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THE FAMILY OF SZÁSZ-DURRMEYER TYPE OPERATORS INVOLVING CHARLIER POLYNOMIALS

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Abstract. In this paper, we consider Szász-Durrmeyer type operators based on Charlier polynomials associated with Srivastava-Gupta operators [17]. For the considered operators, we discuss error of estimation by using first and second order modulus of continuity, Lipchtiz-type space, Ditzian-Totik modulus of smoothness, Voronovskaya type asymptotic formula and weighted modulus of continuity.

1. Introduction

For the Charlier polynomials [8], the generating functions are as follows:

$$e^u\left(1-\frac{u}{a}\right)^t=\sum_{j=0}^{\infty}C_j^{[a]}(t)\frac{u^j}{j!}, \tag{1.1}$$

where $C_j^{[a]}(t)=\sum\limits_{r=0}^{j}\binom{j}{r}(-t)_r\frac{1}{a^r}$ and $(j)_0=1$, $(j)_i=j(j+1)(j+2)\cdots(j+i-1)$ for $i\geq 1$.

Suppose $\gamma>0$, the space $C_\gamma[0,\infty):=\left\{g\in C[0,\infty):|g(t)|\leq Me^{\gamma t}\right\}$ for some $M>0$.

In view of Charlier polynomials, Varma and Tasdelen [19] proposed a sequence of linear positive operators for $g\in C_\gamma[0,\infty)$ as follows:

$$L_n(g;x,a)=e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{j=0}^{\infty}\frac{C_j^{[a]}(-(a-1)nx)}{j!}g\left(\frac{j}{n}\right), \tag{1.2}$$

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where $a > 1$ and $x \in [0, \infty)$. For sufficiently large a, if we replace x by $x - \frac{1}{n}$ the above operators reduce to well-known Szász-Mirakyan operators [18].

In [17], Srivastava and Gupta introduced a new family of linear positive operators as follows:

$$G_n^c(g;x) = (n-c)\sum_{j=0}^{\infty} p_{n,j}(x;c)\int_0^{\infty} p_{n+c,j-1}(u;c)du + p_{n,0}(x;c)g(0), \tag{1.3}$$

where $p_{n,j}(x;c) = \frac{(-x)^j}{j!}\phi_{n,c}^{(j)}(x)$ and

$$\phi_{n,c}(x) = \begin{cases} e^{-nx}, & c = 0, \\ (1+cx)^{\frac{-n}{c}}, & c = 1,2,3,\ldots \end{cases}$$

For the operators (1.3), they also studied the rate of convergence for the functions of bounded variation. Ispir and Yüksel [10] defined the Bézier varient of the Srivastava-Gupta operators and discussed rate of convergence for the functions of bounded variation. Srivastava-Gupta [17] contains several well-known operators for different values of c. Many authors have proposed various forms and modifications of the above operators and studied several local and global approximation results. For more (see [1,3,7,12,14,16,20,21]).

Motivated from the above stated work, we define a linear positive operators for $g \in C_B[0,\infty)$ as follows:

$$\begin{aligned} G_{n,c}^{[a]}(g;x) =&(n-c)e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\left[\sum_{j=1}^{\infty}\frac{C_j^{[a]}(-(a-1)nx)}{j!}\int_0^{\infty} p_{n+c,j-1}(u;c)g(u)du\right. \\ &\left. +C_0^{[a]}g(0)\right]. \end{aligned} \tag{1.4}$$

In above operators, it can easily be seen that if we take $c = 1$, we obtain Szász-Durrmeyer type operators involving Charlier Polynomials which were proposed by Kajla and Agrawal [11] and studied several approximation results like Vorovskaya type asymptotic theorem, local approximation, statistical rate of convergence and functions of bounded variation. For more articles based on Charlier polynomials (see [2,4]).

The main purpose of this article is to define the operators (1.4) and discuss the approximation results using the first and second order modulus of continuity, Lipschitz-type space, Ditzian-Totik modulus of smoothness, Voronovskaya-type formula and weighted approximation.

2. Auxiliary Results

Lemma 2.1 ([19]). *For the operators $L_n(\cdot;x,a)$, we have*

(i) $L_n(1;x,a) = 1$;
(ii) $L_n(u;x,a) = x + \frac{1}{n}$;

(iii) $L_n(u^2;x,a) = x^2 + \frac{x}{n}\left(3+\frac{1}{a-1}\right) + \frac{2}{n^2}$;

(iv) $L_n(u^3;x,a) = x^3 + \frac{x^2}{n}\left(6+\frac{3}{a-1}\right) + \frac{x}{n^2}\left(5+\frac{3}{a-1}+\frac{1}{(a-1)^2}\right) + \frac{5}{n^3}$;

(v)

$$\begin{aligned} L_n(u^4;x,a) =& x^4 + \frac{x^3}{n}\left(10+\frac{6}{a-1}\right) + \frac{x^2}{n^2}\left(31+\frac{30}{a-1}+\frac{11}{(a-1)^2}\right) \\ &+ \frac{x}{n^3}\left(37+\frac{31}{a-1}+\frac{20}{(a-1)^2}+\frac{6}{(a-1)^3}\right) + \frac{15}{n^4}. \end{aligned}$$

Lemma 2.2. *The moments of the operators $G_{n,c}^{[a]}(u^i;x)$, $i=0,1,2,3,4$, are as follows:*

(i) $G_{n,c}^{[a]}(1;x) = 1$;

(ii) $G_{n,c}^{[a]}(u;x) = \frac{1}{(n-2c)}(nx+2)$;

(iii) $G_{n,c}^{[a]}(u^2;x) = \frac{1}{(n-2c)(n-3c)}\left(n^2x^2 + n\left(6+\frac{1}{a-1}\right)x + 7\right)$;

(iv)

$$\begin{aligned} G_{n,c}^{[a]}(u^3;x) =& \frac{1}{(n-2c)(n-3c)(n-4c)}\left(n^3\,x^3 + 3n^2\left(4+\frac{1}{a-1}\right)x^2\right. \\ &\left.+n\left(28+\frac{12}{a-1}+\frac{2}{(a-1)^2}\right)x+34\right); \end{aligned}$$

(v)

$$\begin{aligned} G_{n,c}^{[a]}(u^4;x) =& \frac{1}{(n-2c)(n-3c)(n-4c)(n-5c)}\left(n^4x^4 + 2n^3\left(10+\frac{3}{a-1}\right)x^3\right. \\ &+ n^2\left(126+\frac{60}{a-1}+\frac{11}{(a-1)^2}\right)x^2 \\ &\left.+ n\left(292+\frac{126}{a-1}+\frac{40}{(a-1)^2}+\frac{6}{(a-1)^3}\right)x+209\right). \end{aligned}$$

Lemma 2.3. *The central moments for the defined operators:*

(i) $G_{n,c}^{[a]}(u-x;x) = \frac{2}{(n-2c)}(1+cx)$;

(ii) $G_{n,c}^{[a]}((u-x)^2;x) = \frac{1}{(n-2c)(n-3c)}\left(c(n+6c)x^2 + \left(n\left(2+\frac{1}{a-1}\right)+12c\right)x + 7\right)$;

(iii)

$$\begin{aligned} G_{n,c}^{[a]}((u-x)^4;x) =& \frac{1}{(n-2c)(n-3c)(n-4c)(n-5c)}\Big((3n^2+86cn+126c^2)c^2x^4 \\ &+ \frac{2c(3(2a-1)n^2+4c(43a-28)n+240(a-1)c^2)}{(a-1)}x^3 \\ &+ \frac{(56a^2-100a+47)n^2+2c(91a^2-62a-9)n+840(a-1)^2c^2}{(a-1)^2}x^2 \\ &+\frac{2(78a^3-171a^2+128a-32)n+680c(a-1)^3}{(a-1)^3}x+209\Big). \end{aligned}$$

Lemma 2.4. *For sufficiently large n, we have*

(i) $\lim_{n\to\infty} nG_{n,c}^{[a]}((u-x);x) = 2(1+cx)$;

(ii) $\lim_{n\to\infty} nG_{n,c}^{[a]}((u-x)^2;x) = x\left(cx+\frac{1}{a-1}+2\right)$;

(iii) $\lim_{n\to\infty} n^2G_{n,c}^{[a]}((u-x)^4;x) = x^2\left(3c^2x^2+\frac{3(2a-1)}{(a-1)}x+\frac{56a^2-100a+47}{(a-1)^2}\right)$.

3. Main Result

Theorem 3.1. *Let $g\in C_\gamma[0,\infty)$ and for sufficiently large n the operators $G_{n,c}^{[a]}(g(u);x)$ converges to $g(x)$ uniformly in each compact subset of $[0,\infty)$.*

Proof. From Lemma 2.2, $\lim_{n\to\infty} G_{n,c}^{[a]}(1;x) = 1$, $\lim_{n\to\infty} G_{n,c}^{[a]}(u;x) = x$ and $\lim_{n\to\infty} G_{n,c}^{[a]}(u^2;x) = x^2$. Then by Bohman-Korovokin theorem, $G_{n,c}^{[a]}(g(u);x)$ converges to $g(x)$ uniformly in each compact subset of $[0,\infty)$. ☐

Theorem 3.2. *For $g\in C_\gamma[0,\infty)$ and $g'(x)$, $g''(x)$ exist in $[0,\infty)$, we have*

$$\left[G_{n,c}^{[a]}(g(u);x)-g(x)\right] = 2(1+cx)g'(x)+\frac{x}{2!}\left(cx+\frac{1}{a-1}+2\right)g''(x).$$

Proof. From Taylor's expansion, we have

$$g(u)=g(x)+(u-x)g'(x)+\frac{(u-x)^2g''(x)}{2!}+r(u,x)(u-x)^2,$$

where $r(u,x)$ converges to 0 when $u\to x$.

Applying $G_{n,c}^{[a]}(\cdot;x)$ in above expression, we have

$$n\left[G_{n,c}^{[a]}(g(u);x)-g(x)\right] = nG_{n,c}^{[a]}((u-x);x)g'(x)+\frac{nG_{n,c}^{[a]}((u-x)^2;x)g''(x)}{2!} + nG_{n,c}^{[a]}(r(u,x)(u-x)^2;x). \tag{3.1}$$

Using Cauchy-Schwarz inequality and Lemma 2.4 in last term of above equality, we obtain

$$\lim_{n\to\infty} nG_{n,c}^{[a]}(r(u,x)(u-x)^2;x)=0. \tag{3.2}$$

From (3.1), using (3.2) and Lemma 2.4, we get the required result. ☐

Let $C_B[0,\infty)$ be the space of real valued continuous and bounded functions g on $[0,\infty)$, provided with norm

$$\|g\| = \sup_{x\in[0,\infty)} |g(x)|,$$

and Peetre's K-functional for $g\in C_B[0,\infty)$ is given as:

$$K_2(g;\delta)=\inf_{x\in W_\infty^2}\left\{\|g-h\|+\delta\|h''\|\right\},\quad \delta>0,$$

where $W^2_\infty[0,\infty) = \{h \in C_B[0,\infty) : h', h'' \in C_B[0,\infty)\}$. Devore and Lorentz [5, Theorem 2.4, page 177], provided relation between Peetre's K functional and second order modulus of continuity as follows:

$$K_2(g;\delta) \leq C\omega_2(g;\sqrt{\delta}), \tag{3.3}$$

and the second order modulus of continuity $\omega_2(g;\sqrt{\delta})$ is given as

$$\omega_2(g;\sqrt{\delta}) = \sup_{0<i\leq\delta} \sup_{x\in[0,\infty)} |g(x+2i) - 2g(x+i) + g(x)|.$$

The usual modulus of continuity $\omega(g;\delta)$ for $g \in C_B[0,\infty)$

$$\omega(g;\delta) = \sup_{0<i\leq\delta} \sup_{x\in[0,\infty)} |g(x+i) - g(x)|.$$

Theorem 3.3. *For $g \in C_B[0,\infty)$ and $a > 1$, we have*

$$\left|G^{[a]}_{n,c}(g(u);x) - g(x)\right| \leq C\omega_2\left(g\sqrt{\delta^a_{n,c}(x)}\right) + \omega\left(g;\left|\frac{2(1+cx)}{(n-2c)}\right|\right),$$

where C is positive constant and $\delta^a_{n,c}(x) = \left[G^{[a]}_{n,c}((u-x)^2;x) + \frac{2(1+cx)^2}{(n-2c)^2}\right]$.

Proof. We consider an auxiliary operators:

$$\tilde{G}^{[a]}_{n,c}(g(u);x) = G^{[a]}_{n,c}(g(u);x) - g\left(x + \frac{2(1+cx)}{n-2c}\right) + g(x).$$

The Taylor's expansion for the function $h \in W^2_\infty[0,\infty)$ is given as

$$h(u) = h(x) + (u-x)h'(x) + \int_x^u (u-x)h''(u)du.$$

Applying $\tilde{G}^{[a]}_{n,c}(\cdot;x)$ in above expression

$$\tilde{G}^{[a]}_{n,c}(h(u);x) - h(x) = \tilde{G}^{[a]}_{n,c}((u-x);x)h'(x) + \tilde{G}^{[a]}_{n,c}\left(\int_x^u (u-x)h''(u)du;x\right).$$

Since $\tilde{G}^{[a]}_{n,c}(1;x) = 1$, $\tilde{G}^{[a]}_{n,c}(u;x) = x$ and $\tilde{G}^{[a]}_{n,c}(u-x;x) = 0$, we get

$$\begin{aligned}\left|\tilde{G}^{[a]}_{n,c}(h(u);x) - h(x)\right| &= \left|\tilde{G}^{[a]}_{n,c}\left(\int_x^u (u-x)h''(u)du;x\right)\right| \\ &\leq \left|G^{[a]}_{n,c}\left(\int_x^u (u-x)h''(u)du;x\right)\right| \\ &\quad + \left|\int_x^{x+\frac{2(1+cx)}{n-2c}} \left(x + \frac{2(1+cx)}{n-2c} - u\right)h''du\right|\end{aligned}$$

$$\leq \left[G_{n,c}^{[a]}((u-x)^2;x) + \frac{2(1+cx)^2}{(n-2c)^2} \right] \|h''\|$$

$$\leq \delta_{n,c}^{a}(x) \|h''\| . \tag{3.4}$$

Using auxiliary operators we can write

$$\begin{aligned}\left| G_{n,c}^{[a]}(g(u);x) - g(x) \right| \leq & \left| \tilde{G}_{n,c}^{[a]}(g-h;x) - (g-h)(x) \right| + \left| \tilde{G}_{n,c}^{[a]}(h(u);x) - h(x) \right| \\ & + \left| g\left(x + \frac{2(1+cx)}{n-2c}\right) - g(x) \right| \\ \leq & 2|g-h| + \delta_{n,c}^{a}(x)\|h''\| + \omega\left(g; \frac{2(1+cx)}{n-2c}\right).\end{aligned}$$

Taking infimum on the right hand side of the above inequality for $g \in W_\infty^2[0,\infty)$, we have

$$\left| G_{n,c}^{[a]}(g(u);x) - g(x) \right| = 2K_2\left(g; \delta_{n,c}^{a}(x)\right) + \omega\left(g, \left|\frac{2(1+cx)}{n-2c}\right|\right).$$

From (3.3), we obtain

$$\left| G_{n,c}^{[a]}(g(u);x) - g(x) \right| = C\omega_2\left(g; \sqrt{\delta_{n,c}^{a}(x)}\right) + \omega\left(g, \left|\frac{2(1+cx)}{n-2c}\right|\right).$$

Hence, the proof. ☐

In the next theorem, we estimate global rate of convergence by using Ditzian-Totik modulus of smoothness $\omega_{\phi^\alpha}(g;\delta)$ for $g \in C_B[0,\infty)$, $0 < \alpha \leq 1$ and $\phi(x) = \sqrt{x(1+cx)}$ which is defined as:

$$\omega_{\phi^\alpha}(g;\delta) = \sup_{0\leq s\leq \delta} \sup_{x \pm \frac{s\phi^\alpha(x)}{2} \in [0,\infty)} \left| g\left(x + \frac{s\phi^\alpha(x)}{2}\right) - g\left(x - \frac{s\phi^\alpha(x)}{2}\right) \right|,$$

and the Peetre K-functional is defined as:

$$K_{\phi^\alpha}(g;\delta) = \inf_{g \in W_\alpha} \left\{ \|g-h\| - \delta \|\phi^\alpha g'\| \right\},$$

where W_α is subspaces of those functions which are locally absolutely continuous on $g \in [0,\infty)$ with the normed $\|\phi^\alpha g'\| \leq \infty$. In [6], there exists a constant $C > 0$ such that

$$C^{-1}\omega_{\phi^\alpha}(g;\delta) \leq K_{\phi^\alpha}(g;\delta) \leq C\omega_{\phi^\alpha}(g;\delta).$$

Theorem 3.4. *Suppose $g \in C_B[0,\infty)$ and for sufficiently large n, we have*

$$\left| G_{n,c}^{[a]}(g;x) - g(x) \right| \leq C\omega_{\phi^\alpha}\left(g; \frac{\phi^{1-\alpha}(x)}{\sqrt{n}}\right).$$

Proof. For $h \in W_\alpha$, we have

$$h(u) = h(x) + \int_x^u h'(t)dt.$$

Applying $G_{n,c}^{[a]}(\cdot;x)$ in the above equality and using Hölder's inequality, we obtain

$$\begin{aligned}\left|G_{n,c}^{[a]}(h;x)-h(x)\right| \leq & G_{n,c}^{[a]}\left(\int_x^u h'(t)dt;x\right)\\ \leq & \left\|\phi^\alpha h'\right\| G_{n,c}^{[a]}\left(\int_x^u \frac{dt}{\phi^\alpha(t)};x\right)\\ \leq & \left\|\phi^\alpha h'\right\| G_{n,c}^{[a]}\left(|u-x|^{1-\alpha}\left|\int_x^u \frac{dt}{\phi(t)}\right|^\alpha;x\right). \end{aligned} \tag{3.5}$$

Let $p(u,x)=\left|\int_x^u \frac{dt}{\phi(t)}\right|$, we have

$$\begin{aligned} p(u,x) \leq & \left|\int_x^u \frac{dt}{\phi(t)}\right|\left(\frac{1}{\sqrt{1+cx}}+\frac{1}{\sqrt{1+cu}}\right)\\ \leq & \frac{2\,|u-x|}{\sqrt{x}+\sqrt{u}}\left(\frac{1}{\sqrt{1+cx}}+\frac{1}{\sqrt{1+cu}}\right)\\ \leq & \frac{2\,|u-x|}{\sqrt{x}}\left(\frac{1}{\sqrt{1+cx}}+\frac{1}{\sqrt{1+cu}}\right). \end{aligned}$$

Since $|a+b|^\alpha \leq |a|^\alpha+|b|^\alpha$, $0\leq\alpha\leq 1$, and from the above inequality, we obtain

$$\left|\int_x^u \frac{dt}{\phi(t)}\right|^\alpha \leq \frac{2^\alpha|u-x|^\alpha}{x^{\frac{\alpha}{2}}}\left(\frac{1}{(1+cx)^{\frac{\alpha}{2}}}+\frac{1}{(1+cu)^{\frac{\alpha}{2}}}\right). \tag{3.6}$$

From (3.5), (3.6) and using Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left|G_{n,c}^{[a]}(h;x)-h(x)\right| \leq & \frac{2^\alpha\left\|\phi^\alpha h'\right\|}{x^{\frac{\alpha}{2}}}G_{n,c}^{[a]}\left(|u-x|\left(\frac{1}{(1+cx)^{\frac{\alpha}{2}}}+\frac{1}{(1+cu)^{\frac{\alpha}{2}}}\right);x\right)\\ \leq & \frac{2^\alpha\left\|\phi^\alpha h'\right\|}{x^{\frac{\alpha}{2}}}\left(\frac{1}{(1+cx)^{\frac{\alpha}{2}}}\left(G_{n,c}^{[a]}((u-x)^2;x)\right)^{\frac{1}{2}}\right.\\ & \left.+\left(G_{n,c}^{[a]}((u-x)^2;x)\right)^{\frac{1}{2}}\times\left(G_{n,c}^{[a]}((1+cu)^{-\alpha};x)\right)^{\frac{1}{2}}\right). \end{aligned} \tag{3.7}$$

From Theorem 3.1, $G_{n,c}^{[a]}((1+cu)^{-\alpha}$ converges to $(1+cx)^{-\alpha}$ for sufficiently large n. Thus, for $\epsilon>0$, there exist $n_0\in N$ such that $G_{n,c}^{[a]}((1+cu)^{-\alpha};x)\leq(1+cx)^{-\alpha}+\epsilon$ for all $n\geq n_0$.

Choosing $\varepsilon=(1+cx)^{-\alpha}$, we get

$$G_{n,c}^{[a]}((1+cu)^{-\alpha};x)\leq 2(1+cx)^{-\alpha}, \quad \text{for all } n\geq n_0. \tag{3.8}$$

For sufficiently large n there exists a constant $C>0$, such that

$$G_{n,c}^{[a]}((u-x)^2;x)\leq C\frac{\phi^2(x)}{n}. \tag{3.9}$$

From (3.7) to (3.9), we obtain

$$\left|G_{n,c}^{[a]}(h;x)-h(x)\right|\le 2^{\alpha+1}C\left\|\phi^{\alpha}h'\right\|\frac{\phi^{1-\alpha}(x)}{\sqrt{n}}. \tag{3.10}$$

We can write

$$\begin{aligned}\left|G_{n,c}^{[a]}(g;x)-g(x)\right|\le&\left|G_{n,c}^{[a]}(g-h;x)\right|+\left|G_{n,c}^{[a]}(h;x)-h(x)\right|+|h(x)-g(x)|\\ \le&2\left\|g-h\right\|+\left|G_{n,c}^{[a]}(h;x)-h(x)\right|.\end{aligned}$$

From (3.10), we get

$$\begin{aligned}\left|G_{n,c}^{[a]}(g;x)-g(x)\right|\le&2\left\|g-h\right\|+2^{\alpha+1}C\left\|\phi^{\alpha}h'\right\|\frac{\phi^{1-\alpha}(x)}{\sqrt{n}}\\ \le&C\left\{\left\|g-h\right\|+\frac{\phi^{1-\alpha}(x)}{\sqrt{n}}\left\|\phi^{\alpha}h'\right\|\right\}\\ \le&CK_{\phi}^{\alpha}\left(g;\frac{\phi^{1-\alpha}(x)}{\sqrt{n}}\right)\\ \le&C{\omega_{\phi}}^{\alpha}\left(g;\frac{\phi^{1-\alpha}(x)}{\sqrt{n}}\right).\end{aligned}$$

Hence, the proof. □

In [15], the Lipschitz-type space for positive real numbers β_1, β_2 is defined as:

$$Lip_M^{\beta_1,\beta_2}(\lambda)=\left\{g\in C_{\mathbf{B}}[0,\infty):|g(u)-g(x)|\le M_g\frac{|u-x|^{\lambda}}{(u+\beta_1x^2+\beta_2x)^{\frac{\lambda}{2}}};x,u\in[0,\infty)\right\},$$

where $M_g>0$ and $0<\lambda\le 1$.

Theorem 3.5. *Let $g\in Lip_M^{\beta_1,\beta_2}(\lambda)$ and $0<\lambda\le 1$, then for $x\ge 0$ we have*

$$\left|G_{n,c}^{[a]}(g;x)-g(x)\right|\le M_g\left(\frac{\mu_{n,2}^{a,c}(x)}{(\beta_1x^2+\beta_2x)}\right)^{\frac{\lambda}{2}},$$

where $\mu_{n,2}^{a,c}(x)=G_{n,c}^{[a]}((u-x)^2;x)$.

Proof. First, we discuss the result for $\lambda=1$. For $g\in Lip_M^{\beta_1,\beta_2}(\lambda)$, we have

$$\begin{aligned}\left|G_{n,c}^{[a]}(g;x)-g(x)\right|\le&(n-c)e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{j=0}^{\infty}\frac{C_j^{[a]}(-(a-1)nx)}{j!}\\ &\times\int_0^{\infty}p_{n,j}(u;c)\left|g(u)-g(x)\right|du\\ \le&(n-c)e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{j=0}^{\infty}\frac{C_j^{[a]}(-(a-1)nx)}{j!}\end{aligned}$$

$$\times \int_0^\infty p_{n,j}(u;c) M_g \frac{|u-x|}{(u+\beta_1 x^2+\beta_2 x)^{\frac{1}{2}}} du.$$

Since $\frac{1}{\sqrt{u+\beta_1 x^2+\beta_2 x}} < \frac{1}{\sqrt{\beta_1 x^2+\beta_2 x}}$, applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\left|G_{n,c}^{[a]}(g;x)-g(x)\right| \le & \frac{M_g}{\sqrt{\beta_1 x^2+\beta_2 x}}(n-c)e^{-1}\Big(1-\frac{1}{a}\Big)^{(a-1)nx}\sum_{j=0}^{\infty}\frac{C_j^{[a]}(-(a-1)nx)}{j!} \\
& \times \int_0^\infty p_{n,j}(x;c)\,|u-x|\,du \\
\le & \frac{M_g}{\sqrt{\beta_1 x^2+\beta_2 x}}\sqrt{G_{n,c}^{[a]}((u-x)^2;x)} \\
\le & M_g\sqrt{\frac{\mu_{n,2}^{a,c}(x)}{\beta_1 x^2+\beta_2 x}}.
\end{aligned}$$

The result is true for $\lambda=1$. Now we prove for $0<\lambda<1$. Using Hölder's inequality with $p=\frac{2}{\lambda}$ and $q=\frac{2}{2-\lambda}$, we have

$$\begin{aligned}
& \left|G_{n,c}^{[a]}(g;x)-g(x)\right| \\
\le & \Bigg\{n-c)e^{-1}\Big(1-\frac{1}{a}\Big)^{(a-1)nx}\sum_{j=0}^{\infty}\frac{C_j^{[a]}(-(a-1)nx)}{j!} \\
\le & M_g\Bigg\{(n-c)e^{-1}\Big(1-\frac{1}{a}\Big)^{(a-1)nx}\sum_{j=0}^{\infty}\frac{C_j^{[a]}(-(a-1)nx)}{j!} \\
& \times \int_0^\infty p_{n,j}(x;c)\frac{(u-x)^2}{(u+\beta_1 x^2+\beta_2 x)}du\Bigg\}^{\frac{\lambda}{2}} \\
\le & \frac{M_g}{(\beta_1 x^2+\beta_2 x)^{\frac{\lambda}{2}}}\Bigg\{(n-c)e^{-1}\Big(1-\frac{1}{a}\Big)^{(a-1)nx}\sum_{j=0}^{\infty}\frac{C_j^{[a]}(-(a-1)nx)}{j!} \\
& \times \int_0^\infty p_{n,j}(x;c)(u-x)^2 du\Bigg\}^{\frac{\lambda}{2}} \\
\le & M_g\left(\frac{\mu_{n,2}^{a,c}(x)}{(\beta_1 x^2+\beta_2 x)}\right)^{\frac{\lambda}{2}}.
\end{aligned}$$

Hence, the proof. ∎

Theorem 3.6. *If $g(x)$ is continuously differentiable function on $[0,\infty)$ and $|g'(x)| \leq D$ for some $D > 0$, then we have*

$$\left|G_{n,c}^{[a]}(g;x) - g(x)\right| \leq D\left|\frac{2(1+cx)}{n-2c}\right| + 2\sqrt{\mu_{n,2}^{a,c}(x)}\omega_b\left(g';\sqrt{\mu_{n,2}^{a,c}(x)}\right),$$

where $\omega_b(g;\delta)$, $\delta > 0$, is usual modulus of continuity on $[0,b]$ and

$$\mu_{n,2}^{a,c}(x) = G_{n,c}^{[a]}((u-x)^2;x).$$

Proof. From Lagrange's mean value theorem, we get

$$g(u) - g(x) = (u-x)g'(\eta) = (u-x)g'(x) + (u-x)(g'(\eta) - g'(x)),$$

where η lies between x and u.

now, we apply $G_{n,c}^{[a]}(\cdot;x)$ on both side of the above equation. Since $x < \eta < u$ we have $|\eta - x| < |u-x|$ and

$$\left|G_{n,c}^{[a]}(g;x) - g(x)\right| \leq |g'(x)|\left|G_{n,c}^{[a]}((u-x);x)\right| + \omega_b(g';\delta)\left(|u-x| + \frac{(u-x)^2}{\delta}\right).$$

Applying Cauchy-Schwarz inequality, we obtain

$$\left|G_{n,c}^{[a]}(g;x) - g(x)\right| \leq D\left|\frac{2(1+cx)}{n-2c}\right| + \sqrt{\mu_{n,2}^{a,c}(x)}\omega_b(g';\delta)\left(1 + \frac{\sqrt{\mu_{n,2}^{a,c}(x)}}{\delta}\right).$$

Taking $\delta = \sqrt{\mu_{n,2}^{a,c}(x)}$, we get required result. □

In our next theorem, we study the rate of convergence for the operators (1.4) based on Lipscitz maximal function of order r given by Lenze [13] as

$$\varpi_r(g;x) = \sup_{u \neq x,\, x,u \in [0,\infty)} \frac{|g(u) - g(x)|}{|u-x|^r}, \tag{3.11}$$

where $0 < r \leq 1$.

Theorem 3.7. *For $g \in C_B[0,\infty)$, we have*

$$\left|G_{n,c}^{[a]}(g;x) - g(x)\right| \leq \varpi_r(g;x)\Big(\mu_{n,2}^{a,c}(x)\Big)^r.$$

Proof. From (3.11), we obtain

$$\left|G_{n,c}^{[a]}(g;x) - g(x)\right| \leq \varpi_r(g;x)G_{n,c}^{[a]}\left(|u-x|^r;x\right).$$

Using Hölder's inequality with $p = \frac{2}{r}$ and $q = \frac{2}{2-r}$, we obtain

$$\left|G_{n,c}^{[a]}(g;x) - g(x)\right| \leq \varpi_r(g;x)\Big(G_{n,c}^{[a]}\left((u-x)^2;x\right)\Big)^r \leq \varpi_r(g;x)\Big(\mu_{n,2}^{a,c}(x)\Big)^r.$$

Hence, the proof. □

Let $C_2[0,\infty)$ be the space of all continuous functions on $[0,\infty)$ and defined as:

$$C_2[0,\infty) := \left\{g : |g| \leq M_g(1+x^2)\right\},$$

where M_g is positive constant which may depends on g with the norm

$$\|g\|_2 = \sup_{x>0} \frac{|g(x)|}{1+x^2}.$$

Let $C_2^*[0,\infty) := \left\{g \in C_2[0,\infty) : \lim_{x\to\infty} \frac{g(x)}{1+x^2} \text{ exists and finite}\right\}$. The weighted modulus of continuity [9] $\Omega(g;\delta)$ is given as

$$\Omega(g;\delta) = \sup_{0\leq\beta<\delta} \frac{|g(x+\beta)-g(x)|}{(1+\beta^2)(1+x^2)}.$$

Lemma 3.1. *For every $g \in C_2^*[0,\infty)$, $\Omega(g;\delta)$ has the properties:*

(i) *$\Omega(g;\delta)$ is a monotonically increasing function of δ;*
(ii) *$\lim_{\delta\to 0^+} \Omega(g;\delta) = 0$;*
(iii) *$\Omega(g;k\delta) \leq 2(1+k)(1+\delta^2)\Omega(g;\delta)$, $k>0$ and $\delta>0$.*

Theorem 3.8. *For $g \in C_2^*[0,\infty)$, we have*

$$\sup_{x\in[0,\infty)} \frac{\left|G_{n,c}^{[a]}(g;x)-g(x)\right|}{(1+x^2)^{\frac{5}{2}}} \leq C\Omega\left(g;\frac{1}{\sqrt{n}}\right),$$

where C is positive constant depends on a and c.

Proof. For $x,u \in [0,\infty)$ and from (3.11), we can write

$$\begin{aligned}|g(u)-g(x)| \leq& (1+(u-x)^2)(1+x^2)\Omega\left(g;\frac{|u-x|\,\delta}{\delta}\right)\\ \leq& 2(1+\delta^2)(1+x^2)\left(1+\frac{|u-x|}{\delta}\right)(1+(u-x)^2)\Omega(g;\delta).\end{aligned}$$

Applying $G_{n,c}^{[a]}(\cdot;x)$ in the above inequality, we have

$$\begin{aligned}\left|G_{n,c}^{[a]}(g;x)-g(x)\right| \leq& 2(1+\delta^2)(1+x^2)\Omega(g;\delta)G_{n,c}^{[a]}\left(\left(1+\frac{|u-x|}{\delta}\right)(1+(u-x)^2);x\right)\\ \leq& 2(1+\delta^2)(1+x^2)\Omega(g;\delta)\left\{G_{n,c}^{[a]}(1;x)+G_{n,c}^{[a]}((u-x)^2;x)\right.\\ &+\left.\frac{1}{\delta}G_{n,c}^{[a]}\left(|u-x|\,;x\right)+\frac{1}{\delta}G_{n,c}^{[a]}\left(|u-x|\,(u-x)^2;x\right)\right\}\\ \leq& 2(1+\delta^2)(1+x^2)\Omega(g;\delta)\left\{1+G_{n,c}^{[a]}((u-x)^2;x)\right.\\ &+\frac{1}{\delta}\Big(G_{n,c}^{[a]}\left((u-x)^2;x\right)\Big)^{\frac{1}{2}}\\ &\left.+\frac{1}{\delta}\Big(G_{n,c}^{[a]}\left((u-x)^2;x\right)\Big)^{\frac{1}{2}}\Big(G_{n,c}^{[a]}\left((u-x)^4;x\right)\Big)^{\frac{1}{2}}\right\}.\end{aligned} \tag{3.12}$$

From Lemma 2.3, we have

$$G_{n,c}^{[a]}((u-x)^2;x) \leq \frac{C_1(1+x^2)}{n}$$

and

$$G_{n,c}^{[a]}((u-x)^4;x) \leq \frac{C_2(1+x^2)^2}{n^2},$$

where C_1 and C_2 are positive constants depend on a and c.

Using the above inequality in (3.12) and taking $\delta = \frac{1}{\sqrt{n}}$, we get

$$\begin{aligned}\left|G_{n,c}^{[a]}(g;x)-g(x)\right| \leq & 2\left(1+\frac{1}{n}\right)\Omega\left(g;\frac{1}{\sqrt{n}}\right)(1+x^2)\{1+C_1\,(1+x^2) \\ & +\sqrt{C_1(1+x^2)}+\sqrt{C_1C_2}(1+x^2)^{\frac{3}{2}}\right\}.\end{aligned}$$

Taking $C = 4\left(1+C_1+\sqrt{C_1}+\sqrt{C_1C_2}\right)$, we obtain the result. □

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