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NOVEL WAVELET APPROACH FOR SOLVING FRACTIONAL BAGLEY-TORVIK PROBLEMS

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Abstract. The primary purpose of this study is to construct truncated solutions for fractional Bagley-Torvik problems (FBT) by developing a novel method including newly defined Clique wavelets and collocation points. Clique wavelets are defined by utilizing Clique polynomials on $[0, 1]$. The convergence of this method is investigated and supported by illustrative examples through tables and figures. As a result, the efficiency and effectiveness of the method is proved by theorems and examples.

1. Introduction

In the mathematical modelling of various processes in physics,geophysics, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electrodynamics of complex medium, visco-elasticity, etc., fractional differential equations (FDE's) are utilized in common since they reflects the behaviour of the processes much more better than ordinary differential equations [1–13]. The properties of FDE's such as hereditary properties make them to play a significant role to analyze the present and future development of the processes [14, 15]. Therefore, the complicated processes can be modelled by FDE's with less difficulty compared with nonlinear differential equations [16]. As a result, many researchers use FDE's as an excellent tool to model and analyze a number of processes in science. However, to acquire analytical solutions of FDE's is much more difficult compared with ordinary differential equations. Consequently, a number of numerical methods such as Adomian decomposition method [17], homotopy perturbation method [18], the generalized

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Taylor collocation [19], a fractional linear multi-step method and a predictor-corrector (PC) method of the Adams type method [20], the Haar wavelet method [21, 22], an iterative reproducing kernel algorithm [23], etc. have been established to construct numerical or approximate solutions for them.

Wavelets can be defined as oscillatory functions with compact support. Diverse polynomials such as Chebyshev, Legendre, Hermite, Bernstein and Lucas polynomials [24–32] have been used in wavelet theory to develop some wavelet schemes to construct truncated or analytical solutions of differential equations. Generally orthogonality properties of these polynomials leads to a bases for important spaces in which a wide range of problems in system analysis, optimal control, numerical analysis, signal analysis and time-frequency analysis, etc., are solved analytically or approximately [33, 34].

FBT, proposed by Bagley and Torvik, is used to model visco-elastically damped structures where visco-elasticity is defined by a fractional derivative. FBT allows us to analyze the qualitative behavior of real material. Therefore it plays a prominent role in engineering and applied sciences. The order of the fractional derivative of damping term in FBT leads to the model of various processes but in FBT it is equal to $\frac{3}{2}$ which implies that damping depends on frequency. Moreover, in the modeling of the behavior of rigid plate or a gas embedded in a viscous fluid in a fluid [9,23]

In this study, we take a different approach by constructing Clique Wavelets to solve fractional Bagley-Torvik equation which have been used commonly in visco-elasticity theory. The novel contribution of this research is that the Clique wavelets are defined and used together with collocation points first time in this study. Even though Clique polynomials introduced by Fisher and Solow [35] are not orthogonal polynomials, they form a bases for $L_2[0, 1]$ which allows us to define Clique wavelets.

The following fractional FBT in Caputo sense is taken into consideration:

$$D^2 y(t) + \lambda_1 D^\alpha y(t) + \lambda_2 y(t) = f(t), \tag{1.1}$$

$$y(0) = \mu_1, \quad \kappa_1 y'(0) + \kappa_2 y(1) = \mu_2, \tag{1.2}$$

where $0 < \alpha \leq 2$, the operator D^α denotes the Caputo fractional derivative of order α, λ_1 and λ_2 are constants and the function f is continuous on the interval $[0, 1]$.

The present paper is outline as follows. The fundamental definitions are given in Preliminaries. In Section 3, the establishment of wavelets which is used in suggested method is illustrated. In Section 4, the implementation and algorithm of proposed method is presented. The convergence analysis is given in Section 5. In the final section, four different examples are presented to illustrate the implementation and the effectiveness of the proposed method.

2. Preliminaries

Definition 2.1. The Riemann-Liouville integral for α is [3, 36]:

$$J^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, & \alpha > 0, \\ f(t), & \alpha = 0. \end{cases} \tag{2.1}$$

Definition 2.2. The α^{th} order fractional derivative in Caputo sense is given by [3, 36]

$$D^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, & m-1 < \alpha < m, m \in \mathbb{N}, \\ \frac{d^m f}{dt^m}(t), & \alpha = m. \end{cases} \tag{2.2}$$

Definition 2.3. The Clique polynomials on a bounded interval of the real line $[a, b]$, $b > a \geq 0$, are defined as follows:

$$P_n(t) = \sum_{k=0}^{n} \binom{n}{k} t^k. \tag{2.3}$$

In general, the clique polynomial is given by,

$$P_n(t) = (1+t)^n = \binom{n}{0} + \binom{n}{1} t + \binom{n}{2} t^2 + \cdots + \binom{n}{n} t^n. \tag{2.4}$$

For $n = 0, 1$ in (2.4), we get $P_0(t) = 1$ and $P_1(t) = 1 + t$. This set of polynomials have the following recursive formulation

$$P_{n+1}(t) = (1+t) P_n(t), \quad n = 0, 1, \ldots \tag{2.5}$$

3. Construction of Clique Wavelets

In this section, Clique polynomials on the interval $[0, 1]$ are utilized to establish Clique wavelets $W_{j,i}(t)$ for $j = 0, 1, \ldots, 2^k - 1$ and $i = 0, 1, \ldots, n$ on $[0, 1)$ as in the following form:

$$\Psi_{j,i}(t) = \begin{cases} \frac{1}{\sqrt{\sigma_i}} 2^{\frac{k}{2}} P_i(2^k t - j), & \text{if } t \in \left[\frac{j}{2^k}, \frac{j+1}{2^k}\right), \\ 0, & \text{otherwise}, \end{cases} \tag{3.1}$$

where k and n are non-negative integers and $\sigma_i = \frac{2^{2n}-1}{2n+1}$.

The sets of Clique wavelets are bases for $L_2[0, 1]$. The followings present the related Clique wavelets for $k = 1$ and $n = 2$

$$\Psi_{0,0}(t) = \begin{cases} \frac{1}{\sqrt{\sigma_0}} \sqrt{2} P_0(2t), & \text{if } t \in \left[0, \frac{1}{2}\right), \\ 0, & \text{if } t \in \left[\frac{1}{2}, 1\right), \end{cases} \tag{3.2}$$

$$\Psi_{0,1}(t) = \begin{cases} \frac{1}{\sqrt{\sigma_1}} \sqrt{2} P_1(2t), & \text{if } t \in \left[0, \frac{1}{2}\right), \\ 0, & \text{if } t \in \left[\frac{1}{2}, 1\right), \end{cases} \tag{3.3}$$

$$\Psi_{0,2}(t) = \begin{cases} \frac{1}{\sqrt{\sigma_2}} \sqrt{2} P_2(2t), & \text{if } t \in \left[0, \frac{1}{2}\right), \\ 0, & \text{if } t \in \left[\frac{1}{2}, 1\right), \end{cases} \tag{3.4}$$

$$\Psi_{1,0}(t) = \begin{cases} 0, & \text{if } t \in \left[0, \frac{1}{2}\right), \\ \frac{1}{\sqrt{\sigma_0}}\sqrt{2}P_0(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right), \end{cases} \tag{3.5}$$

$$\Psi_{1,1}(t) = \begin{cases} 0, & \text{if } t \in \left[0, \frac{1}{2}\right), \\ \frac{1}{\sqrt{\sigma_1}}\sqrt{2}P_1(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right), \end{cases} \tag{3.6}$$

$$\Psi_{1,2}(t) = \begin{cases} 0, & \text{if } t \in \left[0, \frac{1}{2}\right), \\ \frac{1}{\sqrt{\sigma_2}}\sqrt{2}P_2(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right). \end{cases} \tag{3.7}$$

Therefore, an approximation of a function y in $L_2[0,1)$ can be constructed in terms of Clique wavelets as:

$$y_n(t) \approx \sum_{j=0}^{2^k-1} \sum_{i=0}^{n} c_{j,i}\Psi_{j,i}(t), \tag{3.8}$$

where the coefficients $c_{j,i}$ are obtained by the following inner product

$$c_{j,i} = \langle \Psi_{j,i}(t), y_n(t) \rangle = \int_0^1 \Psi_{j,i}(t) y_n(t) dt. \tag{3.9}$$

4. Implementation of the Proposed Method

In the establishment of the approximate solution y for the problem (1.1)–(1.2) in terms of special polynomials as

$$\sum_{j=0}^{2^k-1} \sum_{i=0}^{n} c_{j,i}\Psi_{j,i}(t) \tag{4.1}$$

the following steps below are taken.

Step 1. Substituting the n^{th} degree approximation of (4.1) into the (1.1) yields the following equation:

$$\sum_{j=0}^{2^k-1} \sum_{i=0}^{n} c_{j,i}\Psi''_{j,i}(t) + \lambda_1 \sum_{j=0}^{2^k-1} \sum_{i=0}^{n} c_{j,i}D^{\alpha}\Psi_{j,i}(t) + \lambda_2 \sum_{j=0}^{2^k-1} \sum_{i=0}^{n} c_{j,i}\Psi_{j,i}(t) = f(t), \tag{4.2}$$

where $m-1 < \alpha \leqslant m$.

Step 2. Collocating (4.2) at the nodes $t_r = \frac{r}{n}$, $r = 0, 1, 2, \ldots, n$, leads to the following system of fractional ordinary differential equations:

$$\sum_{j=0}^{2^k-1} \sum_{i=0}^{n} c_{j,i}\Psi''_{j,i}(t_r) + \lambda_1 \sum_{j=0}^{2^k-1} \sum_{i=0}^{n} c_{j,i}D^{\alpha}\Psi_{j,i}(t_r) + \lambda_2 \sum_{j=0}^{2^k-1} \sum_{i=0}^{n} c_{j,i}\Psi_{j,i}(t_r) = f(t_r), \tag{4.3}$$

where $m-1 < \alpha \leqslant m$.

Step 3. Plugging the n^{th} degree approximation of (4.2) into in the initial and boundary conditions (1.2) yields the following system of algebraic equations, we can obtain $([\alpha]+1)$ equations as follows:

$$\sum_{j=0}^{2^k-1}\sum_{i=0}^{n} c_{j,i}\Psi_{j,i}(0) = \mu_1, \tag{4.4}$$

$$\kappa_1 \sum_{j=0}^{2^k-1}\sum_{i=0}^{n} c_{j,i}\Psi'_{j,i}(0) + \kappa_2 \sum_{j=0}^{2^k-1}\sum_{i=0}^{n} c_{j,i}\Psi_{j,i}(1) = \mu_2. \tag{4.5}$$

Step 4. Finally, a system of fractional algebraic equations are obtained. The unknown coefficients $c_{j,i}$, $j = 0,1,\dots,2^k-1$, $i = 0,1,\dots,n$, are determined by RPSM which allows us to construct the approximate solution $y_n(x,t)$.

5. Convergence and Error Analysis

The primary purpose of this section is to investigate the convergence analysis of the Clique polynomials in L_2 norm for fractional BTP. In other words, the series for an approximation of a function in terms of Clique polynomials converges to the exact solution as it is expanded by increasing the number of base elements. The definition of $L_2([0,1])$ is given as [37]

$$L_2([0,1]) = \{\ell : [0,1] \to \mathbb{R} \mid \ell \text{ is a measurable and } \|\ell\| < +\infty\}, \tag{5.1}$$

where

$$\|\ell\|^2 = \int_0^1 |\ell(t)|^2 dt \tag{5.2}$$

indicates the induced norm related to the following inner product of the space $L_2([0,1])$:

$$\langle \ell(t), r(t)\rangle = \int_0^1 \ell(t)r(t)dt. \tag{5.3}$$

We first consider a finite-dimensional subspace of $L_2([0,1])$ of the form

$$\begin{aligned}\mathcal{C}_N = \mathrm{Span}\langle &\Psi_{0,0}(t), \Psi_{0,1}(t), \dots, \Psi_{0,N}(t), \Psi_{1,0}(t), \Psi_{1,1}(t), \dots, \Psi_{1,N}(t), , \dots, \\ &\Psi_{2^k-1,0}(t), \Psi_{2^k-1,1}(t), \dots, \Psi_{2^k-1,N}(t)\rangle.\end{aligned} \tag{5.4}$$

Clearly, $\dim(\mathcal{C}_N) = (2^k-1)\times N$ and $\mathcal{C}_N$ is a complete subspace of $L_2([0,1])$ since it is closed and finite-dimensional. Every function $u \in L_2([0,1])$ has a unique best approximation $u^* \in \mathcal{C}_N$ in the following sense

$$\|u(t) - u^*(t)\| \leqslant \|u(t) - v(t)\|, \quad \text{for all } v \in \mathcal{C}_N. \tag{5.5}$$

Theorem 5.1. *Let ℓ_N denote the interpolating function of $u \in C^N([0,1])$ at N Chebyshev nodes in the interval $[0,1]$. Then, for every $t \in [0,1]$, we have*

$$\|u(t) - \ell_N(t)\| \leqslant \frac{\|u\|_\infty}{2^{2N-1}N!}, \tag{5.6}$$

where $\|u\|_\infty := \max_{t\in[0,1]} |u^{(N)}(t)|$.

Since, the sets of Clique wavelets form a bases for $L_2([0,1])$, *every function* $u \in L_2([0,1])$ *can be represented by the series form in terms of Clique wavelet polynomials as*

$$u(t) = \sum_{j=0}^{+\infty}\sum_{i=0}^{+\infty} c_{j,i}\Psi_{j,i}(t). \tag{5.7}$$

The restriction of it to the finite dimensional subspace $\mathcal{C}_N$ *of* $L_2([0,1])$, *a truncated series of* u *can be written as*

$$u(t) \approx u_N(t) = \sum_{j=0}^{2^k-1}\sum_{i=0}^{N} c_{j,i}\Psi_{j,i}(t). \tag{5.8}$$

Proof. See [38,39]. □

Theorem 5.2. *For the best approximation* u_N *to* $u \in C^N([0,1]) \cap L_2([0,1])$ *in the space* $\mathcal{C}_N$, *the following inequality holds*

$$\|E_N(t)\|^2 = \|u(t) - u_N(t)\|^2 \leq \|u(t) - v(t)\|^2, \quad \textit{for all } v \in \mathcal{C}_N. \tag{5.9}$$

The above inequality holds for $v = \ell \in \mathcal{C}_N$. *As a result, we deduce that*

$$\|E_N(t)\|^2 \leq \int_0^1 \left|\frac{\|u\|_\infty}{2^{2N+1}(N+1)!}\right|^2 dt \leq \left[\frac{\|u\|_\infty}{2^{2N+1}(N+1)!}\right]^2. \tag{5.10}$$

As N *tends to infinity, the desired result is obtained.*

6. Illustrative Examples

The primary aim of this section is to present the implementation of the method by illustrative examples and check their accuracy.

Example 6.1. Consider the following fractional BTP as [3,41]:

$$D^{\frac{3}{2}}y(t) + D^2y(t) + y(t) = 7t + t^3 + \frac{8t^{\frac{3}{2}}}{\sqrt{\pi}} + 1, \tag{6.1}$$

$$y(0) = y'(0) = 1, \tag{6.2}$$

for which $y(t) = t^3 + t + 1$ denotes the analytic solution.

The Clique wavelet solution of the problem (6.1)–(6.2) is the excellent truncated solution with higher accuracy for the exact solution compare to the other the existing methods. It is seen that the Figure 1 that as the number of Clique wavelets increases in the subspace $\mathcal{C}_N$, the truncated solutions get closer to the exact solution. As it is presented in Table 1 that the accuracy of the presented method is higher than the other existing methods.

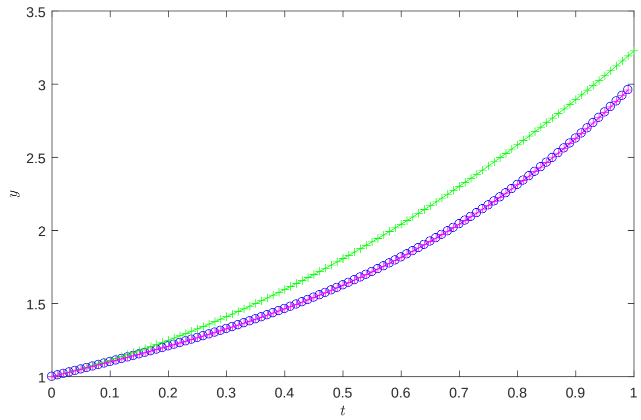

FIGURE 1. A Graphical presentation of truncated solutions (in green and pink) with regard to $n = 2, 3$ and exact solution (in blue) for Example 6.1.

TABLE 1. Comparison of results for $n = 3$.

t	VIM [44]	FIM [44]	SLC[43]	LWS[45]	CWM
0.01	0.08214	2.76e-3	2.57e-02	1.99e-15	2.22e-16
0.25	0.17315	3.42e-3	1.51e-01	1.22e-14	4.44e-16
0.5	0.10515	1.01e-3	5.44e-01	4.90e-14	0
0.75	1.34104	4.98e-3	1.083448	1.10e-13	0
1	4.11359	5.01e-3	1.676130	1.96e-13	0

Example 6.2. Consider the following fractional BTP as [40]:

$$D^2y(t) - \frac{2}{5}D^{\frac{3}{2}}y(t) - \frac{1}{2}y(t) = -\frac{1}{2}t^3 + \frac{3}{4}t^2 + \frac{183}{32}t - 3 - \frac{4}{5}\cdot\frac{\sqrt{t}(-3+4t)}{\sqrt{\pi}}, \tag{6.3}$$

$$y(0) = 0, \quad y'(0) = \frac{9}{16}, \tag{6.4}$$

for which $y(t) = t^3 - \frac{3}{2}t^2 + \frac{9}{16}t$ denotes to the analytic solution.

The Clique wavelet solution of the problem (6.3)–(6.4) is the excellent truncated solution with higher accuracy for the exact solution compare to the other the existing methods. It is seen that the Figure 2 that as the number of Clique wavelets increases in the subspace $\mathcal{C}_N$, the truncated solutions get closer to the exact solution.

As it is presented in Table 2 that the accuracy of the presented method is higher than the other existing methods.

Example 6.3. Consider the following fractional BTP as [41]:

$$D^2y(t) + D^{\frac{3}{2}}y(t) + y(t) = t^3 + 5t + 8\frac{t^{\frac{3}{2}}}{\sqrt{\pi}}, \tag{6.5}$$

$$y(0) = y'(0) = 0, \tag{6.6}$$

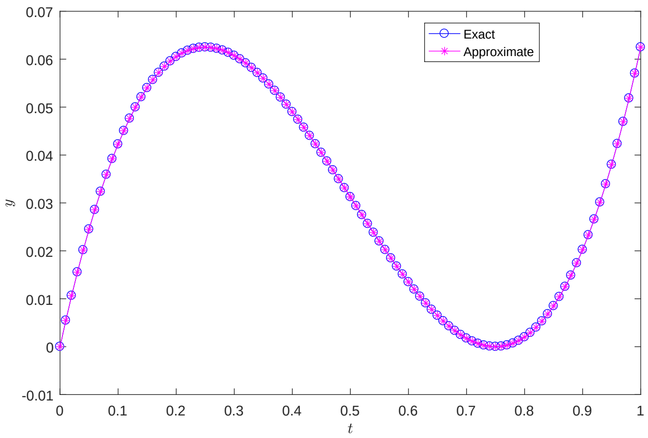

FIGURE 2. A Graphical presentation of truncated solutions (in pink) with regard to $n = 4$ and exact solution (in blue) for Example 6.2.

TABLE 2. Comparison of results for various n.

x	$N = 2$	$N = 3$	$N = 4$
0.1	1.71e-02	7.13e-04	6.25e-16
0.25	9.74e-02	3.70e-03	5.00e-16
0.50	3.27e-01	9.77e-03	1.11e-16
0.75	5.95e-01	1.07e-02	1.67e-16
1	8.09e-01	1.11e-13	2.22e-16

for which $y(t) = t(t^2 - 1)$ denotes the analytic solution.

The Clique wavelet solution of the problem (6.5)–(6.6) is the excellent truncated solution with higher accuracy for the exact solution compare to the other the existing methods. It is seen from the Figure 1, as the number of Clique wavelets increases in the subspace $\mathcal{C}_N$, the truncated solutions get closer to the exact solution. As it is presented in Table 3 that the accuracy of the presented method is higher than the other existing methods.

Example 6.4. Consider the following fractional BTP as [42]:

$$D^2y(t) - D^\alpha y(t) = -1 - E_\alpha((t-1)^\alpha), \quad 0 < \alpha \leqslant 1, \tag{6.7}$$

$$y(0) = y(1) = 0, \tag{6.8}$$

for which $y(t) = t(1 - E_\alpha((t-1)^\alpha))$ denotes to the analytic solution.

The Clique wavelet solution of the problem (6.7)–(6.8) is the excellent truncated solution with higher accuracy for the exact solution compare to the other existing methods. It is seen from the Figure 4, as the number of Clique wavelets increases in the subspace $\mathcal{C}_N$, the truncated solutions get closer the exact solution. As it is presented in Table 4 that the accuracy of the presented method is higher than the other existing methods.

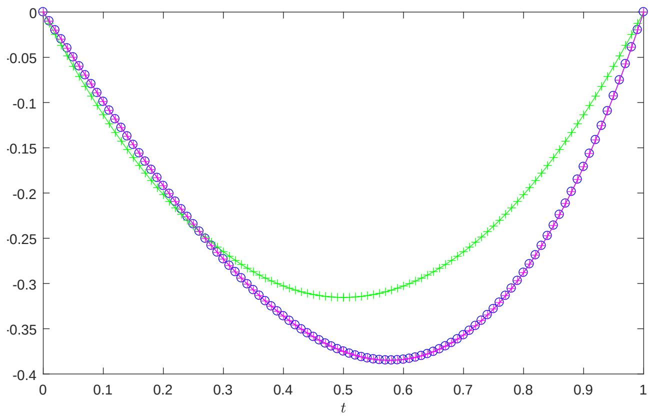

FIGURE 3. A Graphical presentation of truncated solutions (in green and pink) with regard to $n = 2, 3$ and exact solution (in blue) for Example 6.3

TABLE 3. Comparison of results for various n.

x	$N = 3$	[45]
0.2	5.551e-17	9.575e-15
0.4	1.665e-16	3.758e-14
0.6	1.110e-16	8.426e-14
0.8	0	1.497e-13
1	0	0

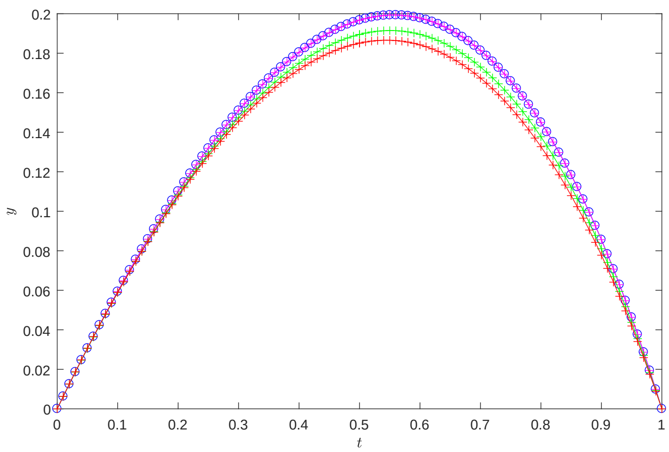

FIGURE 4. A Graphical presentation of truncated solutions (in pink, green and red) with regard to $\alpha = 1, 0.9, 9.8$ and exact solution (in blue) for Example 6.4.

7. CONCLUSION

In this research, a novel method is developed by defining Clique wavelets and using them with collocation points to construct truncated solutions of the Bagley-Torvik

Table 4. Comparison of results for various n.

x	$N=2$	$N=3$	$N=4$	$N=5$
0.1	1.29e-02	2.71e-03	2.47e-04	9.35e-05
0.25	1.86e-02	3.63e-03	5.47e-04	8.75e-05
0.50	4.07e-03	1.20e-01	2.76e-05	4.05e-05
0.75	1.53e-02	9.60e-02	1.05e-04	2.51e-05
1	0	0	4.16e-17	2.22e-16

initial value problems. Its convergence analysis is also presented and supported by illustrative examples. It is shown that obtained truncated solutions have higher accuracy compare to other methods. As a result, the effectiveness and accuracy of this Clique wavelets method is investigated and illustrated. This effective approach can be also utilized to obtain other fractional problems in applied sciences. In the future work, this method will be taken into consideration to construct truncated solutions of time-fractional initial value problems and inverse problems.

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