b^{-1}) &= (\upsilon_P(a*b^{-1}))^k\\
&\leqslant (\upsilon_P(a)\vee\upsilon_P(b))^k\\
&= (\upsilon_P(a))^k\vee(\upsilon_P(b))^k=\upsilon_P^k(a)\vee\upsilon_P^k(b),\quad \text{for all } a,b\in G.
\end{aligned}$$

Consequently, P^k is a PFSG of G. □

Definition 3.4. For three chosen real numbers $\varepsilon_1\in[0,1]$, $\varepsilon_2\in[0,1]$ and $\varepsilon_3\in[0,1]$ with $\varepsilon_1+\varepsilon_2=1$ and $\varepsilon_2+\varepsilon_3=1$, we define restricted PFS P over the set of universe A as $P=\{(a,\mu_P(a),\eta_P(a),\upsilon_P(a)):a\in A\}$, where $\mu_P(a)\in[0,\varepsilon_1]$, $\eta_P(a)\in[0,\varepsilon_2]$ and $\upsilon_P\in[0,\varepsilon_3]$ such that $0\leqslant\mu_P(a)+\eta_P(a)+\upsilon_P(a)\leqslant 1$. For any $a\in A$, $(\mu_P(a),\eta_P(a),\upsilon_P(a))$ is called picture fuzzy value (PFV). In case of here defined restricted PFS, $(\varepsilon_1,\varepsilon_2,0)$ is the largest PFV.

Now, let us define a new type of restricted PFS called normalized restricted PFS as an extension of normalized IFS.

Definition 3.5. Let $P=(\mu_P,\eta_P,\upsilon_P)$ be a restricted PFS in A. Then P is said to be normalized restricted PFS if there exists $a\in A$ such that $\mu_P(a)=\varepsilon_1$, $\eta_P(a)=\varepsilon_2$ and $\upsilon_P(a)=0$.

Depending upon three real numbers $\varepsilon_1\in[0,1]$, $\varepsilon_2\in[0,1]$ and $\varepsilon_3\in[0,1]$ with the proposed conditions $\varepsilon_1+\varepsilon_2=1$ and $\varepsilon_2+\varepsilon_3=1$, many restricted PFSs are obtained and also many corresponding normalized restricted PFSs are obtained. Choose $\varepsilon_1=1$, $\varepsilon_2=0$ and $\varepsilon_3=1$. Then $\mu_P(a)\in[0,1]$, $\eta_P(a)=0$ and $\upsilon_P(a)\in[0,1]$. Thus, the neutral component is removed completely. So, restricted PFS reduces to IFS and it becomes normalized when there exists $a \in A$ such that $\mu_P(a) = \varepsilon_1 = 1$ and $v_P(a) = 0$, which is familiar to the concept of normalized IFS. So, normalized restricted PFS can be treated as an extension of normalized IFS.

Proposition 3.9. *Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a normalized restricted PFS which forms a PFSG of G. Then $\mu_P(e) = \varepsilon_1$, $\eta_P(e) = \varepsilon_2$ and $v_P(e) = 0$, where e is the identity in G.*

Proof. Since P is a normalized restricted PFS therefore there exists some $a \in G$ such that $\mu_P(a) = \varepsilon_1$, $\eta_P(a) = \varepsilon_2$ and $v_P(a) = 0$. Now, by Proposition 3.1, it is known that $\mu_P(e) \geqslant \mu_P(a) = \varepsilon_1$, $\eta_P(e) \geqslant \eta_P(a) = \varepsilon_2$ and $v_P(e) \leqslant v_P(a) = 0$. It follows that $\mu_P(e) = \varepsilon_1$, $\eta_P(e) = \varepsilon_2$ and $v_P(e) = 0$. □

A new kind of group relation called conjugate is defined below for PFSGs.

Definition 3.6. Let $(G, *)$ be a crisp group of G and $P = (\mu_P, \eta_P, v_P)$, $Q = (\mu_Q, \eta_Q, v_Q)$ be two PFSGs G. Then P is conjugate to Q if there exists $a \in G$ such that $\mu_P(u) = \mu_Q(a * u * a^{-1})$, $\eta_P(u) = \eta_Q(a * u * a^{-1})$, $v_P(u) = v_Q(a * u * a^{-1})$ for all $u \in G$.

Proposition 3.10. *Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$, $Q = (\mu_Q, \eta_Q, v_Q)$, $R = (\mu_R, \eta_R, v_R)$, $S = (\mu_S, \eta_S, v_S)$ be four PFSGs of G such that P is conjugate to R and Q is conjugate to S. Then $P \times Q$ is conjugate to $R \times S$.*

Proof. Since P is conjugate to R, therefore $\mu_P(u_1) = \mu_R(a * u_1 * a^{-1})$, $\eta_P(u_1) = \eta_R(a * u_1 * a^{-1})$ and $v_P(u_1) = v_R(a * u_1 * a^{-1})$ for some $a \in G$ and for all $u_1 \in G$. Since Q is conjugate to S, therefore $\mu_Q(u_2) = \mu_S(b * u_2 * b^{-1})$, $\eta_Q(u_2) = \eta_S(b * u_2 * b^{-1})$ and $v_Q(u_2) = v_S(b * u_2 * b^{-1})$ for some $b \in G$ and for all $u_2 \in G$.

Now,

$$\begin{aligned}
\mu_{P\times Q}((u_1, u_2)) &= \mu_P(u_1) \wedge \mu_Q(u_2) = \mu_R(a * u_1 * a^{-1}) \wedge \mu_S(b * u_2 * b^{-1}) \\
&= \mu_{R\times S}((a, b)(u_1, u_2)(a, b)^{-1}), \\
\eta_{P\times Q}((u_1, u_2)) &= \eta_P(u_1) \wedge \eta_Q(u_2) = \eta_R(a * u_1 * a^{-1}) \wedge \eta_S(b * u_2 * b^{-1}) \\
&= \eta_{R\times S}((a, b)(u_1, u_2)(a, b)^{-1}), \\
v_{P\times Q}((u_1, u_2)) &= v_P(u_1) \vee v_Q(u_2) = v_R(a * u_1 * a^{-1}) \vee v_S(b * u_2 * b^{-1}) \\
&= v_{R\times S}((a, b)(u_1, u_2)(a, b)^{-1}),
\end{aligned}$$

for some $(a, b) \in G \times G$ and for all $(u_1, u_2) \in G \times G$. Therefore, $P \times Q$ is conjugate to $R \times S$. □

The following proposition reflects on (θ, ϕ, ψ)-cut of a PFS. It actually tells about the condition imposed on (θ, ϕ, ψ)-cut of a PFS under which a PFS will be a PFSG.

Proposition 3.11. *Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a PFS in G. Then P is a PFSG of G if all (θ, ϕ, ψ)-cuts of P are crisp subgroups of G.*

Proof. Let $a, b \in G$, with $\theta = \mu_P(a) \wedge \mu_P(b)$, $\phi = \eta_P(a) \wedge \eta_P(b)$ and $\psi = \upsilon_P(a) \vee \upsilon_P(b)$. Then $\theta \in [0,1]$, $\phi \in [0,1]$ and $\psi \in [0,1]$ such that $\theta + \phi + \psi \in [0,1]$ is satisfied. It is observed that

$$\begin{aligned} \mu_P(a) &\geqslant \mu_P(a) \wedge \mu_P(b) = \theta, \\ \eta_P(a) &\geqslant \eta_P(a) \wedge \eta_P(b) = \phi, \\ \upsilon_P(a) &\leqslant \upsilon_P(a) \vee \upsilon_P(b) = \psi. \end{aligned}$$

Also,

$$\begin{aligned} \mu_P(b) &\geqslant \mu_P(a) \wedge \mu_P(b) = \theta, \\ \eta_P(b) &\geqslant \eta_P(a) \wedge \eta_P(b) = \phi, \\ \upsilon_P(b) &\leqslant \upsilon_P(a) \vee \upsilon_P(b) = \psi. \end{aligned}$$

Thus,

$$\begin{aligned} &\mu_P(a) \geqslant \theta, \quad \eta_P(a) \geqslant \phi, \quad \upsilon_P(a) \leqslant \psi, \\ &\mu_P(b) \geqslant \theta, \quad \eta_P(b) \geqslant \phi, \quad \upsilon_P(b) \leqslant \psi. \end{aligned}$$

It follows that $a, b \in C_{\theta,\phi,\psi}(P)$. Since $C_{\theta,\phi,\psi}(P)$ is a crisp subgroup of G, therefore $a * b^{-1} \in C_{\theta,\phi,\psi}(P)$. This yields

$$\begin{aligned} \mu_P(a * b^{-1}) &\geqslant \theta = \mu_P(a) \wedge \mu_P(b), \\ \eta_P(a * b^{-1}) &\geqslant \phi = \eta_P(a) \wedge \eta_P(b), \\ \upsilon_P(a * b^{-1}) &\leqslant \psi = \upsilon_P(a) \vee \upsilon_P(b). \end{aligned}$$

Since a, b are arbitrary elements of G, therefore $\mu_P(a * b^{-1}) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\eta_P(a * b^{-1}) \geqslant \eta_P(a) \wedge \eta_P(b)$ and $\upsilon_P(a * b^{-1}) \leqslant \upsilon_P(a) \vee \upsilon_P(b)$ for all $a, b \in G$. Consequently, P is a PFSG of G. $\square$

Proposition 3.12. *Let G be a group and $P = (\mu_P, \eta_P, \upsilon_P)$ be a PFSG of G. Then the set $S = \{a \in G : \mu_P(a) = \mu_P(e), \eta_P(a) = \eta_P(e), \upsilon_P(a) = \upsilon_P(e)\}$ forms a crisp subgroup of G, where e plays the role of identity in the group G.*

Proof. Let S is non-empty because $e \in S$, where e is the identity in G. Let $a, b \in S$. Then $\mu_P(a) = \mu_P(b) = \mu_P(e)$, $\eta_P(a) = \eta_P(b) = \eta_P(e)$ and $\upsilon_P(a) = \upsilon_P(b) = \upsilon_P(e)$.

Since P be a PFSG of G, therefore

$$\begin{aligned} \mu_P(a * b^{-1}) &\geqslant \mu_P(a) \wedge \mu_P(b) = \mu_P(e) \wedge \mu_P(e) = \mu_P(e), \\ \eta_P(a * b^{-1}) &\geqslant \eta_P(a) \wedge \eta_P(b) = \eta_P(e) \wedge \eta_P(e) = \eta_P(e), \\ \upsilon_P(a * b^{-1}) &\leqslant \upsilon_P(a) \vee \upsilon_P(b) = \upsilon_P(e) \vee \upsilon_P(e) = \upsilon_P(e). \end{aligned}$$

From Proposition 3.1, $\mu_P(e) \geqslant \mu_P(a*b^{-1})$, $\eta_P(e) \geqslant \eta_P(a*b^{-1})$ and $\upsilon_P(e) \leqslant \upsilon_P(a*b^{-1})$. Consequently, $\mu_P(e) = \mu_P(a*b^{-1})$, $\eta_P(e) = \eta_P(a*b^{-1})$ and $\upsilon_P(e) = \upsilon_P(a*b^{-1})$. Thus,

$$a, b \in S \Rightarrow a * b^{-1} \in S.$$

Therefore, S is a crisp subgroup of G. $\square$

The following proposition reflects on (θ, ϕ, ψ)-cut of a PFSG. From the definition of (θ, ϕ, ψ)-cut of a PFS, we have noticed that (θ, ϕ, ψ)-cut of a PFS is a crisp set. From the following proposition, we will know (θ, ϕ, ψ)-cut of a PFSG is a crisp subgroup of the universal group.

Proposition 3.13. *Let $(G, *)$ be a group and $P = (\mu_P, \eta_P, v_P)$ be a PFSG of G. Then $C_{\theta,\phi,\psi}(P)$ is crisp subgroup of G.*

Proof. Let $a, b \in C_{\theta,\phi,\psi}(P)$. Then $\mu_P(a) \geqslant \theta$, $\eta_P(a) \geqslant \phi$, $v_P(a) \leqslant \psi$ and $\mu_P(b) \geqslant \theta$, $\eta_P(b) \geqslant \phi$, $v_P(b) \leqslant \psi$. Since P is a PFSG, therefore

$$\begin{aligned}
\mu_P(a * b^{-1}) &\geqslant \mu_P(a) \wedge \mu_P(b) \geqslant \theta \wedge \theta = \theta, \\
\eta_P(a * b^{-1}) &\geqslant \eta_P(a) \wedge \eta_P(b) \geqslant \phi \wedge \phi = \phi, \\
v_P(a * b^{-1}) &\leqslant v_P(a) \vee v_P(b) \leqslant \psi \vee \psi = \psi.
\end{aligned}$$

Thus,

$$a, b \in C_{\theta,\phi,\psi}(P) \Rightarrow a * b^{-1} \in C_{\theta,\phi,\psi}(P).$$

Consequently, $C_{\theta,\phi,\psi}(P)$ is a crisp subgroup of G. □

The following proposition gives the relationship between the r-th power of an element and the element itself of the universal group in case of a PFSG. The relationship is given in terms of picture fuzzy membership values.

Proposition 3.14. *Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a PFSG of G. Then $\mu_P(a^r) \geqslant \mu_P(a)$, $\eta_P(a^r) \geqslant \eta_P(a)$, $v_P(a^r) \leqslant v_P(a)$ for all $a \in G$ and for all integers r, where $a^r = a * a * \cdots * a$ (r times).*

Proof. Case 1. Let r be a positive integer. Then $r \geqslant 1$. Let us suppose $P(r)$: $\mu_P(a^r) \geqslant \mu_P(a)$, $\eta_P(a^r) \geqslant \eta_P(a)$ and $v_P(a^r) \leqslant v_P(a)$ for all $a \in G$. Here, $P(1)$ is trivially true. Now, since P is a PFSG of G, therefore

$$\begin{aligned}
\mu_P(a^2) &= \mu_P(a * a) \geqslant \mu_P(a) \wedge \mu_P(a) = \mu_P(a), \\
\eta_P(a^2) &= \eta_P(a * a) \geqslant \eta_P(a) \wedge \eta_P(a) = \eta_P(a), \\
v_P(a^2) &= v_P(a * a) \leqslant v_P(a) \vee v_P(a) = v_P(a), \quad \text{for all } a \in G.
\end{aligned}$$

So, $P(2)$ is true. Let us assume that $P(r)$ is true for $r = m$, i.e., $\mu_P(a^m) \geqslant \mu_P(a)$, $\eta_P(a^m) \geqslant \eta_P(a)$ and $v_P(a^m) \leqslant v_P(a)$ for all $a \in G$.

Now,

$$\begin{aligned}
\mu_P(a^{m+1}) &= \mu_P(a^m * a) \geqslant \mu_P(a^m) \wedge \mu_P(a) \geqslant \mu_P(a) \wedge \mu_P(a) = \mu_P(a), \\
\eta_P(a^{m+1}) &= \eta_P(a^m * a) \geqslant \eta_P(a^m) \wedge \eta_P(a) \geqslant \eta_P(a) \wedge \eta_P(a) = \eta_P(a), \\
v_P(a^{m+1}) &= v_P(a^m * a) \leqslant v_P(a^m) \vee v_P(a) \leqslant v_P(a) \vee v_P(a) = v_P(a), \quad \text{for all } a \in G.
\end{aligned}$$

So, $P(r)$ is true for $r = m + 1$. Hence, $P(r)$ is true for all positive integers r.

Case 2. Let r be a negative integer. Then $r \leqslant -1$. Say $t = -r$. Then $t \geqslant 1$. Now, $\mu_P(a^r) = \mu_P(a^{-t}) = \mu_P(a^t)$, $\eta_P(a^r) = \eta_P(a^{-t}) = \eta_P(a^t)$ and $v_P(a^r) = v_P(a^{-t}) =$

$\upsilon_P(a^t)$ for all $a \in G$ [by Proposition 3.1, because a^{-t} is the inverse of a^t in G]. As t is a positive integer therefore the case is similar as Case 1. Thus finally, $\mu_P(a^r) \geqslant \mu_P(a)$, $\eta_P(a^r) \geqslant \eta_P(a)$ and $\upsilon_P(a^r) \leqslant \upsilon_P(a)$ for all $a \in G$.

Case 3. When $r = 0$, then it is trivially true as $\mu_P(e) \geqslant \mu_P(a)$, $\eta_P(e) \geqslant \eta_P(a)$ and $\upsilon_P(e) \leqslant \upsilon_P(a)$ for all $a \in G$, by Proposition 3.1. □

Proposition 3.15. *Let $(G, *)$ be a group and $P = (\mu_P, \eta_P, \upsilon_P)$ be a PFSG of G. Then for $a \in G$ $\mu_P(a * b) = \mu_P(b)$, $\eta_P(a * b) = \eta_P(b)$ and $\upsilon_P(a * b) = \upsilon_P(b)$ for all $b \in G$ if and only if $\mu_P(a) = \mu_P(e)$, $\eta_P(a) = \eta_P(e)$ and $\upsilon_P(a) = \upsilon_P(e)$, where e plays the role of identity in G.*

Proof. Let for $a \in G$, $\mu_P(a * b) = \mu_P(b)$, $\eta_P(a * b) = \eta_P(b)$ and $\upsilon_P(a * b) = \upsilon_P(b)$ for all $b \in G$. When $b = e$, then $\mu_P(a) = \mu_P(e)$, $\eta_P(a) = \eta_P(e)$ and $\upsilon_P(a) = \upsilon_P(e)$.

Conversely, let $\mu_P(a) = \mu_P(e)$, $\eta_P(a) = \eta_P(e)$ and $\upsilon_P(a) = \upsilon_P(e)$. It is observed that

$$\begin{aligned}
\mu_P(a * b) &\geqslant \mu_P(a) \wedge \mu_P(b) \quad \text{[because } P \text{ is a PFSG]}\\
&= \mu_P(e) \wedge \mu_P(b) = \mu_P(b) \quad \text{[by Proposition 3.1]},\\
\eta_P(a * b) &\geqslant \eta_P(a) \wedge \eta_P(b) \quad \text{[because } P \text{ is a PFSG]}\\
&= \eta_P(e) \wedge \eta_P(b) = \eta_P(b) \quad \text{[by Proposition 3.1]},\\
\text{and } \upsilon_P(a * b) &\leqslant \upsilon_P(a) \vee \upsilon_P(b) \quad \text{[because } P \text{ is a PFSG]}\\
&= \upsilon_P(e) \vee \upsilon_P(b) = \upsilon_P(b), \quad \text{for all } b \in G \quad \text{[by Proposition 3.1]}.
\end{aligned}$$

Also,

$$\begin{aligned}
\mu_P(b) &= \mu_P(a^{-1} * a * b) = \mu_P(a^{-1} * (a * b))\\
&\geqslant \mu_P(a^{-1}) \wedge \mu_P(a * b) \geqslant \mu_P(a) \wedge \mu_P(a * b)\\
&= \mu_P(e) \wedge \mu_P(a * b)\\
&= \mu_P(a * b) \quad \text{[by Proposition 3.1]},\\
\eta_P(b) &= \eta_P(a^{-1} * a * b) = \eta_P(a^{-1} * (a * b))\\
&\geqslant \eta_P(a^{-1}) \wedge \eta_P(a * b) \geqslant \eta_P(a) \wedge \eta_P(a * b)\\
&= \eta_P(e) \wedge \eta_P(a * b)\\
&= \eta_P(a * b) \quad \text{[by Proposition 3.1]},\\
\upsilon_P(b) &= \upsilon_P(a^{-1} * a * b) = \upsilon_P(a^{-1} * (a * b))\\
&\leqslant \upsilon_P(a^{-1}) \vee \upsilon_P(a * b) \leqslant \upsilon_P(a) \vee \upsilon_P(a * b)\\
&= \upsilon_P(e) \vee \upsilon_P(a * b)\\
&= \upsilon_P(a * b), \quad \text{for all } b \in G \quad \text{[by Proposition 3.1]}.
\end{aligned}$$

Thus, $\mu_P(a * b) = \mu_P(b)$, $\eta_P(a * b) = \eta_P(b)$ and $\upsilon_P(a * b) = \upsilon_P(b)$ for all $b \in G$. □

Proposition 3.16. *Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, \upsilon_P)$ be a restricted PFSG of G for three chosen non-negative real numbers ε_1, ε_2 and ε_3. Let $\{a_k\}$*

be a sequence of elements in G such that $\lim_{k\to\infty}\mu_P(a_k)=\varepsilon_1$, $\lim_{k\to\infty}\eta_P(a_k)=\varepsilon_2$ and $\lim_{k\to\infty}v_P(a_k)=0$. Then $\mu_P(e)=\varepsilon_1$, $\eta_P(e)=\varepsilon_2$ and $v_P(e)=0$, where e is the identity in G.

Proof. From Proposition 3.1, $\mu_P(e)\geqslant\mu_P(a_k)$, $\eta_P(e)\geqslant\eta_P(a_k)$ and $v_P(e)\leqslant v_P(a_k)$ for all $k\in N$. Therefore, $\mu_P(e)\geqslant\lim_{k\to\infty}\mu_P(a_k)=\varepsilon_1$, $\eta_P(e)\geqslant\lim_{k\to\infty}\eta_P(a_k)=\varepsilon_2$ and $v_P(e)\leqslant\lim_{k\to\infty}v_P(a_k)=0$. Thus, $\mu_P(e)\geqslant\varepsilon_1$, $\eta_P(e)\geqslant\varepsilon_2$ and $v_P(e)\leqslant 0$. Consequently, $\mu_P(e)=\varepsilon_1$, $\eta_P(e)=\varepsilon_2$ and $v_P(e)=0$. ☐

Proposition 3.17. *Let $(G,*)$ be a cyclic group and $P=(\mu_P,\eta_P,v_P)$ be a PFSG of G. Let a be any element in G such that it generates the group G with $a\in C_{\theta,\phi,\psi}(P)$. Then $C_{\theta,\phi,\psi}(P)=G$.*

Proof. Here $G=\langle a\rangle$. Let $a\in C_{\theta,\phi,\psi}(P)$. Then $\mu_P(a)\geqslant\theta$, $\eta_P(a)\geqslant\phi$ and $v_P(a)\leqslant\psi$. Let $t\in G$. Then $t=a^k$ for some integer k. Now,

$$\begin{aligned}\mu_P(t)&=\mu_P(a^k)\\&\geqslant\mu_P(a)\quad\text{[by Proposition 3.14]}\\&\geqslant\theta,\\\eta_P(t)&=\eta_P(a^k)\\&\geqslant\eta_P(a)\quad\text{[by Proposition 3.14]}\\&\geqslant\phi,\\v_P(t)&=v_P(a^k)\\&\leqslant v_P(a)\quad\text{[by Proposition 3.14]}\\&\leqslant\psi.\end{aligned}$$

Thus, $t\in G$ implies $t\in C_{\theta,\phi,\psi}(P)$. Therefore, $G\subseteq C_{\theta,\phi,\psi}(P)$. Already, it is known that $C_{\theta,\phi,\psi}(P)\subseteq G$. Consequently, $G=C_{\theta,\phi,\psi}(P)$. ☐

4. Homomorphism of Picture Fuzzy Subgroups

Here, we study some properties of PFSG under the classical group-homomorphism and anti-group homomorphism.

Definition 4.1. Let $(G_1,*)$ and $(G_2,\circ)$ be two crisp groups. Then a mapping $h: G_1\to G_2$ is said to be a group homomorphism if $h(a*b)=h(a)\circ h(b)$ for all $a,b\in G_1$.

Definition 4.2. Let $(G_1,*)$ and $(G_2,\circ)$ be two crisp groups and $h: G_1\to G_2$ be a surjective group-homomorphism. Then for a PFS $P=(\mu_P,\eta_P,v_P)$, the image of P is the PFS $h(P)=(\mu_{h(P)},\eta_{h(P)},v_{h(P)})$ defined by

$$\mu_{h(P)}(b)=\bigvee_{a\in h^{-1}(b)}\mu_P(a),\quad \eta_{h(P)}(b)=\bigwedge_{a\in h^{-1}(b)}\eta_P(a),\quad v_{h(P)}(b)=\bigwedge_{a\in h^{-1}(b)}v_P(a),$$

for all $b\in G_2$.

Proposition 4.1. *$(G_1,*)$ and $(G_2,\circ)$ be two crisp groups and $h: G_1 \to G_2$ be a bijective group homomorphism. Then for a PFSG P in G_1, $h(P)$ is a PFSG of G_2.*

Proof. It is observed that for $b_1 \in G_2$,

$$\mu_{h(P)}(b_1) = \bigvee_{a_1\in h^{-1}(b_1)} \mu_P(a_1), \quad \eta_{h(P)}(b_1) = \bigwedge_{a_1\in h^{-1}(b_1)} \eta_P(a_1), \quad v_{h(P)}(b_1) = \bigvee_{a_1\in h^{-1}(b_1)} v_P(a_1).$$

Since h is bijective, therefore $h^{-1}(b_1)$ is a singleton set. So, it can be written as $h^{-1}(b_1) = a_1$, i.e., $h(a_1) = b_1$ for unique $a_1 \in G_1$. Therefore, $\mu_{h(P)}(b_1) = \mu_{h(P)}(h(a_1)) = \mu_P(a_1)$, $\eta_{h(P)}(b_1) = \mu_{h(P)}(h(a_1)) = \eta_P(a_1)$ and $v_{h(P)}(b_1) = v_{h(P)}(h(a_1)) = v_P(a_1)$ for unique $a_1 \in G_1$.

Now,

$$\begin{aligned}
\mu_{h(P)}(b_1 \circ b_2^{-1}) &= \mu_{h(P)}(h(a_1) \circ (h(a_2))^{-1}) \\
&\quad [\text{because } b_1 = h(a_1) \text{ and } b_2 = h(a_2) \text{ for unique } a_1 \text{ and } a_2 \in G_1] \\
&= \mu_{h(P)}(h(a_1 * a_2^{-1})) \quad [\text{as } h \text{ is group homomorphism}] \\
&= \mu_P(a_1 * a_2^{-1}) \\
&\geqslant \mu_P(a_1) \wedge \mu_P(a_2) \quad [\text{as } P \text{ is a PFSG}] \\
&= \mu_{h(P)}(h(a_1)) \wedge \mu_{h(P)}(h(a_2)) = \mu_{h(P)}(b_1) \wedge \mu_{h(P)}(b_2), \\
\eta_{h(P)}(b_1 \circ b_2^{-1}) &= \eta_{h(P)}(h(a_1) \circ (h(a_2))^{-1}) \\
&= \eta_{h(P)}(h(a_1 * a_2^{-1})) \quad [\text{as } h \text{ is group homomorphism}] \\
&= \eta_P(a_1 * a_2^{-1}) \\
&\geqslant \eta_P(a_1) \wedge \eta_P(a_2) \quad [\text{as } P \text{ is a PFSG}] \\
&= \eta_{h(P)}(h(a_1)) \wedge \eta_{h(P)}(h(a_2)) = \eta_{h(P)}(b_1) \wedge \eta_{h(P)}(b_2), \\
v_{h(P)}(b_1 \circ b_2^{-1}) &= v_{h(P)}(h(a_1) \circ (h(a_2))^{-1}) \\
&= v_{h(P)}(h(a_1 * a_2^{-1})) \quad [\text{as } h \text{ is group homomorphism}] \\
&= v_P(a_1 * a_2^{-1}) \\
&\leqslant v_P(a_1) \vee v_P(a_2) \quad [\text{as } P \text{ is a PFSG}] \\
&= v_{h(P)}(h(a_1)) \vee v_{h(P)}(h(a_2)) \\
&= v_{h(P)}(b_1) \vee v_{h(P)}(b_2), \quad \text{for all } b_1, b_2 \in G_2.
\end{aligned}$$

Consequently, $h(P)$ is a PFSG of G_2. □

Definition 4.3. Let $(G_1,*)$ and $(G_2,\circ)$ be two crisp groups. Then a mapping $h: G_1 \to G_2$ is said to be an anti group homomorphism if $h(a*b) = h(b) \circ h(a)$ for all $a, b \in G_1$.

Definition 4.4. Let $(G_1,*)$ and $(G_2,\circ)$ be two crisp groups and $Q = (\mu_Q, \eta_Q, v_Q)$ be a PFSG of G_2. Then for a mapping $h: G_1 \to G_2$, $h^{-1}(Q)$ is the PFS $h^{-1}(Q) = (\mu_{h^{-1}(Q)}, \eta_{h^{-1}(Q)}, v_{h^{-1}(Q)})$ defined by $\mu_{h^{-1}(Q)}(a) = \mu_Q(h(a))$, $\eta_{h^{-1}(Q)}(a) = \eta_Q(h(a))$ and $v_{h^{-1}(Q)}(a) = v_Q(h(a))$ for all $a \in G_1$.

Proposition 4.2. *Let $(G_1,*)$ and $(G_2,\circ)$ be two crisp groups and $Q=(\mu_Q,\eta_Q,v_Q)$ be a PFSG of G_2. Then for an anti group-homomorphism h, $h^{-1}(Q)$ is a PFSG of G_1.*

Proof. Let $h^{-1}(Q)$=($\mu_{h^{-1}(Q)}$, $\eta_{h^{-1}(Q)}$, $v_{h^{-1}(Q)}$), where $\mu_{h^{-1}(Q)}(a)=\mu_Q(h(a))$, $\eta_{h^{-1}(Q)}(a)$ $=\eta_Q(h(a))$, $v_{h^{-1}(Q)}(a)=v_Q(h(a))$ for all $a\in G_1$. Now, we have

$$\begin{aligned}
\mu_{h^{-1}(Q)}(a*b^{-1}) &= \mu_Q(h(a*b^{-1}))\\
&= \mu_Q(h(b^{-1})\circ h(a)) \quad \text{[because } h \text{ is an anti group-homomorphism]}\\
&= \mu_Q((h(b))^{-1}\circ h(a))\\
&\geqslant \mu_Q((h(b))^{-1})\wedge\mu_Q(h(a)) \quad \text{[because } Q \text{ is a PFSG of } G_2\text{]}\\
&\geqslant \mu_Q(h(b))\wedge\mu_Q(h(a)) \quad \text{[because } Q \text{ is PFSG of } G_2\text{]}\\
&= \mu_Q(h(a))\wedge\mu_Q(h(b))=\mu_{h^{-1}(Q)}(a)\wedge\mu_{h^{-1}(Q)}(b),\\
\eta_{h^{-1}(Q)}(a*b^{-1}) &= \eta_Q(h(a*b^{-1}))\\
&= \eta_Q(h(b^{-1})\circ h(a)) \quad \text{[because } h \text{ is an anti group-homomorphism]}\\
&= \eta_Q((h(b))^{-1})\circ h(a))\\
&\geqslant \eta_Q((h(b))^{-1})\wedge\eta_Q(h(a)) \quad \text{[because } Q \text{ is a PFSG of } G_2\text{]}\\
&\geqslant \eta_Q(h(b))\wedge\eta_Q(h(a)) \quad \text{[because } Q \text{ is a PFSG of } G_2\text{]}\\
&= \eta_Q(h(a))\wedge\eta_Q(h(b))=\eta_{h^{-1}(Q)}(a)\wedge\eta_{h^{-1}(Q)}(b),\\
v_{h^{-1}(Q)}(a*b^{-1}) &= v_Q(h(a*b^{-1}))\\
&= v_Q(h(b^{-1})\circ h(a)) \quad \text{[because } h \text{ is an anti group-homomorphism]}\\
&= v_Q((h(b))^{-1}\circ h(a))\\
&\leqslant v_Q((h(b))^{-1})\vee v_Q(h(a)) \quad \text{[because } Q \text{ is a PFSG of } G_2\text{]}\\
&\leqslant v_Q(h(b))\vee v_Q(h(a)) \quad \text{[because } Q \text{ is a PFSG of } G_2\text{]}\\
&= v_Q(h(a))\vee v_Q(h(b))=v_{h^{-1}(Q)}(a)\vee v_{h^{-1}(Q)}(b), \quad \text{for all } a,b\in G_1.
\end{aligned}$$

Consequently, $h^{-1}(Q)$ is a PFSG of G_1. □

5. Picture Fuzzy Coset and Picture Fuzzy Normal Subgroup

Here, we define different kinds of picture fuzzy cosets (PFCSs) and picture fuzzy normal subgroup (PFNSG). Also, we investigate some related properties.

Definition 5.1. Let $(G,*)$ be a crisp group and $P=(\mu_P,\eta_P,v_P)$ be a PFSG of G. Then for any $a\in G$ the picture fuzzy left coset of P in G is the PFS $aP=(\mu_{aP},\eta_{aP},v_{aP})$ defined by $\mu_{aP}(u)=\mu_P(a^{-1}*u)$, $\eta_{aP}(u)=\mu_P(a^{-1}*u)$ and $v_{aP}(u)=v_P(a^{-1}*u)$ for all $u\in G$.

Definition 5.2. Let $(G,*)$ be a crisp group and $P=(\mu_P,\eta_P,v_P)$ be a PFSG of G. Then for any $a\in G$ the picture fuzzy right coset of P in G is the PFS $Pa=$

$(\mu_{Pa}, \eta_{Pa}, v_{Pa})$ defined by $\mu_{Pa}(u) = \mu_P(u * a^{-1})$, $\eta_{Pa}(u) = \mu_P(u * a^{-1})$ and $v_{Pa}(u) = v_P(u * a^{-1})$ for all $u \in G$.

Definition 5.3. Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a PFSG of G. Then for any $a \in G$ the picture fuzzy middle coset of P in G is the PFS $aPa^{-1} = (\mu_{aPa^{-1}}, \eta_{aPa^{-1}}, v_{aPa^{-1}})$ defined by $\mu_{aPa^{-1}}(u) = \mu_P(a^{-1} * u * a)$, $\eta_{aPa^{-1}}(u) = \eta_P(a^{-1} * u * a)$ and $v_{aPa^{-1}}(u) = v_P(a^{-1} * u * a)$ for all $u \in G$.

In classical sense, any subgroup of a classical group is said to be normal if left coset and right coset of the subgroup for any element of the classical group are equal. In picture fuzzy sense, a PFSG is said to be PFNSG if picture fuzzy membership values of left coset and right coset of PFSG for any element of the universal group are equal.

Definition 5.4. Let $(g, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a PFSG of G. Then P is called a PFNSG of G if $\mu_{Pa}(u) = \mu_{aP}(u)$, $\eta_{Pa}(u) = \eta_{aP}(u)$, $v_{Pa}(u) = v_{aP}(u)$ for all $a, u \in G$.

Proposition 5.1. *Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a PFSG of G. Then P is a PFNSG of G if and only if $\mu_P(a * b) = \mu_P(b * a)$, $\eta_P(a * b) = \eta_P(b * a)$ and $v_P(a * b) = v_P(b * a)$ for all $a, b \in G$.*

Proof. Let $P = (\mu_P, \eta_P, v_P)$ be a PFNSG of G. Therefore, $\mu_{Pa}(u) = \mu_{aP}(u)$, $\eta_{Pa}(u) = \eta_{aP}(u)$ and $v_{Pa}(u) = v_{aP}(u)$ for all $a, u \in G$, i.e., $\mu_P(u * a^{-1}) = \mu_P(a^{-1} * u)$, $\eta_P(u * a^{-1}) = \eta_P(a^{-1} * u)$ and $v_P(u * a^{-1}) = v_P(a^{-1} * u)$ for all $a, u \in G$.

Now, $\mu_P(a * b) = \mu_P(a * (b^{-1})^{-1}) = \mu_P((b^{-1})^{-1} * a) = \mu_P(b * a)$, $\eta_P(a * b) = \eta_P(a * (b^{-1})^{-1}) = \eta_P((b^{-1})^{-1} * a) = \eta_P(b * a)$ and $v_P(a * b) = v_P(a * (b^{-1})^{-1}) = v_P((b^{-1})^{-1} * a) = v_P(b * a)$ for all $a, b \in G$.

Conversely, let $\mu_P(a*b) = \mu_P(b*a)$, $\eta_P(a*b) = \eta_P(b*a)$ and $v_P(a*b) = v_P(b*a)$ for all $a, b \in G$, i.e., $\mu_P(a * (b^{-1})^{-1}) = \mu_P((b^{-1})^{-1} * a)$, $\eta_P(a * (b^{-1})^{-1}) = \eta_P((b^{-1})^{-1} * a)$ and $v_P(a * (b^{-1})^{-1}) = v_P((b^{-1})^{-1} * a)$ for all $a, b \in G$. Letting $z = b^{-1}$ we get $\mu_P(a * z^{-1}) = \mu_P(z^{-1} * a)$, $\eta_P(a * z^{-1}) = \eta_P(z^{-1} * a)$ and $v_P(a * z^{-1}) = v_P(z^{-1} * a)$ for all $a, z \in G$. It follows that $\mu_{Pz}(a) = \mu_{zP}(a)$, $\eta_{Pz}(a) = \eta_{zP}(a)$ and $v_{Pz}(a) = v_{zP}(a)$ for all $a, z \in G$. Consequently, P is a PFNSG of G. $\square$

Proposition 5.2. *Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a PFSG of G. Then P is a PFNSG of G if and only if $\mu_P(a*u*a^{-1}) = \mu_P(u)$, $\eta_P(a*u*a^{-1}) = \eta_P(u)$ and $v_P(a * u * a^{-1}) = v_P(u)$ for all $a, u \in G$.*

Proof. Let P be a PFNSG of G. Then

$$\begin{aligned}
\mu_P(a * u * a^{-1}) &= \mu_P((a * u) * a^{-1}) \\
&= \mu_P(a^{-1} * (a * u)) \quad \text{[by Proposition 5.1, as } P \text{ is a PFNSG of } G] \\
&= \mu_P((a^{-1} * a) * u) = \mu_P(u), \\
\eta_P(a * u * a^{-1}) &= \eta_P((a * u) * a^{-1}) \\
&= \eta_P(a^{-1} * (a * u)) \quad \text{[using Proposition 5.1, as } P \text{ is a PFNSG of } G]
\end{aligned}$$

$$\begin{aligned}
&= \eta_P((a^{-1} * a) * u) = \eta_P(u),\\
v_P(a * u * a^{-1}) &= v_P((a * u) * a^{-1})\\
&= v_P(a^{-1} * (a * u)) \quad \text{[using Proposition 5.1, as } P \text{ is a PFNSG of G]}\\
&= v_P((a^{-1} * a) * u) = v_P(u), \quad \text{for all } a, u \in G.
\end{aligned}$$

Conversely, let the conditions be hold. Then $\mu_P(a*b) = \mu_P(b^{-1}*(b*a)*b) = \mu_P(b^{-1}*(b*a)*(b^{-1})^{-1}) = \mu_P(b*a)$, $\eta_P(a*b) = \eta_P(b^{-1}*(b*a)*b) = \eta_P(b^{-1}*(b*a)*(b^{-1})^{-1}) = \eta_P(b*a)$ and $v_P(a*b) = v_P(b^{-1}*(b*a)*b) = v_P(b^{-1}*(b*a)*(b^{-1})^{-1}) = v_P(b*a)$ for all $a, b \in G$. Therefore, by Proposition 5.1, P is a PFNSG of G. □

Proposition 5.3. *Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a PFNSG of G. Then $S = \{u \in G : \mu_P(u) = \mu_P(e), \eta_P(u) = \eta_P(e), v_P(u) = v_P(e)\}$ is a crisp normal subgroup of G.*

Proof. By Proposition 3.12, S is a crisp subgroup of G. Let $a \in G$ and $u \in S$. Then $\mu_P(u) = \mu_P(e)$, $\eta_P(u) = \eta_P(e)$, $v_P(u) = v_P(e)$. Since P is a PFNSG of G, therefore, by Proposition 5.2, $\mu_P(a*u*a^{-1}) = \mu_P(u)$, $\eta_P(a*u*a^{-1}) = \eta_P(u)$ and $v_P(a*u*a^{-1}) = v_P(u)$. It follows that $\mu_P(a*u*a^{-1}) = \mu_P(e)$, $\eta_P(a*u*a^{-1}) = \eta_P(e)$ and $v_P(a*u*a^{-1}) = v_P(e)$. Thus, $a*u*a^{-1} \in S$. Hence, S is a crisp normal subgroup of G. □

Proposition 5.4. *Let $(G, *)$ and $P = (\mu_P, \eta_P, v_P)$ be a PFNSG of G. Then for any $a \in G$, aPa^{-1} is a PFNSG of G.*

Proof. Let $aPa^{-1}=(\mu_{aPa^{-1}}, \eta_{aPa^{-1}}, v_{aPa^{-1}})$, where $\mu_{aPa^{-1}}(u) = \mu_P(a^{-1}*u*a)$, $\eta_{aPa^{-1}}(u) = \eta_P(a^{-1}*u*a)$ and $v_{aPa^{-1}}(u) = v_P(a^{-1}*u*a)$ for all $u \in G$. Now,

$$\begin{aligned}
\mu_{aPa^{-1}}(u_1 * u_2) &= \mu_P(a^{-1} * (u_1 * u_2) * a)\\
&= \mu_P(a^{-1} * (u_1 * u_2 * a))\\
&= \mu_P((u_1 * u_2 * a) * a^{-1})\\
&\quad \text{[by Proposition 5.1, as } P \text{ is a PFNSG of } G]\\
&= \mu_P((u_1 * u_2) * (a * a^{-1}))\\
&= \mu_P(u_1 * u_2) = \mu_P(u_2 * u_1)\\
&\quad \text{[by Proposition 5.1, as } P \text{ is a PFNSG of } G],\\
\eta_{aPa^{-1}}(u_1 * u_2) &= \eta_P(a^{-1} * (u_1 * u_2) * a)\\
&= \eta_P(a^{-1} * (u_1 * u_2 * a))\\
&= \eta_P((u_1 * u_2 * a) * a^{-1})\\
&\quad \text{[by Proposition 5.1, as } P \text{ is a PFNSG of } G]\\
&= \eta_P((u_1 * u_2) * (a * a^{-1}))\\
&= \eta_P(u_1 * u_2) = \eta_P(u_2 * u_1)\\
&\quad \text{[by Proposition 5.1, as } P \text{ is a PFNSG of } G],
\end{aligned}$$

$$
\begin{aligned}
v_{aPa^{-1}}(u_1 * u_2) =& v_P(a^{-1} * (u_1 * u_2) * a)\\
=& v_P(a^{-1} * (u_1 * u_2 * a))\\
=& v_P((u_1 * u_2 * a) * a^{-1})\\
& \text{[by Proposition 5.1, as } P \text{ is PFNSG of } G]\\
=& v_P((u_1 * u_2) * (a * a^{-1}))\\
=& v_P(u_1 * u_2) = v_P(u_2 * u_1)\\
& \text{[by Proposition 5.1, as } P \text{ is a PFNSG of } G],
\end{aligned}
$$

for all $u_1, u_2 \in G$. Also,

$$
\begin{aligned}
\mu_{aPa^{-1}}(u_2 * u_1) =& \mu_P(a^{-1} * (u_2 * u_1) * a)\\
=& \mu_P(a^{-1} * (u_2 * u_1 * a))\\
=& \mu_P((u_2 * u_1 * a) * a^{-1})\\
& \text{[by Proposition 5.1, as } P \text{ is a PFNSG of } G]\\
=& \mu_P((u_2 * u_1) * (a * a^{-1})) = \mu_P(u_2 * u_1),\\
\eta_{aPa^{-1}}(u_2 * u_1) =& \eta_P(a^{-1} * (u_2 * u_1) * a)\\
=& \eta_P(a^{-1} * (u_2 * u_1 * a))\\
=& \eta_P((u_2 * u_1 * a) * a^{-1})\\
& \text{[by Proposition 5.1, as } P \text{ is a PFNSG of } G]\\
=& \eta_P((u_2 * u_1) * (a * a^{-1})) = \eta_P(u_2 * u_1),\\
v_{aPa^{-1}}(u_2 * u_1) =& v_P(a^{-1} * (u_2 * u_1) * a)\\
=& v_P(a^{-1} * (u_2 * u_1 * a))\\
=& v_P((u_2 * u_1 * a) * a^{-1})\\
& \text{[by Proposition 5.1, as } P \text{ is PFNSG of } G]\\
=& v_P((u_2 * u_1) * (a * a^{-1})) = v_P(u_2 * u_1), \quad \text{for all } u_1, u_2 \in G.
\end{aligned}
$$

Thus, it is obtained that $\mu_{aPa^{-1}}(u_1 * u_2) = \mu_{aPa^{-1}}(u_2 * u_1)$, $\eta_{aPa^{-1}}(u_1 * u_2) = \eta_{aPa^{-1}}(u_2 * u_1)$ and $v_{aPa^{-1}}(u_1 * u_2) = v_{aPa^{-1}}(u_2 * u_1)$ for all $u_1, u_2 \in G$. By Proposition 5.1, aPa^{-1} is a PFNSG of G. $\square$

6. Order of Picture Fuzzy Subgroup

Here, we define the order of a PFSG with the help of the cardinality of a special type of crisp subgroup. Also, we explore some results that correspond to the order of PFSG.

Definition 6.1. Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a PFSG of G. Then the order of the PFSG P is denoted by $O(P)$ and is defined as the cardinality of the crisp set $H_P = \{u \in G : \mu_P(u) = \mu_P(e),\ \eta_P(u) = \eta_P(e),\ v_P(u) = v_P(e)\}$, where e plays the role of identity in G.

Proposition 6.1. *Let $(G, *)$ be a crisp group and $P = (\mu_P, \eta_P, v_P)$ be a PFNSG of G. Then $O(P) = O(aPa^{-1})$ for any $a \in G$.*

Proof. From Definition 6.1, $O(P) = |H_P|$ and $O(aPa^{-1}) = |H_{aPa^{-1}}|$, where $H_P = \{u \in G : \mu_P(u) = \mu_P(e), \eta_P(u) = \eta_P(e), v_P(u) = v_P(e)\}$ and $H_{aPa^{-1}} = \{u \in G : \mu_{aPa^{-1}}(u) = \mu_{aPa^{-1}}(e), \eta_{aPa^{-1}}(u) = \eta_{aPa^{-1}}(e), v_{aPa^{-1}}(u) = v_{aPa^{-1}}(e)\}$. Now,

$$\begin{aligned}
\mu_{aPa^{-1}}(q) = \mu_{aPa^{-1}}(e) &\Leftrightarrow \mu_P(a^{-1} * q * a) = \mu_P(a^{-1} * e * a) \\
&\Leftrightarrow \mu_P((a^{-1} * q) * a) = \mu_P(e) \\
&\Leftrightarrow \mu_P(a * (a^{-1} * q)) = \mu_P(e) \\
&\qquad \text{[by Proposition 5.1, because } P \text{ is a PFNSG of } G\text{]} \\
&\Leftrightarrow \mu_P((a * a^{-1}) * q) = \mu_P(e) \\
&\Leftrightarrow \mu_P(q) = \mu_P(e), \\
\eta_{aPa^{-1}}(q) = \eta_{aPa^{-1}}(e) &\Leftrightarrow \eta_P(a^{-1} * q * a) = \eta_P(a^{-1} * e * a) \\
&\Leftrightarrow \eta_P((a^{-1} * q) * a) = \eta_P(e) \\
&\Leftrightarrow \eta_P(a * (a^{-1} * q)) = \eta_P(e) \\
&\qquad \text{[by Proposition 5.1, because } P \text{ is a PFNSG of } G\text{]} \\
&\Leftrightarrow \eta_P((a * a^{-1}) * q) = \eta_P(e) \\
&\Leftrightarrow \eta_P(q) = \eta_P(e), \\
v_{aPa^{-1}}(q) = v_{aPa^{-1}}(e) &\Leftrightarrow v_P(a^{-1} * q * a) = v_P(a^{-1} * e * a) \\
&\Leftrightarrow v_P((a^{-1} * q) * a) = v_P(e) \\
&\Leftrightarrow v_P(a * (a^{-1} * q)) = v_P(e) \\
&\qquad \text{[by Proposition 5.1, because } P \text{ is PFNSG of } G\text{]} \\
&\Leftrightarrow v_P((a * a^{-1}) * q) = v_P(e) \\
&\Leftrightarrow v_P(q) = v_P(e), \quad \text{for all } q \in G.
\end{aligned}$$

Thus, if $r \in H_{aPa^{-1}}$ then $r \in H_P$ and if $s \in H_P$ then $s \in H_{aPa^{-1}}$. So, $H_{aPa^{-1}} \subseteq H_P$ and $H_P \subseteq H_{aPa^{-1}}$. Consequently, $H_P{=}H_{aPa^{-1}}$ which indicates that H_P and $H_{aPa^{-1}}$ have the same cardinality, i.e., $O(P) = O(aPa^{-1})$. ☐

Proposition 6.2. *Let $(G, *)$ be a crisp abelian group and $P = (\mu_P, \eta_P, v_P)$ be a PFSG which is conjugate to $Q = (\mu_Q, \eta_Q, v_Q)$. Then P and Q have the same order.*

Proof. From Definition 6.1, it is known that $O(P) = |H_P|$ and $O(Q) = |H_Q|$, where $H_P = \{u \in G : \mu_P(u) = \mu_P(e), \eta_P(u) = \eta_P(e), v_P(u) = v_P(e)\}$ and $H_Q = \{u \in G : \mu_Q(u) = \mu_Q(e), \eta_Q(u) = \eta_Q(e), v_Q(u) = v_Q(e)\}$, where e is the identity in G. Since P is conjugate to Q, therefore $\mu_P(u) = \mu_Q(a * u * a^{-1})$, $\eta_P(u) = \eta_Q(a * u * a^{-1})$, $v_P(u) = v_Q(a * u * a^{-1})$ for some $a \in G$ and for all $u \in G$.

Now, it is observed that

$$\begin{aligned} a * u * a^{-1} &= (a * u) * a^{-1} \\ &= a^{-1} * (a * u) \quad \text{[because } G \text{ is abelian]} \\ &= (a^{-1} * a) * u = u, \quad \text{for all } a, u \in G. \end{aligned}$$

It follows that

$$\begin{aligned} &\mu_P(u) = \mu_Q(a * u * a^{-1}) = \mu_Q(u), \\ &\eta_P(u) = \eta_Q(a * u * a^{-1}) = \eta_Q(u), \\ &v_P(u) = v_Q(a * u * a^{-1}) = v_Q(u), \quad \text{for all } u \in G. \end{aligned}$$

Thus, H_P=$\{u \in G : \mu_P(u) = \mu_P(e), \eta_P(u) = \eta_P(e), v_P(u) = v_P(e)\}$=$\{u \in G : \mu_Q(u) = \mu_Q(e), \eta_Q(u) = \eta_Q(e), v_Q(u) = v_Q(e)\}$=$H_Q$. Therefore, H_P and H_Q have the same cardinality. Hence, P and Q have the same order. □

Theorem 6.1 (Lagrange's theorem on PFSG)**.** *Let* $(G, *)$ *be a crisp group and* $P = (\mu_P, \eta_P, v_P)$ *be a PFSG of* G*. Then* $O(P)$ *is a divisor of* $O(G)$*.*

Proof. From Definition 6.1, it is known that $O(P) = |H_P|$, where $H_P = \{u \in G : \mu_P(u) = \mu_P(e), \eta_P(a) = \eta_P(e), v_P(a) = v_P(e)\}$, e plays the role of identity in G. Now, by Proposition 3.12, it is known that H_P is a crisp subgroup of G. By Lagrange's theorem on crisp group, $|H_P|$ is a divisor of $O(G)$, i.e., $O(P)$ is a divisor of $O(G)$. □

7. Conclusion

Investigation of the structure of algebraic system leads a significant in the field of Mathematics, Computer Science and other different areas. Here we have studied the theory of subgroup in the context of picture fuzzy set. In this paper, notion of PFSG has been established and different properties of PFSG have been investigated. Also, different notions related to PFSG such as PFCS, PFNSG, the order of PFSG have been brought into the light of our study. We expect that this paper will be fruitful to the researchers for further study of the theory of subgroup under some other types of set environment.

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