Kragujevac Journal of Mathematics
Volume 51(5) (2027), Pages 863–873.

ON HYBRID NUMBERS WITH GAUSSIAN GENERALIZED THIRD-ORDER JACOBSTHAL COEFFICIENTS

GAMALIEL MORALES

Abstract. In this paper, we consider hybrid numbers with Gaussian generalized third-order Jacobsthal coefficients and investigate their interesting properties such as the generating function, Binet formula, Cassini and d'Ocagne identities. Moreover, we illustrate the results with some examples.

1. Introduction

In 2018, Özdemir introduced the hybrid numbers and gave some properties of these numbers (see [9]). The set of hybrid numbers is

$$\mathbb{R}[\mathbf{i}, \boldsymbol{\varepsilon}, \mathbf{h}] = \{\phi_0 + \phi_1\mathbf{i} + \phi_2\boldsymbol{\varepsilon} + \phi_3\mathbf{h} : \phi_r \in \mathbb{R}, r = 0, 1, 2, 3\}.$$

The hybrid product is obtained by distributing the terms to the right, preserving the order of multiplication of the units and then writing the values of the following substituting each product of units by the equalities $\mathbf{i}^2 = -1$, $\boldsymbol{\varepsilon}^2 = 0$, $\mathbf{h}^2 = 1$ and $\mathbf{ih} = -\mathbf{hi} = \boldsymbol{\varepsilon} + \mathbf{i}$. Table 1 shows us that the multiplication operation with the hybrid numbers is not commutative.

In the literature, many researchers investigate some remarkable properties of some well-known sequences using this system. Horadam hybrid sequence is a generalization of some sequences such as Fibonacci, Pell, Jacobsthal and Balancing hybrid sequences. These sequences and their generalizations have applications in number theory, geometry and algebra. Hence, these sequences have been studied by Bród [1], Kilic [2],

Key words and phrases. Binet formula, Cassini identity, d'Ocagne identity, generating function, hybrid number.
2020 *Mathematics Subject Classification.* 05A15; 11B39; 11R52.
DOI
Received: July 01, 2025.
Accepted: January 12, 2026.

Morales [3–5], Şentürk et al. [12], Szynal-Liana et al. [15–17], Tan and Ait-Amrane [18], and many others.

Table 1. The multiplication table for basis of $\mathbb{R}[\mathbf{i},\boldsymbol{\varepsilon},\mathbf{h}]$.

$\times$	1	$\mathbf{i}$	$\boldsymbol{\varepsilon}$	$\mathbf{h}$
1	1	$\mathbf{i}$	$\boldsymbol{\varepsilon}$	$\mathbf{h}$
$\mathbf{i}$	$\mathbf{i}$	-1	$1-\mathbf{h}$	$\boldsymbol{\varepsilon}+\mathbf{i}$
$\boldsymbol{\varepsilon}$	$\boldsymbol{\varepsilon}$	$1+\mathbf{h}$	0	$-\boldsymbol{\varepsilon}$
$\mathbf{h}$	$\mathbf{h}$	$-(\boldsymbol{\varepsilon}+\mathbf{i})$	$\boldsymbol{\varepsilon}$	1

In this paper, the Gaussian generalized third-order Jacobsthal sequence, denoted by $gH_n^{(3)}$, is defined by the recurrence relation

$$gH_n^{(3)} = gH_{n-1}^{(3)} + gH_{n-2}^{(3)} + 2gH_{n-3}^{(3)}, \quad n \geq 4,$$

with the initial conditions $gH_1^{(3)} = b + a\mathbf{i}$, $gH_2^{(3)} = c + b\mathbf{i}$ and $gH_3^{(3)} = (2a+b+c) + c\mathbf{i}$, where a, b and c are integers not all zero.

For $n \geq 1$, note that the recurrence relation of Gaussian generalized third-order Jacobsthal sequence can be rewritten as follows:

$$gH_n^{(3)} = H_n^{(3)} + H_{n-1}^{(3)}\mathbf{i}, \quad \mathbf{i}^2 = -1,$$

where $H_n^{(3)}$ is the nth generalized third-order Jacobsthal numbers with initial conditions $H_0^{(3)} = a$, $H_1^{(3)} = b$ and $H_2^{(3)} = c$. Note that the definition is reduced to the certain special cases depending on the choice of the parameters a, b and c as shown in Table 2.

Table 2. Special numbers associated with choices of a, b and c.

Number	a	b	c	Symbol
Gaussian third-order Jacobsthal	0	1	1	$JG_n^{(3)} = J_n^{(3)} + J_{n-1}^{(3)}\mathbf{i}$
Gaussian third-order Jacobsthal-Lucas	2	1	5	$jG_n^{(3)} = j_n^{(3)} + j_{n-1}^{(3)}\mathbf{i}$
Gaussian modified third-order Jacobsthal	3	1	3	$KG_n^{(3)} = K_n^{(3)} + K_{n-1}^{(3)}\mathbf{i}$

Recently, Morales studied some properties of Gaussian third-order Jacobsthal numbers and their generalizations in [6, 7]. Furthermore, in [13], Soykan et al. defined Gaussian generalized Tribonacci numbers and investigated Gaussian Tribonacci and Gaussian Tribonacci-Lucas numbers with their properties. In [19], Taşci investigated Gaussian numbers with Leonardo coefficients.

In this paper, we consider the hybrid numbers with Gaussian generalized third-order Jacobsthal coefficients. Then we get some characteristic relations of them.

2. Main Results

In this section, we give the definition of hybrid numbers with Gaussian generalized third-order Jacobsthal coefficients, denoted by $g\mathbb{H}_n^{(3)}$.

Definition 2.1. Let us define the hybrid numbers with Gaussian generalized third-order Jacobsthal coefficients as below:

$$g\mathbb{H}_n^{(3)} = gH_n^{(3)} + gH_{n+1}^{(3)}\mathbf{i} + gH_{n+2}^{(3)}\boldsymbol{\varepsilon} + gH_{n+3}^{(3)}\mathbf{h}, \quad n \geq 1,$$

where $gH_n^{(3)}$ denotes the nth Gaussian generalized third-order Jacobsthal.

By using the definition of the $g\mathbb{H}_n^{(3)}$, we can write:

$$\begin{aligned} g\mathbb{H}_n^{(3)} =& gH_n^{(3)} + gH_{n+1}^{(3)}\mathbf{i} + gH_{n+2}^{(3)}\boldsymbol{\varepsilon} + gH_{n+3}^{(3)}\mathbf{h} \\ =& gH_{n-1}^{(3)} + gH_{n-2}^{(3)} + 2gH_{n-3}^{(3)} + \left(gH_n^{(3)} + gH_{n-1}^{(3)} + 2gH_{n-2}^{(3)}\right)\mathbf{i} \\ &+ \left(gH_{n+1}^{(3)} + gH_n^{(3)} + 2gH_{n-1}^{(3)}\right)\boldsymbol{\varepsilon} + \left(gH_{n+2}^{(3)} + gH_{n+1}^{(3)} + 2gH_n^{(3)}\right)\mathbf{h}. \end{aligned}$$

In other words, for $n \geq 4$, the hybrid numbers with Gaussian generalized third-order Jacobsthal coefficients can be rewritten by following recurrence

$$g\mathbb{H}_n^{(3)} = g\mathbb{H}_{n-1}^{(3)} + g\mathbb{H}_{n-2}^{(3)} + 2g\mathbb{H}_{n-3}^{(3)}, \tag{2.1}$$

with initial conditions

$$\begin{aligned} g\mathbb{H}_1^{(3)} &= c + (3a + b + 2c)\mathbf{i} + (4a + 2b + 2c)\boldsymbol{\varepsilon} + (2a + 3b + c)\mathbf{h}, \\ g\mathbb{H}_2^{(3)} &= 2a + b + c + (4a + 5b + 3c)\mathbf{i} + (4a + 6b + 4c)\boldsymbol{\varepsilon} + (2a + 3b + 4c)\mathbf{h}, \\ g\mathbb{H}_3^{(3)} &= 2a + 3b + 2c + (6a + 7b + 8c)\mathbf{i} + (8a + 8b + 10c)\boldsymbol{\varepsilon} + (8a + 6b + 7c)\mathbf{h}. \end{aligned}$$

Using Table 2, for special values of $g\mathbb{H}_n^{(3)}$ we obtain the definitions of

(a) nth Gaussian third-order Jacobsthal hybrid number $J\mathbb{H}_n^{(3)}$

$$J\mathbb{H}_n^{(3)} = JG_n^{(3)} + JG_{n+1}^{(3)}\mathbf{i} + JG_{n+2}^{(3)}\boldsymbol{\varepsilon} + JG_{n+3}^{(3)}\mathbf{h},$$

(b) nth Gaussian third-order Jacobsthal-Lucas hybrid number $j\mathbb{H}_n^{(3)}$

$$j\mathbb{H}_n^{(3)} = jG_n^{(3)} + jG_{n+1}^{(3)}\mathbf{i} + jG_{n+2}^{(3)}\boldsymbol{\varepsilon} + jG_{n+3}^{(3)}\mathbf{h},$$

(c) nth Gaussian modified third-order Jacobsthal hybrid number $K\mathbb{H}_n^{(3)}$

$$K\mathbb{H}_n^{(3)} = KG_n^{(3)} + KG_{n+1}^{(3)}\mathbf{i} + KG_{n+2}^{(3)}\boldsymbol{\varepsilon} + KG_{n+3}^{(3)}\mathbf{h}.$$

Before giving the main results, let us introduce generalized third-order Jacobsthal and Gaussian generalized third-order Jacobsthal numbers are of the form

$$H_n^{(3)} = \frac{1}{7}\left[\lambda 2^n + \lambda_1 X_n + \lambda_2 X_{n+1}\right], \tag{2.2}$$

$$gH_n^{(3)} = \frac{1}{7}\left[\lambda 2^{n-1}(2+\mathbf{i}) + (\lambda_1 + (\lambda_2 - \lambda_1)\mathbf{i})X_n + (\lambda_2 - \lambda_1 \mathbf{i})X_{n+1}\right], \tag{2.3}$$

$$X_n = \frac{\omega_1^n - \omega_2^n}{\omega_1 - \omega_2} = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 3, \\ 1, & \text{if } n \equiv 1 \pmod 3, \\ -1, & \text{if } n \equiv 2 \pmod 3, \end{cases} \tag{2.4}$$

where $\lambda = a+b+c$, $\lambda_1 = 4a+4b-3c$, $\lambda_2 = 6a-b-c$ and $\omega_1 = \omega_2^2 = \frac{-1+\mathbf{i}\sqrt{3}}{2}$. Using (2.4), it is easy to see that $gH_n^{(3)}$ satisfies the following relation

$$gH_{n+3}^{(3)} = gH_n^{(3)} + \lambda 2^{n-1}(2+\mathbf{i}), \quad n \geq 1.$$

Theorem 2.1. *For any integer $n \geq 1$, we have*

$$g\mathbb{H}_{n+3}^{(3)} = g\mathbb{H}_n^{(3)} + \lambda 2^{n-1}\Theta,$$

where $\Theta = 4 + 13\boldsymbol{i} + 16\boldsymbol{\varepsilon} + 12\boldsymbol{h}$.

Proof. Considering Definition 2.1, (2.2) and (2.3), we get

$$\begin{aligned} g\mathbb{H}_n^{(3)} =& gH_n^{(3)} + gH_{n+1}^{(3)}\mathbf{i} + gH_{n+2}^{(3)}\boldsymbol{\varepsilon} + gH_{n+3}^{(3)}\mathbf{h} \\ =& \left(gH_{n+3}^{(3)} - \lambda 2^{n-1}(2+\mathbf{i})\right) + \left(gH_{n+4}^{(3)} - \lambda 2^{n}(2+\mathbf{i})\right)\mathbf{i} \\ &+ \left(gH_{n+5}^{(3)} - \lambda 2^{n+1}(2+\mathbf{i})\right)\boldsymbol{\varepsilon} + \left(gH_{n+6}^{(3)} - \lambda 2^{n+2}(2+\mathbf{i})\right)\mathbf{h} \\ =& gH_{n+3}^{(3)} + gH_{n+4}^{(3)}\mathbf{i} + gH_{n+5}^{(3)}\boldsymbol{\varepsilon} + gH_{n+6}^{(3)}\mathbf{h} - \lambda 2^{n-1}(2+\mathbf{i})(1+2\mathbf{i}+4\boldsymbol{\varepsilon}+8\mathbf{h}) \\ =& g\mathbb{H}_{n+3}^{(3)} - \lambda 2^{n-1}\Theta. \end{aligned}$$

The last equation gives the result. □

A Binet-like formula for the Gaussian generalized third-order Jacobsthal hybrid number is given in the following theorem.

Theorem 2.2 (Binet-like formula)**.** *For any integer $n \geq 1$, the nth Gaussian generalized third-order Jacobsthal hybrid number is*

$$\begin{aligned} g\mathbb{H}_n^{(3)} =& \frac{1}{7}\Big(\lambda 2^{n-1}\Theta + \left[-\lambda_2 + (\lambda_2 - 2\lambda_1)\boldsymbol{i} + 2(\lambda_2 - \lambda_1)\boldsymbol{\varepsilon} + (\lambda_1 + \lambda_2)\boldsymbol{h}\right] X_n \\ &+ \left[\lambda_1 - \lambda_2 - (\lambda_1 + \lambda_2)\boldsymbol{i} - 2\lambda_1\boldsymbol{\varepsilon} + (2\lambda_2 - \lambda_1)\boldsymbol{h}\right] X_{n+1}, \Big) \end{aligned}$$

where $X_n = \frac{\omega_1^n - \omega_2^n}{\omega_1 - \omega_2}$ and $\Theta = 4 + 13\boldsymbol{i} + 16\boldsymbol{\varepsilon} + 12\boldsymbol{h}$.

Proof. Using (2.3), (2.4) and Definition 2.1, we can write

$$\begin{aligned}7g\mathbb{H}_n^{(3)} =& 7gH_n^{(3)} + 7gH_{n+1}^{(3)}\mathbf{i} + 7gH_{n+2}^{(3)}\boldsymbol{\varepsilon} + 7gH_{n+3}^{(3)}\mathbf{h}\\ =& \lambda 2^{n-1}(2+\mathbf{i}) + (\lambda_1 + (\lambda_2-\lambda_1)\mathbf{i})X_n + (\lambda_2 - \lambda_1\mathbf{i})X_{n+1}\\ &+ \left[\lambda 2^n(2+\mathbf{i}) + (-\lambda_2 + \lambda_1\mathbf{i})X_n + ((\lambda_1-\lambda_2)+\lambda_2\mathbf{i})X_{n+1}\right]\mathbf{i}\\ &+ \left[\lambda 2^{n+1}(2+\mathbf{i}) + ((-\lambda_1+\lambda_2)-\lambda_2\mathbf{i})X_n + (-\lambda_1 + (\lambda_1-\lambda_2)\mathbf{i})X_n\right]\boldsymbol{\varepsilon}\\ &+ \left[\lambda 2^{n+2}(2+\mathbf{i}) + (\lambda_1 + (\lambda_2-\lambda_1)\mathbf{i})X_n + (\lambda_2 - \lambda_1\mathbf{i})X_{n+1}\right]\mathbf{h}\\ =& \lambda 2^{n-1}(2+\mathbf{i})\,(1 + 2\mathbf{i} + 4\boldsymbol{\varepsilon} + 8\mathbf{h})\\ &+ \left[-\lambda_2 + (\lambda_2 - 2\lambda_1)\mathbf{i} + 2(\lambda_2-\lambda_1)\boldsymbol{\varepsilon} + (\lambda_1+\lambda_2)\mathbf{h}\right]X_n\\ &+ \left[\lambda_1 - \lambda_2 - (\lambda_1+\lambda_2)\mathbf{i} - 2\lambda_1\boldsymbol{\varepsilon} + (2\lambda_2-\lambda_1)\mathbf{h}\right]X_{n+1}.\end{aligned}$$

where $\lambda = a+b+c$, $\lambda_1 = 4a+4b-3c$ and $\lambda_2 = 6a-b-c$. Using $\Theta = 4+13\mathbf{i}+16\boldsymbol{\varepsilon}+12\mathbf{h}$, we have

$$\begin{aligned}g\mathbb{H}_n^{(3)} =& \frac{1}{7}\Big(\lambda 2^{n-1}\Theta + \left[-\lambda_2 + (\lambda_2 - 2\lambda_1)\mathbf{i} + 2(\lambda_2-\lambda_1)\boldsymbol{\varepsilon} + (\lambda_1+\lambda_2)\mathbf{h}\right]X_n\\ &+ \left[\lambda_1 - \lambda_2 - (\lambda_1+\lambda_2)\mathbf{i} - 2\lambda_1\boldsymbol{\varepsilon} + (2\lambda_2-\lambda_1)\mathbf{h}\right]X_{n+1}\Big).\end{aligned}$$

The last equation gives the theorem. □

TABLE 3. Binet formulas associated with choices of a, b and c.

Symbol	λ_1	λ_2	Binet formula
$J\mathbb{H}_n^{(3)}$	1	−2	$\frac{1}{7}\left[2^n\Theta + (2-4\mathbf{i}-6\boldsymbol{\varepsilon}-\mathbf{h})X_n + (3+\mathbf{i}-2\boldsymbol{\varepsilon}-5\mathbf{h})X_{n+1}\right]$
$j\mathbb{H}_n^{(3)}$	−3	6	$\frac{1}{7}\left[2^{n+2}\Theta - (6-12\mathbf{i}-18\boldsymbol{\varepsilon}-3\mathbf{h})X_n - (9+3\mathbf{i}-6\boldsymbol{\varepsilon}-15\mathbf{h})X_{n+1}\right]$
$K\mathbb{H}_n^{(3)}$	7	14	$2^{n-1}\Theta - (2-2\boldsymbol{\varepsilon}-3\mathbf{h})X_n - (1+3\mathbf{i}+2\boldsymbol{\varepsilon}-3\mathbf{h})X_{n+1}$

For simplicity of notation, we will use $\Omega_0 = -\lambda_2+(\lambda_2-2\lambda_1)\mathbf{i}+2(\lambda_2-\lambda_1)\boldsymbol{\varepsilon}+(\lambda_1+\lambda_2)\mathbf{h}$ and $\Omega_1 = \lambda_1-\lambda_2-(\lambda_1+\lambda_2)\mathbf{i}-2\lambda_1\boldsymbol{\varepsilon}+(2\lambda_2-\lambda_1)\mathbf{h}$, we obtain the following result

$$g\mathbb{H}_n^{(3)} = \frac{1}{7}\left[\lambda 2^{n-1}\Theta + \Omega_0 X_n + \Omega_1 X_{n+1}\right] \tag{2.5}$$

and $g\mathbb{H}_0^{(3)} = \frac{1}{7}\left[\lambda 2^{-1}\Theta + \Omega_1\right]$.

The following theorem gives the Gaussian generalized third-order Jacobsthal hybrid numbers with negative indices.

Theorem 2.3. *For any integer $n \geq 0$, we have*

$$g\mathbb{H}_{-n}^{(3)} = \frac{1}{7}\left[\lambda 2^{-(n+1)}\Theta + (\Omega_1 - \Omega_0)X_n + \Omega_1 X_{n+1}\right],$$

where Θ is as Theorem 2.1, Ω_0 and Ω_1 are as in (2.5).

Proof. Using (2.4), $X_{-n} = -X_n$ and $X_n + X_{n+1} + X_{n+2} = 0$, we obtain

$$\begin{aligned} g\mathbb{H}_{-n}^{(3)} &= \frac{1}{7}\left[\lambda 2^{-n-1}\Theta + \Omega_0 X_{-n} + \Omega_1 X_{-n+1}\right] \\ &= \frac{1}{7}\left[\lambda 2^{-(n+1)}\Theta - \Omega_0 X_n - \Omega_1 X_{n-1}\right] \\ &= \frac{1}{7}\left[\lambda 2^{-(n+1)}\Theta - \Omega_0 X_n + \Omega_1 \left(X_n + X_{n+1}\right)\right] \\ &= \frac{1}{7}\left[\lambda 2^{-(n+1)}\Theta + \left(\Omega_1 - \Omega_0\right) X_n + \Omega_1 X_{n+1}\right], \end{aligned}$$

where $\Omega_0 = -\lambda_2 + (\lambda_2 - 2\lambda_1)\mathbf{i} + 2(\lambda_2 - \lambda_1)\boldsymbol{\varepsilon} + (\lambda_1 + \lambda_2)\mathbf{h}$ and $\Omega_1 = \lambda_1 - \lambda_2 - (\lambda_1 + \lambda_2)\mathbf{i} - 2\lambda_1\boldsymbol{\varepsilon} + (2\lambda_2 - \lambda_1)\mathbf{h}$. The result follows from the last equation. ☐

Example 2.1. For $n = 1$, the Gaussian generalized third-order Jacobsthal hybrid number with negative indices is

$$\begin{aligned} g\mathbb{H}_{-1}^{(3)} &= \frac{1}{7}\left[\lambda 2^{-2}\Theta + \Omega_0 X_{-1} + \Omega_1 X_{-1+1}\right] \\ &= \frac{1}{7}\left[\lambda 2^{-2}\Theta - \Omega_0 X_1 + \Omega_1 X_0\right] = \frac{1}{7}\left[\lambda 2^{-2}\Theta - \Omega_0\right] \\ &= \frac{1}{7}\left[\lambda 2^{-2}\Theta + \left(\Omega_1 - \Omega_0\right) X_1 + \Omega_1 X_2\right]. \end{aligned}$$

Now we give some summation formula for the Gaussian generalized third-order Jacobsthal hybrid numbers.

Theorem 2.4. *For any nonnegative integer n, we have*

$$\sum_{l=0}^{n} g\mathbb{H}_l^{(3)} = \frac{1}{3}\left[g\mathbb{H}_{n+2}^{(3)} + 2g\mathbb{H}_n^{(3)} - \frac{1}{7}\left(\frac{3}{2}\lambda\Theta - (\Omega_0 + \Omega_1)\right)\right],$$

where Ω_0 and Ω_1 are as in (2.5).

Proof. By using Theorem 2.2 and (2.1), we have

$$\begin{aligned} \sum_{l=0}^{n} g\mathbb{H}_l^{(3)} =& g\mathbb{H}_0^{(3)} + g\mathbb{H}_1^{(3)} + g\mathbb{H}_2^{(3)} + \sum_{l=3}^{n} g\mathbb{H}_l^{(3)} \\ =& g\mathbb{H}_0^{(3)} + g\mathbb{H}_1^{(3)} + g\mathbb{H}_2^{(3)} \\ &+ \sum_{l=3}^{n} g\mathbb{H}_{l-1}^{(3)} + \sum_{l=3}^{n} g\mathbb{H}_{l-2}^{(3)} + 2\sum_{l=3}^{n} g\mathbb{H}_{l-3}^{(3)} \\ =& 4\sum_{l=0}^{n} g\mathbb{H}_l^{(3)} + g\mathbb{H}_2^{(3)} - g\mathbb{H}_0^{(3)} - 2g\mathbb{H}_n^{(3)} - g\mathbb{H}_{n+2}^{(3)}. \end{aligned}$$

Then, we have

$$\begin{aligned}\sum_{l=0}^{n} g\mathbb{H}_l^{(3)} &= \frac{1}{3}\left[g\mathbb{H}_{n+2}^{(3)} + 2g\mathbb{H}_n^{(3)} + g\mathbb{H}_0^{(3)} - g\mathbb{H}_2^{(3)}\right] \\ &= \frac{1}{3}\left[g\mathbb{H}_{n+2}^{(3)} + 2g\mathbb{H}_n^{(3)} + \frac{1}{7}\left(-\frac{3}{2}\lambda\Theta + \Omega_0 + \Omega_1\right)\right],\end{aligned}$$

which is the desired result. Note that $\Omega_0+\Omega_1 = \lambda_1-2\lambda_2-3\lambda_1\mathbf{i}+(2\lambda_2-4\lambda_1)\boldsymbol{\varepsilon}+3\lambda_2\mathbf{h}$. □

Table 4. Summation formulas associated with choices of a, b and c.

Symbol	λ_1	λ_2	Summation formula
$J\mathbb{H}_n^{(3)}$	1	−2	$\frac{1}{3}\left[J\mathbb{H}_{n+2}^{(3)} + 2J\mathbb{H}_n^{(3)} - (1+6\mathbf{i}+8\boldsymbol{\varepsilon}+6\mathbf{h})\right]$
$j\mathbb{H}_n^{(3)}$	−3	6	$\frac{1}{3}\left[j\mathbb{H}_{n+2}^{(3)} + 2j\mathbb{H}_n^{(3)} - (9+21\mathbf{i}+24\boldsymbol{\varepsilon}+18\mathbf{h})\right]$
$K\mathbb{H}_n^{(3)}$	7	14	$\frac{1}{3}\left[K\mathbb{H}_{n+2}^{(3)} + 2K\mathbb{H}_n^{(3)} - \left(9+\frac{45}{2}\mathbf{i}+24\boldsymbol{\varepsilon}+12\mathbf{h}\right)\right]$

A generating function for the Gaussian generalized third-order Jacobsthal hybrid numbers can be found in the following theorem.

Theorem 2.5 (Generating function)**.** *A generating function for the Gaussian generalized third-order Jacobsthal hybrid number is*

$$\sum_{l=0}^{+\infty} g\mathbb{H}_l^{(3)}x^l = \frac{\frac{1}{7}\left(\lambda 2^{-1}\Theta + \Omega_1 + (\lambda 2^{-1}\Theta + \Omega_0 - 2\Omega_1)\,x + (\lambda 2^{-1}\Theta - 2\Omega_0)\,x^2\right)}{1-x-x^2-2x^3}.$$

Proof. Let us define $g(x) = \sum_{l=0}^{+\infty} g\mathbb{H}_l^{(3)}x^l$. Multiplying this equation by 1, $-x$, $-x^2$ and $-2x^3$, respectively, and summing the last equations, we obtain

$$\begin{aligned}(1-x-x^2-2x^3)g(x) &= g\mathbb{H}_0^{(3)} + \left(g\mathbb{H}_1^{(3)} - g\mathbb{H}_0^{(3)}\right)x + \left(g\mathbb{H}_2^{(3)} - g\mathbb{H}_1^{(3)} - g\mathbb{H}_0^{(3)}\right)x^2 \\ &= g\mathbb{H}_0^{(3)} + \left(g\mathbb{H}_1^{(3)} - g\mathbb{H}_0^{(3)}\right)x + 2g\mathbb{H}_{-1}^{(3)}x^2.\end{aligned}$$

Also, using (2.5), we have

$$2g\mathbb{H}_{-1}^{(3)} = \frac{1}{7}\left[\lambda 2^{-1}\Theta - 2\Omega_0\right], \quad g\mathbb{H}_0^{(3)} = \frac{1}{7}\left[\lambda 2^{-1}\Theta + \Omega_1\right], \quad g\mathbb{H}_1^{(3)} = \frac{1}{7}\left[\lambda\Theta + \Omega_0 - \Omega_1\right].$$

The theorem is proven using the previous equalities. □

3. Type of d'Ocagne's Identity for the Sequence $g\mathbb{H}_n^{(3)}$

We give some properties for the Gaussian generalized third-order Jacobsthal hybrid numbers. We calculate these identities by using the multiplication rules for hybrid numbers in Table 1.

For simplicity of notation, let

$$Z_n = \Omega_0 X_n + \Omega_1 X_{n+1}, \tag{3.1}$$

$$Z_{n+1} = -\Omega_1 X_n + (\Omega_0 - \Omega_1)\,X_{n+1}, \tag{3.2}$$

where $\Omega_0 = -\lambda_2 + (\lambda_2 - 2\lambda_1)\mathbf{i} + 2(\lambda_2 - \lambda_1)\boldsymbol{\varepsilon} + (\lambda_1 + \lambda_2)\mathbf{h}$, $\Omega_1 = \lambda_1 - \lambda_2 - (\lambda_1 + \lambda_2)\mathbf{i} - 2\lambda_1\boldsymbol{\varepsilon} + (2\lambda_2 - \lambda_1)\mathbf{h}$ and X_n is the sequence defined in Eq. (2.4).

Then, the formula of the hybrid numbers $g\mathbb{H}_n^{(3)}$ is given by

$$g\mathbb{H}_n^{(3)} = \frac{1}{7}\left[\lambda 2^{n-1}\Theta + Z_n\right], \tag{3.3}$$

where

$$Z_n = \frac{1}{\omega_1 - \omega_2}\Big[(\Omega_0 + \Omega_1\omega_1)\omega_1^n - (\Omega_0 + \Omega_1\omega_2)\omega_2^n\Big].$$

Note that $Z_{n+2} = -Z_{n+1} - Z_n$, with initial conditions $Z_0 = \Omega_1$ and $Z_1 = \Omega_0 - \Omega_1$.

Using (3.1) and (3.2), we have

$$\begin{aligned}
Z_{m+1}Z_n - Z_m Z_{n+1} =& \left[-\Omega_1 X_m + (\Omega_0 - \Omega_1)X_{m+1}\right]\left[\Omega_0 X_n + \Omega_1 X_{n+1}\right] \\
& - \left[\Omega_0 X_m + \Omega_1 X_{m+1}\right]\left[-\Omega_1 X_n + (\Omega_0 - \Omega_1)X_{n+1}\right] \\
=& (\Omega_0\Omega_1 - \Omega_1\Omega_0)\left[X_m X_n + X_{m+1}X_{n+1}\right] \\
& + \left(\Omega_0^2 - \Omega_1\Omega_0 - \Omega_1^2\right)X_{m+1}X_n - \left(\Omega_0^2 - \Omega_0\Omega_1 - \Omega_1^2\right)X_m X_{n+1},
\end{aligned}$$

with $\Omega_0\Omega_1 \neq \Omega_1\Omega_0$. In particular, if $m = n$, we obtain

$$Z_{n+1}Z_n - Z_n Z_{n+1} = (\Omega_0\Omega_1 - \Omega_1\Omega_0)\left[X_n^2 + X_{n+1}^2 - X_n X_{n+1}\right].$$

The first type of d'Ocagne's identity for the sequence $g\mathbb{H}_n^{(3)}$ is given in the next theorem.

Theorem 3.1. *Let $n \geq m \geq 0$, be integers. Then, we have*

$$\begin{aligned}
g\mathbb{H}_{m+1}^{(3)} g\mathbb{H}_n^{(3)} - g\mathbb{H}_m^{(3)} g\mathbb{H}_{n+1}^{(3)} =& \lambda 2^{m-1}\Theta\left[(\Omega_1 + 2\Omega_0)X_n + (3\Omega_1 - \Omega_0)X_{n+1}\right] \\
& - \lambda 2^{n-1}\left[(\Omega_1 + 2\Omega_0)X_m + (3\Omega_1 - \Omega_0)X_{m+1}\right]\Theta \\
& + (\Omega_0\Omega_1 - \Omega_1\Omega_0)\left[X_m X_n + X_{m+1}X_{n+1}\right] \\
& + \left(\Omega_0^2 - \Omega_1\Omega_0 - \Omega_1^2\right)X_{m+1}X_n \\
& - \left(\Omega_0^2 - \Omega_0\Omega_1 - \Omega_1^2\right)X_m X_{n+1},
\end{aligned}$$

where X_n as in (2.4).

Proof. By using Eq. (3.3), we have

$$\begin{aligned}
& g\mathbb{H}_{m+1}^{(3)} g\mathbb{H}_n^{(3)} - g\mathbb{H}_m^{(3)} g\mathbb{H}_{n+1}^{(3)} \\
=& \frac{1}{49}\left[(\lambda 2^m\Theta + Z_{m+1})\left(\lambda 2^{n-1}\Theta + Z_n\right) - \left(\lambda 2^{m-1}\Theta + Z_m\right)(\lambda 2^n\Theta + Z_{n+1})\right] \\
=& \lambda 2^{m-1}\Theta(2Z_n - Z_{n+1}) - \lambda 2^{n-1}(2Z_m - Z_{m+1})\Theta + Z_{m+1}Z_n - Z_m Z_{n+1}.
\end{aligned}$$

Using the relation $2Z_n - Z_{n+1} = (\Omega_1 + 2\Omega_0) X_n + (3\Omega_1 - \Omega_0) X_{n+1}$, we have

$$\begin{aligned} & g\mathbb{H}_{m+1}^{(3)} g\mathbb{H}_n^{(3)} - g\mathbb{H}_m^{(3)} g\mathbb{H}_{n+1}^{(3)} \\ =& \lambda 2^{m-1}\Theta \left[(\Omega_1 + 2\Omega_0) X_n + (3\Omega_1 - \Omega_0) X_{n+1}\right] \\ & - \lambda 2^{n-1}\left[(\Omega_1 + 2\Omega_0) X_m + (3\Omega_1 - \Omega_0) X_{m+1}\right]\Theta \\ & + (\Omega_0\Omega_1 - \Omega_1\Omega_0)\left[X_m X_n + X_{m+1}X_{n+1}\right] \\ & + \left(\Omega_0^2 - \Omega_1\Omega_0 - \Omega_1^2\right) X_{m+1}X_n - \left(\Omega_0^2 - \Omega_0\Omega_1 - \Omega_1^2\right) X_m X_{n+1}, \end{aligned}$$

where X_n is the sequence defined in (2.4). Thus, the proof is completed. □

If $m = n - 1$ in Theorem 3.1, then the Cassini identity is obtained.

Corollary 3.1. *Let $n \geq 1$, be integer. Then, we have*

$$\begin{aligned} \left(g\mathbb{H}_n^{(3)}\right)^2 - g\mathbb{H}_{n-1}^{(3)} g\mathbb{H}_{n+1}^{(3)} =& \lambda 2^{n-2}\Theta \left[(\Omega_1 + 2\Omega_0) X_n + (3\Omega_1 - \Omega_0) X_{n+1}\right] \\ & - \lambda 2^{n-1}\left[(\Omega_1 + 2\Omega_0) X_{n-1} + (3\Omega_1 - \Omega_0) X_n\right]\Theta \\ & + (\Omega_0\Omega_1 - \Omega_1\Omega_0)\left[X_{n-1} + X_{n+1}\right] X_n \\ & + \left(\Omega_0^2 - \Omega_1\Omega_0 - \Omega_1^2\right) X_n^2 \\ & - \left(\Omega_0^2 - \Omega_0\Omega_1 - \Omega_1^2\right) X_{n-1}X_{n+1}. \end{aligned}$$

If $m = n$ in Theorem 3.1, then the next identity is obtained.

Corollary 3.2. *Let $n \geq 0$, be integer. Then, we have*

$$\begin{aligned} g\mathbb{H}_{n+1}^{(3)} g\mathbb{H}_n^{(3)} - g\mathbb{H}_n^{(3)} g\mathbb{H}_{n+1}^{(3)} =& \lambda 2^{n-1}\Theta \left[(\Omega_1 + 2\Omega_0) X_n + (3\Omega_1 - \Omega_0) X_{n+1}\right] \\ & - \lambda 2^{n-1}\left[(\Omega_1 + 2\Omega_0) X_n + (3\Omega_1 - \Omega_0) X_{n+1}\right]\Theta \\ & + (\Omega_0\Omega_1 - \Omega_1\Omega_0)\left[X_n^2 + X_{n+1}^2 + X_n X_{n+1}\right]. \end{aligned}$$

Example 3.1. For $n = 1$, the Gaussian generalized third-order Jacobsthal hybrid number satisfies the following relations

$$\begin{aligned} \left(g\mathbb{H}_1^{(3)}\right)^2 - g\mathbb{H}_0^{(3)} g\mathbb{H}_2^{(3)} =& \lambda 2^{-1}\Theta\left[3\Omega_0 - 2\Omega_1\right] - \lambda\left[3\Omega_1 - \Omega_0\right]\Theta + \Omega_0^2 - \Omega_0\Omega_1 - \Omega_1^2, \\ g\mathbb{H}_2^{(3)} g\mathbb{H}_1^{(3)} - g\mathbb{H}_1^{(3)} g\mathbb{H}_2^{(3)} =& \lambda\Theta\left[3\Omega_0 - 2\Omega_1\right] - \lambda\left[3\Omega_0 - \Omega_1\right]\Theta + \Omega_0\Omega_1 - \Omega_1\Omega_0. \end{aligned}$$

4. Conclusion

We defined new hybrid numbers by using definitions of generalized third-order Jacobsthal sequence and third-order Jacobsthal numbers. The properties of those hybrid numbers were examined. Some theorems about these numbers were presented. In a later work, we will obtain curious properties of modified third-order Jacobsthal hybrid numbers using the usual modified third-order Jacobsthal numbers studied by Morales in [5, 8]. Future studies could explore the application of hybrid numbers to other special sequences (see [10, 20]), to investigate whether similar properties and patterns emerge.

Acknowledgments

The author would like to thank the anonymous reviewer for his/her valuable comments and suggestions to improve the quality of the paper.

References

[1] D. Bród, A. Szynal-Liana and I. Włoch, *Balancing hybrid numbers, their properties and some identities*, Indian J. Math. **63**(1) (2021), 143–157.
[2] N. Kilic, *Introduction to k-Horadam hybrid numbers*, Kuwait J. Sci. **49**(4) (2022), 1–9. https://doi.org/10.48129/kjs.14929
[3] G. Morales, *On the Lichtenberg hybrid quaternions*, Math. Morav. **29**(1) (2025), 31–41. https://doi.org/10.5937/MatMor2501031M
[4] G. Morales, *Investigation of generalized Fibonacci hybrid numbers and their properties*, Applied Mathematics E-Notes **21** (2021), 110–118.
[5] G. Morales, *Introduction to third-order Jacobsthal and modified third-order Jacobsthal hybrinomials*, Discuss. Math., Gen. Algebra Appl. **41**(1) (2021), 139–152. http://dx.doi.org/10.7151/dmgaa.1349
[6] G. Morales, *On Gauss third-order Jacobsthal numbers and their applications*, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) LXVII f.2 (2021), 231–241.
[7] G. Morales, *Gaussian third-order Jacobsthal and Gaussian third-order Jacobsthal-Lucas polynomials and their properties*, Asian-Eur. J. Math. **14**(5) (2021), Article ID 2150076, 13 pages. https://doi.org/10.1142/S1793557121500765
[8] G. Morales, *A note on modified third-order Jacobsthal numbers*, Proyecciones **39**(2) (2020), 731–747. http://dx.doi.org/10.22199/issn.0717-6279-2020-02-0025
[9] M. Özdemir, *Introduction to hybrid numbers*, Adv. Appl. Clifford Algebra **28**(11) (2018), 1–32. https://doi.org/10.1007/s00006-018-0833-3
[10] E. Özkan, E. Eser and M. Uysal, *On a generalization of the Gadovan numbers*, Math. Morav. **28**(1) (2024), 109–119. https://doi.org/10.5937/MatMor2401109O
[11] E. Polatlı, *Hybrid numbers with Fibonacci and Lucas hybrid number coefficients*, Univers. J. Math. Appl. **6** (2023), 106–113. https://doi.org/10.32323/ujma.1339603
[12] T. Şentürk, G. Bilgici, A. Daşdemir and Z. Ünal, *A study on Horadam hybrid numbers*, Turkish J. Math. **44** (2020), 1212–1221. https://doi.org/10.3906/mat-1908-77
[13] Y. Soykan, E. Taşdemir, I. Okumuş and M. Göcen, *Gaussian generalized Tribonacci numbers*, Journal of Progressive Research in Mathematics **14**(2) (2018), 2373–2387.
[14] Y. Soykan, E. Taşdemir and M. Göcen, *Sum of the squares of terms of Gaussian generalized Tribonacci sequences: closed form formulas of* $\sum_{k=1}^{n} Gw_k^2$, Electron. J. Math. Anal. Appl. **10**(1) (2022), 71–83.
[15] A. Szynal-Liana, *The Horadam hybrid numbers*, Discuss. Math., Gen. Algebra Appl. **38**(1) (2018), 91–98. https://doi.org/10.7151/dmgaa.1287
[16] A. Szynal-Liana and I. Włoch, *The Fibonacci hybrid numbers*, Util. Math. **110** (2019), 3–10.
[17] A. Szynal-Liana and I. Włoch, *On Jacobsthal and Jacobsthal-Lucas hybrid numbers*, Ann. Math. Sil. **33** (2019), 276–283. https://doi.org/10.2478/amsil-2018-0009
[18] E. Tan and N. R. Ait-Amrane, *On a new generalization of Fibonacci hybrid numbers*, Indian J. Pure Appl. Math. **54**(2) (2023), 428–438. https://doi.org/10.1007/s13226-022-00264-3
[19] D. Taşci, *On Gaussian Leonardo numbers*, Contrib. Math. **7** (2023), 34–40. https://doi.org/10.47443/cm.2022.064
[20] N. Yilmaz, *Some new results for the generalized bivariate Fibonacci and Lucas polynomials*, Math. Morav. **28**(1) (2024), 97–108. https://doi.org/10.5937/MatMor2401097Y

Instituto de Matemáticas,
Pontificia Universidad Católica de Valparaíso,
Blanco Viel 596, Valparaíso, Chile.
Email address: gamaliel.cerda.m@mail.pucv.cl
ORCID iD: https://orcid.org/0000-0003-3164-4434