Kragujevac Journal of Mathematics
Volume 45(1) (2021), Pages 21–37.

CHAIN CONDITION AND FUNDAMENTAL RELATION ON (Δ, G)-SETS DERIVED FROM Γ-SEMIHYPERGROUPS

S. OSTADHADI-DEHKORDI

Abstract. The aim of this research work is to define a new class of hyperstructure as a generalization of semigroups, semihypergroups and Γ-semihypergroups that we call (Δ, G)-sets. Also, we define fundamental relation on (Δ, G)-sets and prove some results in this respect. Then, we introduce the notions of quotient (Δ, G)-sets by using a congruence relations. Finally, we introduce the concept of complete parts and Noetherian(Artinian) (Δ, G)-sets.

1. Introduction

The hypergroup notion was introduced in 1934 by a French mathematician F. Marty [17], at the 8^{th} Congress of Scandinavian Mathematicians. He published some notes on hypergroups, using them in different contexts: algebraic functions, rational fractions, non commutative groups. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Since then, hundreds of papers and several books have been written on this topic, see [4–6].

The concept of Γ-semigroup defined by Sen and Saha [18] in 1986 that is a generalization of a semigroup. Many classical notions of semigroups have been extended to Γ-semigroups and a lot of results on Γ-semigroups are published by a lot of mathematicians, for instance, Chattopadhyay [2, 3], Hila [15, 16] and [18].

Recently, the notion of Γ-hyperstructure introduced and studied by many researchers and represent an intensively studied field of research, for example, see

Key words and phrases. Γ-semihypergroup, left(right) (Δ, G)-set, twist product, flat Γ-semihypergroup, absolutely flat Γ-semihypergroup.

2010 *Mathematics Subject Classification.* 20N15.

Received: June 28, 2018.

Accepted: August 22, 2018.

[1,7,8,11–14]. The concept of Γ-semihypergroups was introduced by Davvaz et al. [1,14] and is a generalization of semigroups, a generalization of semihypergroups and a generalization of Γ-semigroups. Also, the concept of (Δ, G)-set was introduced by S. Ostadhadi-Dehkordi [9,10]. He using them in different contexts such as twist product, flat Γ-semihypergroup, absolutely flat Γ-semihypergroup and direct limit that is important tools in the theory of homological algebra.

In this paper, by using a special scalar hyperoperations on Γ-semihypergroups we denote the notions left(right) (Δ, G)-set, (G_1, Δ, G_2)-biset. Also, we introduced regular and strongly regular relations on (Δ, G)-sets and by using fundamental relation we define quotient (Δ, G)-sets. Finally, we define the concept of complete part and Noetherian(Artinian) (Δ, G)-sets and prove some results in respect.

2. Introduction and preliminaries

In this section, we present some basic notions of Γ-semihypergroup. These definitions and results are necessary for the next sections.

Let H be a non-empty set. Then, the map $\circ : H \times H \to P^*(H)$ is called *hyperoperation* or *join operation* on the set H, where $P^*(H)$ denotes the set of all non-empty subsets of H. A hypergroupoid is a set H together with a (binary)hyperoperation. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all $a, b, c \in H$, we have $a \circ (b \circ c) = (a \circ b) \circ c$. A hypergroupoid $(H, \circ)$ is called *quasihypergroup* if for all $a \in H$, we have $a \circ H = H \circ a = H$. A hypergroupoid $(H, \circ)$ which is both a semihypergroup and a quasihypergroup is called a *hypergroup*.

Definition 2.1 ([14])**.** Let G and Γ be nonempty sets and $\alpha : G \times G \to P^*(G)$ be a hyperoperation, where α is an arbitrary element in the set Γ. Then, G is called Γ-*hypergroupoid*.

For any two nonempty subsets G_1 and G_2 of G, we define

$$G_1\alpha G_2 = \bigcup_{g_1 \in G_1, g_2 \in G_2} g_1 \alpha g_2, \quad G_1\alpha\{x\} = G_1\alpha x, \quad \{x\}\alpha G_2 = x\alpha G_2.$$

A Γ-hypergroupoid G is called Γ-*semihypergroup* if for all $x, y, z \in G$ and $\alpha, \beta \in \Gamma$ we have

$$(x\alpha y)\beta z = x\alpha(y\beta z).$$

Example 2.1. Let $\Gamma \subseteq \mathbb{N}$ be a nonempty set. We define

$$x\alpha y = \{z \in \mathbb{N} : z \geq \max\{x, \alpha, y\}\},$$

where $\alpha \in \Gamma$ and $x, y \in \mathbb{N}$. Then, $\mathbb{N}$ is a Γ-semihypergroup.

Example 2.2. Let $\Gamma = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Then, we define hyperoperations $x\alpha_k y = xyk\mathbb{Z}$. Hence, $\mathbb{Z}$ is a Γ-semihypergroup.

Example 2.3. Let G be a nonempty set and Γ be a nonempty set of G. Then, we define $x\alpha y = \{x, \alpha, y\}$. Hence, G is a Γ-semihypergroup.

Example 2.4. Let $(\Gamma, \cdot)$ be a semigroup and $\{A_\alpha\}_{\alpha\in\Gamma}$ be a collection of nonempty disjoint sets and $G = \bigcup_{\alpha\in\Gamma} A_\alpha$, for every $g_1, g_2 \in G$ and $\alpha \in \Gamma$, we define $g_1\widehat{\alpha}g_2 = A_{\alpha_1\alpha\alpha_2}$, where $g_1 \in A_{\alpha_1}$ and $g_2 \in A_{\alpha_2}$. Then, G is a $\widehat{\Gamma}$-semihypergroup, $\widehat{\Gamma} = \{\widehat{\alpha} : \alpha \in \Gamma\}$.

Let G be a Γ-semihypergroup. Then, an element $e_\alpha \in G$ is called *α-identity* if for every $x \in G$, we have $x \in e_\alpha\alpha x \cap x\alpha e_\alpha$ and e_α is called *scalar α-identity* if $x = e_\alpha\alpha x = x\alpha e_\alpha$. We note that if for every $\alpha \in \Gamma$, e is a scalar α-identity, then $x\alpha y = x\beta y$, where $\alpha, \beta \in \Gamma$ and $x, y \in G$. Indeed,

$$x\alpha y = (x\beta e)\alpha y = x\beta(e\alpha y) = x\beta y.$$

Let G be a Γ-semihypergroup and for every $\alpha \in \Gamma$ has an α-identity. Then, G is called a Γ-semihypergroup with identity. In a same way, we can define Γ-semihypergroup with scalar identity.

A Γ-semihypergroup G is *commutative* when

$$x\alpha y = y\alpha x,$$

for every $x, y \in G$ and $\alpha \in \Gamma$.

Definition 2.2. Let G be a Γ-semihypergroup and ρ be an equivalence relation on G. Then, ρ is called *right regular relation* if $x\rho y$ and $g \in G$ implies that for every $t_1 \in x\alpha g$ there is $t_2 \in y\alpha g$ such that $t_1\rho t_2$ and for every $s_1 \in y\alpha g$ there is $s_2 \in x\alpha g$ such that $s_1\rho s_2$. In a same way, we can define *left regular relation.* An equivalence relation ρ is called *strong regular* when $x\rho y$ and $g \in G$ implies that for every $t_1 \in x\alpha g$ and $t_2 \in y\alpha g$, $t_1\rho t_2$, for every $\alpha \in \Gamma$.

Example 2.5. Let $\mathbb{R} = \bigcup_{n\in\mathbb{Z}} A_n$, where $A_n = [n, n+1)$ and $x, y \in \mathbb{R}$ such that $x \in A_n$, $y \in A_m$ and $\alpha \in \mathbb{Z}$. Then, $\mathbb{R}$ is a $\widehat{\mathbb{Z}}$-semihypergroup such that $x\widehat{\alpha}y = A_{n\alpha m}$, where $\widehat{\alpha} \in \widehat{\mathbb{Z}} = \{\widehat{\alpha} : \alpha \in \mathbb{Z}\}$. Let

$$x\rho y \leftrightarrow 2|n - m, \quad x \in A_n, y \in A_m.$$

Then, the relation ρ is strong regular. Also, $x \in \mathbb{R}$, implies that

$$\rho(x) = \{z \in \mathbb{R} : z \in \cdots [n-4, n-3) \cup [n-2, n-1) \cup [n, n+1) \cup [n+2, n+3) \cdots\},$$

where $x \in [n, n+1)$.

Proposition 2.1. *Let G be a Γ-semihypergroup and ρ be a regular relation on G. Then, $[G : \rho] = \{\rho(x) : x \in G\}$ is a $\widehat{\Gamma}$-semihypergroup with respect the following hyperoperation:*

$$\rho(x)\widehat{\alpha}\rho(y) = \{\rho(z) : z \in \rho(x)\alpha\rho(y)\},$$

where $\widehat{\Gamma} = \{\widehat{\alpha} : \alpha \in \Gamma\}$.

Proof. The proof is straightforward. □

Corollary 2.1. *Let G be a Γ-semihypergroup and ρ be an equivalence relation G. Then, ρ is regular (strong regular) if and only if $[G:\rho]$ is $\widehat{\Gamma}$-semihypergroup ($\widehat{\Gamma}$-semigroup).*

Definition 2.3 ([9])**.** Let G be a Γ-semihypergroup with identity and X, Δ be nonempty sets. Then, we say that X is a *left* (Δ, G)*- set* if there is a scalar hyperaction $\delta : G \times X \to P^*(X)$ with the following properties:

$$\begin{aligned}(g_1\alpha g_2)\delta x =& g_1\delta(g_2\delta x),\\ e_\alpha\delta x =& x,\end{aligned}$$

for every $g_1, g_2 \in G$, $\alpha \in \Gamma$, $x \in X$ and $\delta \in \Delta$.

When $\delta : G \times X \to X$, then X is called *scalar left* (Δ, G)*-set.*

Example 2.6. Let G be a Γ-semihypergroup with scalar identity, X and Δ be nonempty sets such that $x_0 \in X$ is a fixed element and $\delta : G \times X \to P^*(X)$ defined by $\delta(g, x) = \{x_0\}$, where $\delta \in \Delta$ and $x \in X$. Then, G is left (Δ, G)-set.

Example 2.7. Let $(G, \circ)$ be a semihypergroup and H be a subsemihypergroup of G. Then, H is a left (Δ, G)-set where $\Delta = \{\circ\}$.

In a same way, we can define a *right* (Δ, G)*-set.* Let G_1 and G_2 be Γ-semihypergroups and X be a nonempty set. Then, we say that X is a (G_1, Δ, G_2)*-bisets* if it is a left (Δ, G_1)-set, right (Δ, G_2)-set and

$$(g_1\delta_1 x)\delta_2 g_2 = g_1\delta_1(x\delta_2 g_2),$$

for every $\delta_1, \delta_2 \in \Delta$, $g_1 \in G_1$, $g_2 \in G_2$ and $x \in X$. When X is a (G_1, Δ, G_2)-bisets and $G_1 = G_2 = G$, we sat that X is a (Δ, G)-bisets.

If G is a commutative Γ-semihypergroup, then there is no distinction between a left and a right (Δ, G)-sets. A *left* (Δ, G)*-subset* Y of X such that $Y\Delta X \subseteq Y$ is called *left* (Δ, G)*-subset* of X. Let X be a left (Δ, G)-set and $\Gamma \subseteq \Delta$. Then, X is also (Γ, G)-set where $\delta : G \times X \to P^*(X)$ and $\delta \in \Gamma$.

Definition 2.4. Let X be a left (Δ, G)-set and Y be a left (Δ, G)-subset of X. Then, we say that Y *closed*, if for all $y \in Y$ and $g \in G$ from $y \in g\delta b$ implies that $b \in Y$.

Definition 2.5. Let X be a (G, Δ, G)-biset and Y be a (G, Δ, G)-subbiset of X. Then, Y is called *invertible* on a right(on a left) if for all $y_1, y_2 \in Y$ and $g \in G$ from $y_1 \in y_2\delta G(y_1 \in G\delta y_2)$ it follows that $y_2 \in y_1\delta G$ $(y_2 \in G\delta y_1)$.

Proposition 2.2. *Let G be a Γ-semihypergroup and X be a (Δ, G)-biset such that Y be a (Δ, G)-subbiset. Then, Y is invertible on the right if and only if $\{y\delta G\}_{y\in Y}$ is a partition of X, for every $y \in Y$.*

Proof. Suppose that Y is invertible on the right and $y \in y_1\delta G \cap y_2\delta G$. Then, $y_1, y_2 \in y\delta G$. This implies that $y_1\delta G \subseteq y\delta G$ and $y_2\delta G \subseteq y\delta G$. Also,

$$y\delta G \subseteq (y_1\delta G)\delta G \subseteq y_1\delta(G\Gamma G) \subseteq y_1\delta G,$$

and $y\delta G \subseteq (y_2\delta G)\delta G = y_2\delta(G\Gamma G) \subseteq y_2\delta G$. Then, $y\delta G = y_1\delta G = y_2\delta G$. On the other hand, $y \in y_1\delta G = y\delta G$. Then, for every $y \in Y$, we have $y \in y\delta G$.

Conversely, let $\{y\delta G\}_{y\in Y}$ be a partition of Y and $y_1 \in y_2\delta G$. Then,

$$y_1\delta G \subseteq (y_2\delta G)\delta G \subseteq y_2\delta(G\Gamma G) \subseteq y_2\delta G,$$

whence $y_1\delta G = y_2\delta G$ and so $y_1 \in y_2\delta G = y_1\delta G$. Then, for all $y \in Y$ we have $y \in y\delta G$. Therefore, $y_2 \in y_2\delta G = y_1\delta G$. □

Definition 2.6. Let X be a left (Δ, G)-set and Y be a left (Δ, G)-subset of X. Then, Y is called *ultraclosed* if for all $g \in G$ and $\delta \in \Delta$, we have $g\delta Y \cap g\delta(X - Y) = \emptyset$.

Proposition 2.3. *Let X be a left (Δ, G)-set and Y be a invertible (Δ, G)-subset. Then, X is closed.*

Proof. Suppose that $y, x \in Y$, $\delta \in \Delta$ and $g \in G$ such that $y \in g\delta x$. Hence $x \in g\delta y \subseteq Y$ and we obtain $x \in Y$. □

Definition 2.7. Let X be a left (Δ, G)-set and H be a Γ-subsemihypergroup of G. Then, we define the following relation:

$$x_1 \equiv x_2 \Leftrightarrow x_1 \in H\delta x_2.$$

This relation is denoted by $x_1H^*x_2$.

Definition 2.8. Let X be a left (G, Δ)-set and ρ be a regular relation on X. Then, ρ is called *regular* if $x_1\rho x_2$ implies that for every $s_1 \in g\delta x_1$ there is $s_2 \in g\delta x_2$ such that $s_1\rho s_2$ and for every $t_2 \in g\delta x_2$ there is $t_1 \in g\delta x_1$ such that $t_1\rho t_2$, where $x_1, x_2 \in X$ and $\delta \in \Delta$. Also, an equivalence relation ρ is called *strongly regular*, when for every $s_1 \in g\delta x_1$ and $s_2 \in g\delta x_2$ implies that $s_1\rho s_2$.

Proposition 2.4. *Let X be an invertible left (Δ, G)-set such that G is commutative. Then, the relation H^* is regular.*

Proof. Suppose that $x \in X$. Then, $x = e_\alpha\delta x \in H\delta x$. It follows that xH^*x, i.e., H^* is reflexive. Let $x_1H^*x_2$. Then, there exist $\delta \in \Delta$ and $h \in H$ such that $x_1 \in h\delta x_2$ which implies that $x_2 \in h\delta x_1 \subseteq H\delta x_1$ which meanies that $x_2H^*x_1$ and so H^* is symmetric. Let $x_1, x_2, x_3 \in X$ such that $x_1H^*x_2$ and $x_2H^*x_3$. Then, there exist $h_1, h_2 \in H$ such that $x_1 \in h_1\delta x_2$ and $x_2 \in h_2\delta x_3$. Hence $x_1 \in h_1\delta(h_2\delta x_3) = (h_1\alpha h_2)\delta x_3 \subseteq H\delta x_3$. This implies that $x_1 \in H\delta x_3$ and so H^* is transitive.

Let x_1, x_2 be an arbitrary elements of X such that $x_1H^*x_2$. It follows that $x_1 \in H\delta x_2$. Hence there exist $h_1 \in H$ such that $x_1 \in h_1\delta x_2$. Let $g \in G$ and $t_1 \in g\delta x_1$. Then,

$$t_1 \in g\delta x_1 \subseteq g\delta(h_1\delta x_2) = (g\alpha h_1)\delta x_2 = (h_1\alpha g)\delta x_2 = h_1\delta(g\delta x_2).$$

Hence there exists $t_2 \in g\delta x_2$ such that $t_1 \in h_1\delta t_2 \subseteq H\delta t_2$. Thus, $t_1H^*t_2$. In a same way, we can see for every $s_2 \in g\delta x_2$ there is $s_1 \in g\delta x_1$ such that $s_1H^*s_2$. Therefore, H^* is a regular relation. □

Proposition 2.5. *Let X be a left (Δ, G)-set and H be a Γ-subsemihypergroup of G. Then, $H^*(x) = H\delta x$.*

Proof. The proof is straightforward. □

Theorem 2.1. *Let X be a left (Δ, G)-set and H be a Γ-subsemihypergroup of G. Then, the set of all classes $[X : H^*] = \{H^*(x) : x \in X\}$ is a left $(\widehat{\Delta}, G)$-set by the following scalar hyperoperation:*

$$g\widehat{\delta}H^*(x) = \{H^*(y) : y \in g\delta H^*(x)\}.$$

Proof. Suppose that $H^*(x_1) = H^*(x_2)$, $g \in G$ and $y \in g\delta H^*(x_1)$. This implies that $x_1 \in H\delta x_2$. Hence, there are $h_1, h_2 \in H$ such that $y \in g\delta(h_1\delta x_1)$ and $x_1 \in h_2\delta x_2$. We have

$$y \in g\delta(h_1\delta x_1) \subseteq g\delta(h_1\delta(h_2\delta x_2)) = g\delta(h_1\alpha h_2)\delta x_2 \subseteq g\delta(H\delta x_2) = g\delta H^*(x_2).$$

Then, $g\delta H^*(x_1) \subseteq g\delta H^*(x_2)$. In a same way, we can see, $g\delta H^*(x_2) \subseteq g\delta H^*(x_1)$. Hence,

$$g\widehat{\delta}H^*(x_1) = g\widehat{\delta}H^*(x_2).$$

Therefore, the scalar hyperoperation $\widehat{\alpha}$ is well-defined. It is easy to see that

$$(g_1\alpha g_2)\widehat{\delta}H^*(x) = g_1\widehat{\delta}(g_2\widehat{\delta}H^*(x)).$$ □

Let X be a left (Δ, G)-set. Then, we define an equivalence relation on X such that smallest strongly regular relation on X. Suppose that X be a left (Δ, G)-set and n be a nonzero natural number. We say that

$$a\beta_n b \Leftrightarrow (\exists \delta_1, \delta_2, \ldots, \delta_n \in \Delta, x \in X, g_1, g_2, \ldots, g_n \in G)\ \{a, b\} \subseteq g_1\delta_1 g_2\delta_2, \ldots, g_n\delta_n x.$$

Let $\beta = \bigcup_{n \geq 1} \beta_n$. Clearly, the relation β is reflexive and symmetric. Denote by β^* the transitive closure.

We say that $x\beta_{\delta^n} y$ when

$$a\beta_{\delta^n} b \Leftrightarrow (\exists x \in X, g_1, g_2, \ldots, g_n \in G)\ \{a, b\} \subseteq g_1\delta g_2\delta, \ldots, g_n\delta x.$$

Let $\beta_\delta = \bigcup_{n \geq 1} \beta_{\delta^n}$ and β^*_δ be transitive closure. Obviously, $\beta^*_\delta \subseteq \beta^*$.

Let X be a (Δ, G)-biset. Then, the relation β_n defined on X as follows:

$$a\beta_n b \Leftrightarrow (\exists x \in X, \delta_i, \gamma_i \in \Delta, g_i, s_i \in G)\ \{a, b\} \subseteq \prod_{i=1}^{n}(g_i\delta_i x)\gamma_i s_i.$$

In a same way, we can define β_δ and transitive closure β^*_δ.

Example 2.8. Let $\mathbb{R}$ be a $\widehat{\mathbb{Z}}$-semihyperring Example 2.5, $x, y \in \mathbb{R}$ such that $\beta(x) = \beta(y)$ and $t_1 = [x]$, $t_2 = [y]$. Then, there exist $g_1, g_2, \ldots, g_m \in \mathbb{R}$ and $\widehat{\delta}_1, \widehat{\delta}_2, \ldots, \widehat{\delta}_m \in \widehat{\mathbb{Z}}$ such that $\{x, y\} \subseteq g_1\widehat{\delta}_1 g_2\widehat{\delta}_2 g_3 \ldots g_{m-1}\widehat{\delta}_{m-1} g_m$. This implies that $t_1 = t_2 = \prod_{i=1}^{m} g_i\delta_i g_{i+1}$. Therefore, $\beta(x) = \beta(y)$ if and only there exists $n \in \mathbb{Z}$ such that $x, y \in [n, n+1)$. Hence $\beta^*(x) = \beta^*(y)$ implies that $x, y \in [n, n+1)$ for some $n \in \mathbb{Z}$.

Theorem 2.2. *Let X be a left (Δ, G)-set. Then, β^* is the smallest strongly regular relation on X.*

Proof. Suppose that $a\beta^* b$ be an arbitrary element of X. It follows that there exist $x_0 = a, x_1, \ldots, x_n = b$ such that for all $i \in \{0, 1, 2, \ldots, n\}$ we have $x_i \beta x_{i+1}$. Let $u_1 \in g\delta a$ and $u_2 \in g\delta b$, where $g \in G$, $\delta \in \Delta$. From $x_i \beta x_{i+1}$ it follows that there exists a hyperproduct P_i, such that $\{x_i, x_{i+1}\} \subseteq P_i$ and so $g\delta x_i \subseteq g\delta P_i$ and $g\delta x_{i+1} \subseteq g\delta P_{i+1}$, which meanies that $g\delta x_i \overline{\overline{\beta}} g\delta x_{i+1}$. Hence for all $i \in \{0, 1, 2, \ldots, n-1\}$ and for all $s_i \in g\delta x_i$ we have $s_i \beta s_{i+1}$. We consider $s_0 = u_1$ and $s_n = u_2$ then we obtain $u_1 \beta^* u_2$. Then β^* is strongly regular on a left.

Let ρ be a strongly regular relation on X. Then, we have

$$\beta_1 = \{(x, x) : x \in X\} \subseteq \rho,$$

since ρ is reflexive. Let $\beta_{n-1} \subseteq \rho$ and $a\beta_n b$. Then, there exist $g_1, g_2, \ldots, g_n \in G$, $\delta_1, \delta_2, \ldots, \delta_n \in \Delta$ and $x \in X$ such that $\{a, b\} \subseteq \prod_{i=1}^{n} g_i\delta_i x = g_1\delta_1 \prod_{i=2}^{n} g_i\delta_i x$. This implies that there exits $u, v \in \prod_{i=2}^{n} g_i\delta_i x$ such that $a \in g_1\delta_1 u$ and $v \in g_1\delta_1 v$. We have $u\beta_{n-1}v$ and according to the hypothesis, we obtain $u\rho v$. Since ρ is regular it follows that $a\rho b$ and $\beta_n \subseteq \rho$. By induction, it follows that $\beta \subseteq \rho$. Therefore, $\beta^* \subseteq \rho$. □

Proposition 2.6. *Let X_1 and X_2 be left (Δ, G)- and right (Δ, G)-sets, respectively and $\beta^*_{X_1}$, $\beta^*_{X_2}$ and $\beta^*_{X_1 \times X_2}$ be relations on X_1, X_2 and $X_1 \times X_2$, respectively. Then,*

$$(a, b)\beta^*_{X_1 \times X_2}(c, d) \Leftrightarrow a\beta^*_{X_1} c, \; b\beta^*_{X_2} d.$$

Proof. Suppose that $(a, b)\beta^*_{X_1 \times X_2}(c, d)$. Then,

$$\{(a, b), (c, d)\} \subseteq \prod_{i=1}^{n} g_i \widehat{\delta_i}(x, y)\widehat{\gamma_i} s_i = \left(\prod_{i=1}^{n} g_i\delta_i x, \prod_{i=1}^{n} y\gamma_i s_i\right).$$

This implies that $\{a, c\} \subseteq \prod_{i=1}^{n} g_i\delta_i x$ and $\{b, d\} \subseteq \prod_{i=1}^{n} y\gamma_i s_i$. Then, $a\beta^*_{X_1} c$ and $b\beta^*_{X_2} d$. One can see that $a\beta^*_{X_1} c$ and $b\beta^*_{X_2} d$ implies that $(a, b)\beta^*_{X_1 \times X_2}(c, d)$. □

Corollary 2.2. *Let X_1 and X_2 be left (Δ, G)- and right (Δ, G)-sets, respectively and $\beta^*_{X_1}$, $\beta^*_{X_2}$ and $\beta^*_{X_1 \times X_2}$ be relations on X_1, X_2 and $X_1 \times X_2$, respectively. Then,*

$$[X_1 \times X_2 : \beta^*_{X_1 \times X_2}] \simeq [X_1 : \beta^*_{X_1}] \times [X_2 : \beta^*_{X_2}].$$

Definition 2.9. A map $\varphi : X \to Y$ from a left (Δ, G)-set X into a left (Δ, G)-set Y is called *morphism* (*G-morphism*) if

$$\varphi(g\delta x) = g\delta\varphi(x),$$

for every $x \in X, \delta \in \Delta$ and $g \in G$.

Example 2.9. Let $(G, \circ)$ be a semihypergroup with scalar identity and G_1 be a subsemihypergroup of $(G, \circ)$. Then, G_1 is a (Γ, G_1)-biset in the obvious way, where $\Gamma = \{\circ\}$.

Example 2.10. Let ρ be a left regular relation on Γ-semihypergroup G. Then, there is a well-defined action of G on $[G:\rho]$ given by

$$g\widehat{\alpha}(\rho(x)) = \{\rho(t) : t \in g\alpha x\},$$

where $\widehat{\alpha} \in \widehat{\Gamma}$ such that $\widehat{\Gamma} = \{\widehat{\alpha} : \alpha \in \Gamma\}$. Hence, with this definition $[G:\rho]$ is a left $(\widehat{\Gamma}, G)$-system.

It is easy to see that the cartesian product $X \times Y$ of a left (Δ, G_1)-set X and a right (Δ, G_2)-set Y becomes $(G_1, \widehat{\Delta}, G_2)$-biset if we make the obvious definitions

$$g_1\widehat{\delta}_1(x,y) = \{(t,y) : t \in g_1\delta_1 x\},\ (x,y)\widehat{\delta}_2 g_2 = \{(x,t) : t \in y\delta_2 g_2\},$$

where $\widehat{\delta}_1, \widehat{\delta}_2 \in \widehat{\Delta}$, $x \in X$, $y \in Y$ and $g_1 \in G_1$, $g_2 \in G_2$.

Let X and Y be (G_1, Δ, G_2)- and (G_2, Δ, G_3)-bisets, respectively and Z be a (G_1, Δ, G_3)-biset. Then, the cartesian product $X \times Y$ is (G_1, Δ, G_3)-biset. A (G_1, Δ, G_3)-map $\varphi_\delta : X \times Y \to Z$ is called δ-*bimap* if

$$\varphi(x\delta g_2, y) = \varphi(x, g_2\delta y),$$

where $x \in X$, $y \in Y$, $g_2 \in G_2$ and $\delta \in \Delta$.

Definition 2.10 ([9])**.** A pair (P, ψ) consisting of (G_1, Δ, G_3)-biset P and a δ-bimap $\psi : X \times Y \to P$ will be called a *twist product* of X and Y over G_2 if for every (G_1, Δ, G_3)-biset Z and for every bimap $\omega : X \times Y \to Z$ there exists a unique bimap $\overline{\omega} : P \to Z$ such that $\overline{\omega} \circ \psi = \omega$.

Suppose that ρ is an equivalence relation on $X \times Y$ as follows:

$$\rho = \{(t_1, t_2) : t_1 \in x\delta g, t_2 \in g\delta y, x \in X, y \in Y, g \in G_2\}.$$

Let us define $X \ominus Y$ to be $[X \times Y : \rho^*]$, where ρ^* is a transitive closure of ρ. We denote a typical element $\rho^*(x,y)$ by $x \ominus y$. By definition of ρ^*, we have $x\delta g \ominus y = x \ominus g\delta y$, where $\delta \in \Delta$.

Proposition 2.7 ([9])**.** *Let X and Y be (G_1, Δ, G_2)- and (G_2, Δ, G_3)-bisets, respectively. Then, two element $x \ominus y$ and $x' \ominus y'$ are equal if and only if $(x,y) = (x',y')$ or there exist $x_1, x_2, \ldots, x_{n-1}$ in X, $h_1, h_2, \ldots, h_{n-1} \in G_2$ and $\delta \in \Delta$ such that*

$$\begin{aligned} x \in x_1\delta g_1, x_1\delta h_1 = x_2\delta g_2, \ldots, x_i\delta g_i = x_{i+1}\delta g_{i+1}, x_{n-1}\delta h_{n-1} &= x'\delta g_n, \\ g_1\delta y = h_1\delta y_1, g_2\delta y_1 = h_2\delta y_2, \ldots, g_{i+1}\delta y_i &= h_{i+1}\delta y_{i+1} \\ &= g_n\delta y_{n-1} \\ &= y'. \end{aligned}$$

Theorem 2.3 ([9])**.** *Let X and Y be (G_1, Δ, G_2)- and (G_2, Δ, G_3)-bisets. Then, the twist product X and Y over G_2 is unique up to isomorphism.*

Proposition 2.8. *Let X and Y be a scalar (Δ, G)-bisets. Then, $X \ominus Y$ is a (Δ, G)-biset by following scalar hyperoperations:*

$$g\widehat{\delta}(x \ominus y) = g\delta x \ominus y, \quad (x \ominus y)\widehat{\delta} g = x \ominus y\delta g,$$

where $\widehat{\delta} \in \widehat{\Delta}$ *and* $x \in X$, $y \in Y$.

Proof. Suppose that $x \ominus y = x' \ominus y'$. By Proposition 2.7, there exist $\delta \in \Delta$, $x_1, x_2, \ldots, x_{n-1} \in X$ and $h_1, h_2, \ldots, h_{n-1} \in G$, such that

$$\begin{aligned} x = x_1\delta g_1, x_1\delta h_1 = x_2\delta g_2, \cdots x_i\delta h_i &= x_{i+1}\delta g_{i+1} \\ x_{n-1}\delta h_{n-1} &= x'\delta g_n, \\ g_1\delta y = h_1\delta y_1,\ g_2\delta y_1 = h_2\delta y_2, \ldots, g_{i+1}\delta y_i &= h_{i+1}\delta y_{i+1} \\ &= g_n\delta y_{n-1} \\ &= y'. \end{aligned}$$

Hence,

$$\begin{aligned} g\delta x = g\delta(x_1\delta g_1), g\delta(x_1\delta h_1) = g\delta(x_2\delta g_2), \ldots, g\delta(x_i\delta h_i) =&\ \ g\delta(x_{i+1}\delta g_{i+1}) \\ g\delta(x_{n-1}\delta h_{n-1}) =&\ \ g\delta(x'\delta g_n). \end{aligned}$$

We have

$$\begin{aligned} g\delta x \ominus y = t_1 \ominus g_1\delta y = t_1 \ominus h_1\delta y = t_1\delta h_1 \ominus y_1 &= t_2\delta g_2 \ominus y_1 \\ &\vdots \\ &= t'\delta g_n \ominus y_{n-1} \\ &= t' \ominus g_n\delta y_{n-1} \\ &= g\delta x' \ominus y', \end{aligned}$$

where $t_i \in X$. Then, the left scalar operation $\widehat{\delta}$ is well-defined. Moreover,

$$(g_1\alpha g_2)\widehat{\delta}(x \ominus y) = (g_1\alpha g_2)\delta x \ominus y = g_1\delta(g_2\delta x) \ominus y = g_1\widehat{\delta}(g_2\widehat{\delta}(x \ominus y)),$$

where $x \in X$, $y \in Y$ and $g \in G$. Hence $X \ominus Y$ is a left $(\widehat{\Delta}, G)$-set. In a same way, we can see $X \ominus Y$ is also right (Δ, G)-set. $\square$

3. Complete Parts and Regular Relations

In this section we define the concept of complete parts and present some results.

Definition 3.1. Let X be a left (Δ, G)-set and Y be a nonempty subset of X. We say that Y is a *complete part* of X if for any nonzero natural number n and $g_1, g_2, \ldots, g_n \in G$, $\delta_1, \delta_2, \ldots, \delta_n \in \Delta$, $x \in X$, the following implication holds:

$$Y \cap \prod_{i=1}^{n} g_i\delta_i x \neq \emptyset \Rightarrow \prod_{i=1}^{n} g_i\delta_i x_i \subseteq Y.$$

Proposition 3.1. *Let X be a left (Δ, G)-set and ρ be a strongly regular relation on X. Then, the equivalence class x is a complete part of X.*

Proof. Suppose that $g_1, g_2, \ldots, g_n \in G$, $\delta_1, \delta_2, \ldots, \delta_n \in \Delta$ and $x \in X$ such that
$$\rho(x) \cap \prod_{i=1}^{n} g_i \widehat{\delta}_i x \neq \emptyset.$$
Then, there exists $y \in \prod_{i=1}^{n} g_i \delta_i x$ such that $y \rho x$. The morphism $\pi : X \to [X : \rho]$ is good and the scalar hyperoperation $\widehat{\delta}$ defined on $[X : \rho]$ is scalar operation. It follows that
$$\pi(y) = \rho(y) = \rho(x) = \pi\left(\prod_{i=1}^{n} g_i \delta_i x\right) = \prod_{i=1}^{n} \pi(g_i \delta_i x) = \prod_{i=1}^{n} g_i \widehat{\delta}_i \pi(x).$$
This implies that $\prod_{i=1}^{n} g_i \widehat{\delta}_i x \subseteq \rho(x)$. □

Proposition 3.2. *Let X and Y be scalar (Δ, G)-bisets such that $X_1 \subseteq X$ be a complete part. Then, $X_1 \ominus Y$ is also complete part in $X \ominus Y$.*

Proof. The proof is straightforward. □

Let A be a nonempty subset of (Δ, G)-sets X. Then, denoted by $C(A)$ the *complete closure* of A, which is the smallest complete part of X, that contain A.

Denote $K_1(A) = A$ and for all $n \geq 1$ denote
$$K_{n+1}(A) = \left\{ x \in X : (\exists t \in \mathbb{N})\ x \in \prod_{i=1}^{t} g_i \delta_i x, K_n(A) \cap \prod_{i=1}^{t} g_i \delta_i x \right\}.$$
Let $K(A) = \bigcup_{n \geq 1} K_n(A)$.

Theorem 3.1. *Let X be a left (Δ, G)-set and A be a nonempty subset of A. Then, $C(A) = K(A)$.*

Proof. Suppose that $K(A) \cap \prod_{i=1}^{t} g_i \delta_i x \neq \emptyset$. Then, there exits $n \geq 1$ such that $K_n(A) \cap \prod_{i=1}^{t} g_i \delta_i x \neq \emptyset$ which meanies that $\prod_{i=1}^{t} g_i \delta_i x \subseteq K_{n+1}(A)$. This implies that $K(A)$ is a complete part of X.

Let C_1 be a complete pat of X such that $A \subseteq C_1$. Then, by induction we prove that $K(A) \subseteq C_1$. We have $K_1(A) \subseteq C_1$ and suppose that $K_n(A) \subseteq C_1$. Let $x \in K_{n+1}(A)$. Then, there exists $t \in \mathbb{N}$ such that $a \in \prod_{i=1}^{t} g_i \delta_i x$ and $K_n(A) \cap \prod_{i=1}^{t} g_i \delta_i x \neq \emptyset$. Hence, $C_1 \cap \prod_{i=1}^{t} g_i \delta_i x \neq \emptyset$ implies that $\prod_{i=1}^{t} g_i \delta_i x \subseteq C_1$. We obtain $a \in C_1$. Therefore, $C(A) = K(A)$. □

Proposition 3.3. *Let X be a left (Δ, G)-set and x be an arbitrary element of X. Then,*

(1) *for all $n \geq 2$ we have $K_n(K_2(x)) = K_{n+1}(x)$;*
(2) *for every $x, y \in X$, $x \in K_n(y) \Leftrightarrow y \in K_n(x)$.*

Proof. (1) We prove the equality by induction. We have
$$K_2(K_2(x)) = \left\{ x \in X : (\exists t \in \mathbb{N})\ x \in \prod_{i=1}^{t} g_i \delta_i x, K_2(x) \cap \prod_{i=1}^{t} g_i \delta_i x \neq \emptyset \right\} = K_3(x).$$

Let $K_{n-1}(K_2(x)) = K_n(x)$. Then,

$$\begin{aligned} K_n(K_2(x)) &= \left\{x \in X : (\exists t \in \mathbb{N})\ x \in \prod_{i=1}^t g_i\delta_i x, K_{n-1}(K_2(x)) \cap \prod_{i=1}^t g_i\delta_i x \neq \emptyset\right\} \\ &= K_{n+1}(x). \end{aligned}$$

(2) We check the equivalence by induction. For $n = 2$, we have

$$x \in K_2(y) = \left\{x \in X : (\exists t \in \mathbb{N})\ x \in \prod_{i=1}^{t} g_i\delta_i x, K_1(y) \cap \prod_{i=1}^{t} g_i\delta_i x \neq \emptyset\right\}.$$

This implies that $\{y, x\} \subseteq \prod_{i=1}^t g_i\delta_i x$ and $y \in K_2(x)$. Suppose that the following equivalence holds:

$$x \in K_{n-1}(y) \Leftrightarrow y \in K_{n-1}(x).$$

We check that $x \in K_n(y) \Leftrightarrow y \in K_n(x)$. Let $x \in K_n(y)$. Then, there exists $\prod_{i=1}^t g_i\delta_i a$ with $x \in \prod_{i=1}^t g_i\delta_i a$ and there exists $b \in \prod_{i=1}^t g_i\delta_i a \cap K_{n-1}(y)$. It follows that $b \in K_2(x)$ and $y \in K_{n-1}(b)$. Hence, $y \in K_{n-1}(K_2(x)) = K_n(x)$. Similarly, we obtain the converse implication. □

Definition 3.2. Let X be a left (Δ, G)-set. Then, we define the relation ω as follows:

$$(x, y) \in \omega \Leftrightarrow (\exists n \geq 1)\ x \in K_n(y).$$

Theorem 3.2. *Let X be a left (Δ, G)-set. Then, the relation ω is an equivalence and coincide with β^*.*

Proof. By Proposition 3.3, the relation ω is an equivalence. Let $(x_1, x_2) \in \beta$. Then, $\{x_1, x_2\} \subseteq \prod_{i=1}^n g_i\delta_i x$, where $g_i \in G$, $\delta_i \in \Delta$ and $t \in \mathbb{N}$. Hence, x_1, x_2 belong to the same scalar hyperoperation and so, $x_1 \in K_2(x_2) \subseteq K(x_2)$. This implies that $\beta \subseteq \omega$ and $\beta^* \subseteq \omega$. Let $(x, y) \in K$ and $x \neq y$. Then, there exists $n \geq 1$, such that $(x, y) \in K_{n+1}$, which means that there exists a scalar hyperproduct P_1, such that $x \in P_1$ and $P_1 \cap K_n(y) \neq \emptyset$. Let $x_1 \in P_1 \cap K_n(y)$. Then, $\{x, x_1\} \subseteq P_1$. Hence $(x, x_1) \in \beta$. Since $x_1 \in K_n(y)$ it follows that there exists a scalar hyperproduct P_2 such that $x_1 \in P_2$ and $P_2 \cap K_{n-1}(y) \neq \emptyset$. Let $x_2 \in P_2 \cap K_{n-1}(y)$. Then, $x_2 \in K_{n-1}(y)$ and $\{x_1, x_2\} \subseteq P_2$. After finite number of steps, we obtain there exists a scalar hyperoperation P_n such that $\{x_{n-1}, x_n\} \subseteq P_n$ and $x_n \in K_{n-(n-1)}(y) = \{y\}$. □

4. Fundamental, Noetherian and Artinian (Δ, G)-Sets

In this section, we introduce the notion of right Noetherian and Artinian (Δ, G)-sets and define fundamental (Δ, G)-sets.

Let X be a left (Δ, G)-set such that G be a Γ-semihypergroup and $\Gamma \subseteq \Delta$. We define a relation ρ on $\Delta \times X$ as follows:

$$((\delta_1, x_1), (\delta_2, x_2)) \in \rho \Leftrightarrow g\delta_1 x_1 = g\delta_2 x_2, \quad \text{for all } g \in G,$$

where $\delta_1, \delta_2 \in \Delta$ and $x_1, x_2 \in X$. Obviously, ρ is an equivalence.

Let $\Theta[X] = [\Delta \times X : \rho]$ denote the set of all equivalence classes. We denote the equivalence class (δ, x) by $[\delta, x]$. We define a relation ϵ on $\Gamma \times G$ as follows:

$$((\delta_1, g_1), (\delta_2, g_2)) \in \epsilon \Leftrightarrow g\delta_1 g_1 = g\delta_2 g_2, \quad \text{for all } g \in G,$$

where $g_1, g_2 \in G$ and $\delta_1, \delta_2 \in \Gamma$. Obviously, ϵ is an equivalence relation and $[\delta, g]$ denote the equivalence class containing (δ, g). We denote $\Theta[G] = \{[\delta, g] : g \in G, \delta \in \Gamma\}$. We define a hyperoperation $\circ$ on $\Theta[G]$ as follows:

$$[\delta_1, g_1] \circ [\delta_2, g_2] = \{[\delta_1, z] : z \in g_1\delta_2 g_2\},$$

where $\delta_1, \delta_2 \in \Delta$ and $g_1, g_2 \in G$. This hyperoperation is well-defined. Indeed, let $[\delta_1, g_1] = [\gamma_1, h_1]$ and $[\delta_2, g_2] = [\gamma_2, , h_2]$, where $\delta_1, \delta_2, \gamma_1, \gamma_2 \in \Gamma$ and $g_1, g_2, h_1, h_2 \in G$. Then,

$$g\delta_1 g_1 = g\gamma_1 h_1, \quad g\delta_2 g_2 = g\gamma_2 h_2, \quad \text{for all } g \in G.$$

Hence,

$$(g\delta_1 g_1)\delta_2 g_2 = (g\gamma_1 h_1)\gamma_2 h_2, \quad \text{for all } g \in G,$$

and

$$g\delta_1(g_1\delta_2 g_2) = g\gamma_1(h_1\gamma_2 h_2).$$

Thus,

$$[\delta_1, g_1] \circ [\delta_2, g_2] = [\gamma_1, h_1] \circ [\gamma_2, h_2].$$

Also

$$\begin{aligned}
([\delta_1, g_1] \circ [\delta_2, g_2]) \circ [\delta_3, g_3] &= (\{[\delta_1, z] : z \in g_1\delta_2 g_2\}) \circ [\delta_3, g_3] \\
&= \bigcup_{z \in g_1\delta_2 g_2} [\delta_1, z] \circ [\delta_3, x] \\
&= \bigcup_{z \in g_1\delta_2 g_2} \{[\delta_1, t] : t \in z\delta_3 g_3\} \\
&= \bigcup_{t \in (g_1\delta_2 g_2)\delta_3 g_3} [\delta_1, t] \\
&= \bigcup_{t \in g_1\delta_2(g_2\delta_3 g_3)} [\delta_1, t] \\
&= [\delta_1, g_1] \circ ([\delta_2, g_2] \circ [\delta_3, g_3]).
\end{aligned}$$

Therefore, $(\Theta[G], \circ)$ is a semihypergroup.

Let $\circ$ be a scalar hyperoperation $\circ : \Theta[G] \times \Theta[X] \to P^*(\Theta[X])$ such that

$$[\delta_1, g] \circ [\delta_2, x] = \{[\delta_1, z] : z \in g\delta_2 x\}.$$

This scalar hyperoperation is well-defined. Indeed, let $[\delta_1, g_1] = [\delta_2, g_2]$ and $[\delta_3, x_1] = [\delta_4, x_2]$ such that $g_1, g_2 \in G$, $\delta_1, \delta_2 \in \Delta$, $x_1, x_2 \in X$ and $\delta_3, \delta_4 \in \Delta$. Then,

$$g\delta_1 g_1 = g\delta_2 g_2, \quad g\delta_3 x_1 = g\delta_4 x_2, \quad \text{for all } g \in G.$$

This implies that $(g\delta_1 g_1)\delta_3 x_1 = (g\delta_2 g_2)\delta_4 x_2$. Hence,

$$[\delta_1, g_1] \circ [\delta_3, x_1] = [\delta_2, g_2] \circ [\delta_4, x_2].$$

Thus the scalar hyperoperation $\circ$ is well-defined. Let $[\delta_1, g_1], [\delta_2, g_2] \in \Theta[G]$ and $[\delta_3, x] \in \Theta[X]$, where $\delta_1, \delta_2 \in \Gamma$. Then,

$$\begin{aligned}
([\delta_1, g_1] \circ [\delta_2, g_2]) \circ [\delta_3, x] &= (\{[\delta_1, z] : z \in g_1\delta_2 g_2\}) \circ [\delta_3, x] \\
&= \bigcup_{z \in g_1\delta_2 g_2} [\delta_1, z] \circ [\delta_3, x] \\
&= \bigcup_{z \in g_1\delta_2 g_2} \{[\delta_1, t] : t \in z\delta_3 x\} \\
&= \bigcup_{t \in (g_1\delta_2 g_2)\delta_3 x} [\delta_1, t] \\
&= \bigcup_{t \in g_1\delta_2 (g_2\delta_3 x)} [\delta_1, t] \\
&= [\delta_1, g_1] \circ ([\delta_2, g_2] \circ [\delta_3, x]).
\end{aligned}$$

Therefore, $\Theta[X]$ is a left $\Theta[G]$-set and is called *fundamental left* (Δ, G)*-set*.

Let $\Theta[X]$ be a fundamental left (Δ, G)-set, $H \subseteq \Theta[X]$ and $T \subseteq X$. Then, we define

$$\begin{aligned}
[H] =& \{x \in X : [\delta, x] \in H \text{ for all } \delta \in \Delta\}, \\
[[T]] =& \{[\delta, x] \in \Theta[X] : g\delta x \subseteq T \text{ for all } g \in G\}.
\end{aligned}$$

A nonempty subset T of a left (Δ, G)-set X is called left (Δ, G)-subset of X when $G\Delta T \subseteq T$. A nonempty subset H of $\Theta[X]$ is called left $\Theta[G]$-subset if $\Theta[G] \circ H \subseteq H$.

Proposition 4.1. *Let X be a left (Δ, G)-set and $H \subseteq \Theta[X]$ be a complete part. Then, $[H]$ is a complete part of X.*

Proof. Suppose that

$$[H] \cap \prod_{i=1}^{n} g_i\delta_i x \neq \emptyset.$$

This implies that there exists $a \in X$ such that $a \in [H] \cap \prod_{i=1}^{n} g_i\delta_i x$. Then, for every $\delta \in \Delta$, $[\delta, a] \in H$. This implies that

$$[\delta, a] \in H \cap \prod_{i=1}^{n} [\delta, g_i] \circ [\delta_i, x].$$

Since $[H]$ is a complete part, $\prod_{i=1}^{n} [\delta, g_i] \circ [\delta_i, x] \subseteq H$. Then,

$$\left\{ b \in \prod_{i=1}^{n} g_i\delta_i x : \forall \delta \in \Delta, [\delta, b] \right\} \subseteq H.$$

Therefore, $[H]$ is a complete part. □

Proposition 4.2. *Let X be a left (Δ, G)-set and $T \subseteq X$ is a complete part. Then, $[[T]]$ is also a complete part of $\Theta[X]$.*

Proof. Suppose that

$$[[T]] \cap \prod_{i=1}^{n} [\delta_i, g_i] \circ [\delta, x] \neq \emptyset.$$

This implies that

$$\begin{aligned}\left\{[\delta_1, z] : z \in \prod_{i=1}^{n} g_i\delta x\right\} \cap [[T]] \neq \emptyset &\Rightarrow \left(\exists z \in \prod_{i=1}^{n} g_i\delta x\right)\ [\delta_1, z] \in [[T]]\\ &\Rightarrow \left(\exists z \in \prod_{i=1}^{n} g_i\delta x\right) (\forall g \in G)\ \ g\delta_1 z \subseteq T\\ &\Rightarrow g\delta \prod_{i=1}^{n} g_i\delta x \cap T \neq \emptyset\\ &\Rightarrow (\forall g \in G)\ g\delta \prod_{i=1}^{n} g_i\delta x \subseteq T\\ &\Rightarrow \prod_{i=1}^{n} [\delta_i, g_i] \circ [\delta, x] \neq \emptyset \subseteq [[T]].\end{aligned}$$

Therefore, $[[T]]$ is also complete part of $\Theta[X]$. □

Proposition 4.3. *Let X be a left (Δ, X)-set such that $T \subseteq X$. Then, $C[[T]] = [[C(T)]]$.*

Proof. Since $C(T)$ is a complete part by Proposition 4.2, $[[C(T)]]$ is also complete part of $\Theta[X]$. Also, $[[T]] \subseteq [[C(T)]]$. Let T_1 be a complete part contain $[[T]]$. Hence, $C[[T]] \subseteq T_1$. Thus, $[[C(T)]]$ is a smallest compte part contain $[[T]]$. Therefore, $C[[T]] = [[C(T)]]$. □

Theorem 4.1. *Let X be a left (Δ, G)-set and $\Theta[X]$ be a fundamental left (Δ, G)-set. Then,*

(i) *If H is a left $\Theta[G]$-subset of $\Theta[X]$, then $[H]$ is a left (Δ, G)-subset of X;*
(ii) *If T is a left (Δ, G)-subset of X, then $[[T]]$ is a left $\Theta[G]$ of $\Theta[X]$.*

Proof. (i) Suppose that $x \in [H]$. Then, for every $\delta \in \Delta$ we have $[\delta, x] \in H$. Since H is a left $\Theta[G]$-set of $\Theta[X]$, thus $[\delta_1, g] \circ [\delta, x] \subseteq H$. So $\{[\delta_1, t] : t \in g\delta x\} \subseteq H$. This implies that $g\delta x \subseteq [H]$. Therefore, $[H]$ is a left (Δ, G)-set of X.
(ii) Let $[\delta, x] \in [[T]]$ and $[\delta_1, g] \in \Theta[G]$. Then, for all $g \in G$, $g\delta x \subseteq T$. Now,

$$[\delta_1, g] \circ [\delta, x] = \{[\delta_1, t] : t \in g\delta x\} \subseteq [[T]].$$

Therefore, $[[T]]$ is a left $\Theta[G]$-subset of $\Theta[X]$. □

Let X be a left (Δ, G)-set and T be a nonempty subset of X. Then,

$$[[[T]]] = \{x \in X : \forall\delta \in \Delta, [\delta, x] \in [[T]]\} = \{x \in X : g\delta x \subseteq T \text{ for all } \delta \in \Delta, g \in G\}.$$

This implies that T is a left (Δ, G)-subset of $[[[T]]]$. Also, when $H \subseteq \Theta[X]$, we have

$$\begin{aligned}[[[H]]] &= \{[\delta, x] \in \Theta[X] : g\delta x \subseteq [H] \text{ for all } g \in G\}\\ &= \{[\delta, x] \in \Theta[X] : [\delta_1, t] \in H \text{ for all } g \in G, \delta_1 \in \Delta, t \in g\delta x\}.\end{aligned}$$

Let H be a left $\Theta[G]$-subset of $\Theta[X]$. Then, for every $\delta_1 \in \Gamma$, $g \in G$ and $[\delta, x] \in H$ we have

$$[\delta_1, g] \circ [\delta, x] = \{[\delta_1, t] : t \in g\delta x\} \subseteq H.$$

When H is a left $\Theta[G]$-subset of $\Theta[X]$, we have $H \subseteq [[[H]]]$.

Let X be a left (Δ, G)-set such that e_α is a unit element of G where $\alpha \in \Gamma$. Then,

$$[\delta, e_\alpha] \circ [\delta, x] = [\delta, e_\alpha \delta x] = [\delta, x].$$

This implies that $[\delta, e_\alpha]$ is a left unity of $\Theta[X]$.

Proposition 4.4. *Let X be a left (Δ, G)-set and T be a left (Δ, G)-subset of X. Then, $[[[T]]] = T$.*

Proof. The proof is straightforward. □

Definition 4.1. Let X be a left (Δ, G)-set. Then, X is said Noetherian, when X satisfies the ascending chain condition on left (Δ, G)-subsets and X is said Artinian when X satisfies the descending chain condition.

Theorem 4.2. *Let X be a left (Δ, G)-set such that $\Theta[X]$ is Noetherian (Artinian) $\Theta[G]$-set. Then, X is Noetherian left (Δ, G)-set.*

Proof. Suppose that $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots \subseteq X_n \subseteq \cdots$ be an ascending chain of left (Δ, G)-set of X. Hence $[X_1] \subseteq [X_2] \subseteq [X_3] \subseteq \cdots \subseteq [X_n] \cdots$ is an ascending chain in $\Theta[X]$. Since $\Theta[X]$ is Noetherian thus there exists a positive integer n such that $[X_n] = [X_{n+k}]$ for every $k \in \mathbb{N}$. This implies that $X_n = [[[X_n]] = [[X_{n+k}]]] = X_{n+k}$ for every $k \in \mathbb{N}$. Therefore, X is Noetherian left (Δ, G)-set. In a same way, when X is Artinian left (Δ, G)-set, then $\Theta[X]$ is also $\Theta[G]$-set. □

Corollary 4.1. *Let X be a left (Δ, G)-set and $\Theta[X]$ is Artinian $\Theta[G]$-set. Then, X is Artinian left (Δ, G)-set.*

Definition 4.2. Let X be a left (Δ, G)-set and A be a nonempty subset of X. Then, intersection of all ideals of X containing A is a left (Δ, G)-set generated by A and denoted by $< A >$.

Proposition 4.5. *Let X be a left (Δ, G)-set and $A \subseteq X$. Then, $< A >= G\Delta A$.*

Proof. Suppose that $H = G\Delta A$. Obviously, $A \subseteq H$ and H is a left (Δ, G)-set of X. Indeed,

$$G\Delta H = G\Delta(G\Delta A) = (G\Gamma G)\Delta A \subseteq G\Delta A = H.$$

Let C be a left (Δ, G)-subset of X such that $A \subseteq C$. Then,

$$H = G\Delta A \subseteq G\Delta C \subseteq C.$$

Therefore, H is a smallest left (Δ, G)-set contain A and $H =< A >$. □

Let X be a left (Δ, G)-set and every nonempty of left (Δ, G)-subset of X partially ordered by inclusion has a maximal element. Then, we say that maximum condition holds for left (Δ, G)-sets.

Theorem 4.3. *Let X be a left (Δ, G)-set. Then, the following conditions are equivalent:*

(i) *X is Noetherian;*
(ii) *X satisfies the maximum condition for left (Δ, G)-sets;*
(iii) *every left (Δ, G)-subset of X is finitely generated.*

Proof. (i)$\Rightarrow$(ii) Suppose that Λ is a nonempty set of left (Δ, G)-subsets which has no maximal element. Let $\Lambda_1 \in \Lambda$. Then, there exists an element $\Lambda_2 \in \Lambda$ such that $\Lambda_1 \subset \Lambda_2$. Also, there exists an element $\Lambda_3 \in \Lambda$ such that $\Lambda_2 \subset \Lambda_3$. By continuing this process we have the accenting chain $\Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \subset \cdots$. This is impossible.

(ii)$\Rightarrow$(iii) Let X_1 be a left (Δ, G)-set and $\Omega = \{< A >: A$ is a finite subset of $X_1\}$. By (ii), Ω has a maximal element $< A_0 >$. Now, if $x \in X_1$, then $< A_0 \cup \{x\} > \in \Omega$. By Maximality of $< A_0 >$ we have $x \in < A_0 >$. Therefore, X_1 is finite generated.

(iii)$\Rightarrow$(i) Suppose that $X_1 \subseteq X_2 \subseteq \cdots$ is a accenting chain of left (Δ, G)-sets and $T = \bigcup_{n \geq 1} X_n$. One can see that T is a left (Δ, G)-set of X. By (iii), T is finite generated. Then, there exist $x_1, x_2, \ldots, x_n \in X$ such that $T = < x_1, x_2, \ldots, x_n >$. Hence for $1 \leq k \leq n$ there exists X_k such that $x_k \in X_{i_k}$. We put $m := \max\{i_1, i_2, \ldots, i_n\}$. Hence, for every $t \geq m$ we have $I_m = I_t$. ☐

Theorem 4.4. *Let Ω be a partition (Δ, G)-set such that $\Omega = \bigcup_{t \in X} A_t$. Then, H is a left (Δ, G)-subset of X if and only if $\Omega_H = \bigcup_{t \in H} A_t$ is a left (Δ, G) of Ω.*

Proof. Suppose that H is a left (Δ, G)-set of X. Then,

$$G\widehat{\Delta}\Omega_H = G\widehat{\Delta} \bigcup_{t \in H} A_t = \bigcup_{t \in H} G\widehat{\Delta} A_t = \bigcup_{t \in G\Delta H} A_t \subseteq \bigcup_{t \in H} A_t = \Omega_H.$$

Hence Ω_H is a left (Δ, G)-subset of Ω.

Conversely, suppose that Ω_H is a left (Δ, G)-subset of Ω, $g \in G$, $\delta \in \Delta$ and $h \in H$. Choose $x \in A_h$. Since Ω_H is a left (Δ, G)-subset of Ω_H, we have

$$g\widehat{\delta}x = \{A_z : z \in g\delta h\} \subseteq \Omega_H.$$

Hence ,$g\delta h \subseteq H$. ☐

Corollary 4.2. *Let Ω be a partition (Δ, G)-set such that X is Noetherian (Artinian) (Δ, G)-set. Then, Ω is Noetherian (Artinian).*

References

[1] S. M. Anvariyeh, S. Mirvakili and B. Davvaz, *On Γ-hyperideals in Γ-semihypergroups*, Carpathian J. Math. **26** (2010), 11–23.
[2] S. Chattopadhyay, *Right inverse Γ-semigroup*, Bull. Calcutta Math. Soc. **93** (2001), 435–442.
[3] S. Chattopadhyay, *Right orthodox Γ-semigroup*, Southeast Asian Bull. Math. **29** (2005), 23–30.
[4] P. Corsini and V. Leoreanu, *Applications of Hyperstructure Theory*, Advances in Mathematics **5**, Kluwer Academic Publishers, Dordrecht, 2003.
[5] P. Corsini, *Prolegomena of Hypergroup Theory*, Second Edition, Aviani Editore, Tricesimo, 1993.
[6] B. Davvaz and V. Leoreanu-Fotea, *Hyperring Theory and Applications*, International Academic Press, Palm Harbor, FL, USA, 2007.

[7] S. O. Dehkordi and B. Davvaz, *A strong regular relation on* Γ*-semihyperrings*, J. Sci. Islam. Repub. Iran **22**(3) (2011), 257–266.
[8] S. O. Dehkordi and B. Davvaz, Γ*-semihyperrings: approximations and rough ideals*, Bull. Malays. Math. Sci. Soc. (2) **35**(4) (2012), 1035–1047.
[9] S. Ostadhadi-Dehkordi, *Twist product derived from* Γ*-semihypergroup*, Kragujevac J. Math. **42**(4) (2018), 607–617.
[10] S. Ostadhadi-Dehkordi, *Direct limit derived from* Γ*-semihypergroup*, Kragujevac J. Math. **40**(1) (2016), 61–72.
[11] S. O. Dehkordi and B. Davvaz, *Ideal theory in* Γ*-semihyperrings*, Iran. J. Sci. Technol. **37A3** (2013), 251–263.
[12] S. Ostadhadi-Dehkordi and B. Davvaz, Γ*-semihyperrings: ideals, homomorphisms and regular relations*, Afr. Mat. **26** (2015), 849–861.
[13] S. Ostadhadi-Dehkordi and B. Davvaz, *Quotient* (Γ, R)*-hypermodules and homology algebra*, Afr. Mat. **27**(3-4) (2016), 353–364.
[14] D. Heidari, S. O. Dehkordi and B. Davvaz, Γ*-semihypergroups and their properties*, UPB Scientific Bulletin, Series A: Applied Mathematics and Physics **72**(1) (2010), 195–208.
[15] K. Hila, *On regular, semiprime and quasi-refexive* Γ*-semigroup and minimal quasi ideals*, Lobachevski J. Math. **29** (2008), 141–152.
[16] K. Hila, *On some classes of le-*Γ*-semigroups*, Algebras, Groups, and Geometries **24** (2007), 485–495.
[17] F. Marty, *Sur une generalization de la notion de group*, in: *Proceedings of* 8^{th} *Congres of Mathematics*, Scandinaves, 1934, 45–49.
[18] A. Seth, Γ*-group congruences on regular* Γ*-semigroups*, Int. J. Math. Math. Sci. **15** (1992), 103–106.

Department of mathematics,
University of Hormozgan,
P. O. Box 3991, Bandar Abbas, Iran
Email address: ostadhadi-Dehkordi@hotmail.com
Email address: ostadhadi@hormozgana.ac.ir