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ON A CLASS OF GENERALIZED CAPILLARITY PHENOMENA INVOLVING FRACTIONAL ψ-HILFER DERIVATIVE WITH $p(\cdot)$-LAPLACIAN OPERATOR

ELHOUSSAIN ARHRRABI[1,*] AND HAMZA EL-HOUARI[1]

Abstract. This research delves into a comprehensive investigation of a class of ψ-Hilfer generalized fractional nonlinear eigenvalue equation originated from a capillarity phenomenon with Dirichlet boundary conditions. The nonlinearity of the problem, in general, do not satisfies the Ambrosetti-Rabinowitz (AR) type condition. Using critical point theorem with variational approach and the (S_+) property of the operator, we establish the existence of positive solutions of our problem with respect to every positive parameter ξ in appropriate fractional ψ-Hilfer spaces. Our main results is novel and its investigation will enhance the scope of the literature on differential equation of fractional ψ-Hilfer generalized capillary phenomena.

1. Introduction

Differential equations are essential in present-day physics, engineering, and various scientific fields, as highlighted in [5]. The ongoing evolution in modern physics and mechanics, as discussed in [1,8–11,16], is leading to changes in traditional areas, which necessitates the development of new mathematical models (see [12–22]). For instance, a reevaluation of the study of capillary phenomena in current literature underscores the pressing need for such advancements. Capillary action, also referred to as capillarity, capillary motion, capillary effect, or wicking, is the ability of a liquid to flow through narrow spaces without external forces, including gravity, and sometimes even against them. This phenomenon can be observed in various situations, such as the upward

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flow of liquids between the bristles of a paintbrush, through thin tubes, within porous materials like paper and plaster, and certain non-porous substances like sand and liquefied carbon fiber, as well as within biological cells. Capillary action is propelled by the intermolecular forces between the liquid and the surrounding solid surfaces. When the diameter of a tube is sufficiently small, a combination of surface tension arising from cohesion within the liquid and adhesive forces between the liquid and the container's wall collaborates to propel the liquid upward. The ascent of water in narrow tubes, and the formation of liquid drops or bubbles, can be effectively analyzed using variational calculus. This method, which hinges on the energy-minimizing nature of observed equilibrium configurations, offers a comprehensive framework for addressing mathematical questions pertinent to diverse phenomena. Recently, the study of capillary phenomena has attracted increased attention, spurred not only by the intrigue surrounding naturally occurring events such as the motion of drops, bubbles, and waves, but also by its practical importance in various applied fields, including industrial, biomedical, pharmaceutical, and microfluidic systems. Given the vast scope of this subject, we will focus on select examples to illustrate key concepts for those interested (see [2, 27, 29, 39]).

The most recent examination of this problem involves applying the $p(\cdot)$-Laplacian and incorporating fractional derivatives in the time dimension. As an example, the authors in [26] employed Ricceri's variational principle, originally due to Bonanno and Molica Bisci, to establish the existence of at least one weak solution and infinitely many weak solutions for the Neumann problem derived from capillary phenomena.

$$
\begin{cases} -\operatorname{div}\left(\left(1+\dfrac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right)+\alpha(x)|u|^{p(x)-2}u=\lambda f(x,u), & \text{in } \Omega,\\ \dfrac{\partial u}{\partial v}=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}
$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary of class C^1, ν is the outer unit normal to $\partial\Omega$, $\alpha \in L^\infty(\Omega)$, f is an L^1-Carathéodory function.

In [30], the authors studied the existence and multiplicity of solutions for nonlinear eigenvalue problems involving $p(x)$-Laplacian-like operators

$$
\begin{cases} -\operatorname{div}\left(\left(1+\dfrac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right)=\lambda f(x,u), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}
$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $p \in C(\Omega)$ and $p(x) > 2$, for all $x \in \Omega$, $\lambda > 0$ and f satisfies some growth condition and Ambrosetti-Rabinowitz type condition (AR).

An interesting question arises: Can capillary phenomena be practically applied using fractional derivatives operators? In one specific example, Pata describes how oxygen is conveyed from capillaries to the surrounding tissues. This process is modeled by a subdiffusion equation that incorporates two fractional derivatives in time. The problem is described as $\mathbf{D}_t^{\nu_1}\mathfrak{C} - \tau\mathbf{D}_t^{\nu_2}\mathfrak{C} = \operatorname{div}(\rho\nabla\mathfrak{C}) - k$, with $0 < \nu_2 < \nu_1 \leq 1$, $\mathfrak{C}$ is a function of space and time, representing the concentration of oxygen, τ is the time lag in concentration of oxygen along the capillary, k is the rate of consumption per volume of tissue, and is the diffusion coefficient of oxygen, which possibly dependent on $\mathfrak{C}$. In particular, the term $\mathbf{D}_t^{\nu_1}\mathfrak{C} - \tau\mathbf{D}_t^{\nu_2}\mathfrak{C}$ details the net diffusion of oxygen to all tissues and $\mathbf{D}_t^{\theta}$ stands for the usual Caputo fractional derivative of order $\theta \in (0,1)$ with respect to time. We refer to an other example [31] where the authors utilized a fractal structure to describe variational formulation of viscoelastic deformation problem in Capillary-Porous materials. There are various options for introducing fractional integro-differentiation operations, in particular, the Riemann-Liouville, Caputo, Grunwald-Letnikov approaches, and their various modifications. Today, it is necessary to account for the dynamic evolution in modern physics and mechanics when constructing mathematical models. In this context, inspired by the studies mentioned above, we aim to highlight such areas to enrich the theoretical knowledge of these problems. Therefore, in this paper, we utilize the generalized ψ-Hilfer fractional derivative to study a nonlinear eigenvalue equation. This equation, which originates from a capillarity phenomenon, includes Dirichlet boundary conditions of the following form:

$$\tag{1.3}
\begin{cases}
\mathrm{D}_T^{\gamma,\beta;\psi}\left(\left(1+\dfrac{|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{p(x)}}{\sqrt{1+|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{2p(x)}}}\right)|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{p(x)-2}\ \mathrm{D}_{0^+}^{\gamma,\beta;\psi}u\right) = \xi g(x,u), & \text{in } \Omega,\\
u = 0, & \text{on } \partial\Omega,
\end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $g \in C(\bar{\Omega}\times\mathbb{R})$ is superlinear and do not satisfy the (AR)-condition, ξ is a positive parameter, $1 < p^- := \operatorname{ess\,inf}_{x\in\bar{\Omega}} p(x) \leq p(x) \leq p^+ := \operatorname{ess\,sup}_{x\in\bar{\Omega}} p(x) < +\infty$. The operators $\mathrm{D}_T^{\gamma,\beta;\psi}$ and $\mathrm{D}_{0^+}^{\gamma,\beta;\psi}$, defined in Section 2.2, are ψ-Hilfer fractional partial derivatives of order $\frac{1}{p(x)} < \gamma < 1$ and type $0 \leq \beta \leq 1$. For more applications of this type of operator, we refer to [25,32–38]. Our objective is to establish the existence of weak solutions to the problem as described by equation (1.3), using a critical point approach and various variational techniques. To achieve this, it is essential to impose specific assumptions on the nonlinear term $g(x,u)$.

(g_1) $g : \Omega\times\mathbb{R}\to\mathbb{R}$ satisfies the Carathéodory condition and

$$|g(x,u)| \leq c_1 + c_2|u|^{r(x)-1}, \quad \text{for all } (x,t)\in\Omega\times\mathbb{R},$$

where $r \in C(\bar{\Omega})$, $c_1, c_2 > 0$ and

$$p^+ < r^- \leq r(x) < p^\star(x) = \begin{cases} \dfrac{Np(x)}{N-\gamma p(x)}, & \text{if } \gamma p(x) < N,\\ +\infty, & \text{if } \gamma p(x) \geq N, \quad \text{for all } x\in\Omega. \end{cases}$$

(g_2) The following limit holds uniformly for a.e. $x \in \Omega$

$$\lim_{|t|\to+\infty} \frac{G(x,t)}{|t|^{p^+}} = +\infty, \quad \text{where } G(x,t) = \int_0^t g(x,\tau)d\tau.$$

(g_3) $g(x,t) = o\left(|t|^{p^+-1}\right)$, $t \to 0$, for $x \in \Omega$ uniformly.

(g_4) There exists a constant $c_3 > 0$ such that

$$\overline{G}(x,t) \leq \overline{G}(x,s) + c_3,$$

for any $x \in \Omega, 0 < t < s$ or $s < t < 0$, where $\overline{G}(x,t) := tg(x,t) - 2p^+G(x,t)$.

We note that condition (g_4) can be derived from the following condition:

(g_4)$'$ there exists $u_0 > 0$ such that $t \mapsto \frac{g(x,t)}{|t|^{2p^+-1}}$ is increasing in $t \geq t_0$ and decreasing in $t \leq -t_0$.

An example of a function that satisfies conditions (g_1)-(g_4) and does not satisfy the (AR)-condition is as follows:

$$g(x,u) = 2p^+|u|^{2p^+-2}u\log(|u|+1).$$

Definition 1.1. Let X be a real Banach space and $\Upsilon \in C^1(X,\mathbb{R})$. We say that Υ satisfies the Cerami condition at the level c or $(C)_c$ for short, if any sequence $u_n \subset X$ such that $\Upsilon(u_n) \to c$ and $(1+\|u_n\|)\Upsilon'(u_n) \to 0$ called a $(C)_c$ sequence, has a convergent subsequence.

Our approach to proving the existence results for problems (1.3) relies on proving that our operator is the type (S_+) and employing recent critical points theorems, for the convenience of the readers, we recall the critical theorems in [3]. It is important to note that these theorems play a crucial role in our strategy. Let us recall the following.

Theorem 1.1 ([3])**.** *Let X be a real Banach space and let $E \in C^1(X,\mathbb{R})$ satisfy the (C_c) condition for any $c > 0$, $E(0) = 0$ and the following conditions hold.*

(i) *There exist two positive constants ρ and R such that $E(u) \geq R$ for any $u \in X$ with $\|u\| = \rho$.*

(ii) *There exists a function $\xi \in X$ such that $\|\xi\| > \rho$ and $E(\xi) < 0$.*

Then, the functional E has a critical value $c \geq R$, i.e., there exists $u \in X$ such that $E'(u) = 0$ and $E(u) = c$.

Remark 1.1. In the case where $\beta \to 1$ and $\psi(x) = x$, our problem (1.3) reduces to the integer case (for more details, refer to [32]). For this reason, we observe that our problem generalizes many papers in the literature.

This work is organized as follows. In Section 2, we provide a brief overview of the key features of variable exponent Lebesgue spaces and ψ-fractional derivative spaces. Moving on to Section 3, we present the existing solutions to problems (1.3), along with their corresponding proofs.

2. Preliminary

In this section we collect preliminary concepts of the theory of variable exponent Lebesgue space, classical and fractional ψ-Hilfer derivative space (see [4,6,7,23–25,28]).

2.1. Variable exponent Lebesgue space.

In the following, we define

$$C(\bar{\Omega}) = \Big\{ s \in C(\Omega) : \; 1 < s^- \leq s^+ < +\infty \Big\},$$

where

$$s^- := \inf_{x \in \bar{\Omega}} s(x) \quad \text{and} \quad s^+ := \sup_{x \in \bar{\Omega}} s(x).$$

Denote by $\mathbf{U}(\Omega)$ the set of all measurable real-valued functions defined in Ω. For any $s \in C^+(\Omega)$, we denote the variable exponent Lebesgue space by

$$L^{s(x)}(\Omega) = \left\{ u \in \mathbf{U}(\Omega) : \int_\Omega |u(x)|^{s(x)} dx < +\infty \right\},$$

equipped with the Luxemburg norm

$$\|u\|_{L^{s(x)}} = \inf \left\{ \lambda > 0 : \; \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{s(x)} dx \leq 1 \right\},$$

then, the variable exponent Lebesgue space $\left(L^{s(x)}(\Omega), \| \cdot \|_{L^{s(x)}} \right)$ becomes a Banach space.

We have the following generalized Hölder inequality

$$\left| \int_\Omega u(x) v(x) dx \right| \leq \; 2 \|u\|_{L^{s(x)}} \|v\|_{L^{\bar{s}(x)}},$$

for $u \in L^{s(x)}(\Omega)$, $v \in L^{\bar{s}(x)}(\Omega)$ such that $\frac{1}{s(x)} + \frac{1}{\bar{s}(x)} = 1$.

At this point, let define the following map $\sigma_{s(x)} : L^{s(x)}(\Omega) \to \mathbb{R}$ by

$$\sigma_{s(x)}(u) = \int_\Omega |u(x)|^{s(x)} dx.$$

Then, we can see the important relationship between the norm $|| \cdot ||_{L^{s(x)}}$ and the corresponding modular function $\sigma_{s(x)}(\cdot)$ given in the next proposition.

Proposition 2.1 ([25])**.** *If u and $(u_n)_{n \in \mathbb{N}} \in L^{s(x)}(\Omega)$, we have*

(i) $\|u\|_{L^{s(x)}} < 1 \; (= 1, > 1)$ *if and only if* $\sigma_{s(x)}(u) < 1 \; (= 1, > 1)$*;*

(ii) $\|u\|_{L^{s(x)}} > 1,$ *then* $\|u\|^{s^-}_{L^{s(x)}} \leq \sigma_{s(x)}(u) \leq \|u\|^{s^+}_{L^{s(x)}}$*;*

(iii) $\|u\|_{L^{s(x)}} < 1,$ *then* $\|u\|^{s^+}_{L^{s(x)}} \leq \sigma_{s(x)}(u) \leq \|u\|^{s^-}_{L^{s(x)}}$*;*

(iv) $\lim_{n \to +\infty} \|u_n - u\|_{L^{s(x)}} = 0,$ *if and only if* $\lim_{n \to +\infty} \sigma_{s(x)}(u_n - u) = 0.$

2.2. **ψ-Hilfer fractional derivative space.** Let $A := [c, d]$, $-\infty \leq c < d \leq +\infty$, $n-1 < \gamma < n$, $n \in \mathbb{N}$, $\mathbf{f}$, $\psi \in C^n(A, \mathbb{R})$ such that ψ is increasing and $\psi'(x) \neq 0$, for all $x \in A$.

• The left-sided fractional ψ-Hilfer integrals of a function $\mathbf{f}$ is given by

$$\mathbf{I}_c^{\gamma;\psi}\mathbf{f}(x) = \frac{1}{\Gamma(\gamma)} \int_c^x \psi'(y)(\psi(x) - \psi(y))^{\gamma-1}\mathbf{f}(y)dy. \tag{2.1}$$

• The right-sided fractional ψ-Hilfer integrals of a function $\mathbf{f}$ is given by

$$\mathbf{I}_d^{\gamma;\psi}\mathbf{f}(x) = \frac{1}{\Gamma(\gamma)} \int_x^d \psi'(y)(\psi(y) - \psi(x))^{\gamma-1}\mathbf{f}(y)dy. \tag{2.2}$$

• The left-sided ψ-Hilfer fractional derivatives for a function $\mathbf{f}$ of order γ and type $0 \leq \beta \leq 1$ is defined by

$$\mathrm{D}_c^{\gamma,\beta;\psi}\mathbf{f}(x) = \mathbf{I}_c^{\beta(n-\gamma);\psi}\left(\frac{1}{\psi'(x)} \cdot \frac{d}{dx}\right)^n \mathbf{I}_c^{(1-\beta)(n-\gamma);\psi}\mathbf{f}(x).$$

• The right-sided ψ-Hilfer fractional derivatives for a function $\mathbf{f}$ of order γ and type $0 \leq \beta \leq 1$ is defined by

$$\mathrm{D}_d^{\gamma,\beta;\psi}\mathbf{f}(x) = \mathbf{I}_d^{\beta(n-\gamma);\psi}\left(-\frac{1}{\psi'(x)} \cdot \frac{d}{dx}\right)^n \mathbf{I}_d^{(1-\beta)(n-\gamma);\psi}\mathbf{f}(x).$$

Choosing $\beta \to 1$, we obtain ψ-Caputo fractional derivatives left-sided and right-sided, given by

$$\mathrm{D}_c^{\gamma;\psi}\mathbf{f}(x) = \mathbf{I}_c^{(n-\gamma);\psi}\left(\frac{1}{\psi'(x)} \cdot \frac{d}{dx}\right)^n \mathbf{f}(x), \tag{2.3}$$

$$\mathrm{D}_d^{\gamma;\psi}\mathbf{f}(x) = \mathbf{I}_d^{(n-\gamma);\psi}\left(-\frac{1}{\psi'(x)} \cdot \frac{d}{dx}\right)^n \mathbf{f}(x). \tag{2.4}$$

Remark 2.1. The ψ-Hilfer fractional derivatives defined as above can be written in the following form

$$\mathrm{D}_c^{\gamma,\beta;\psi}\mathbf{f}(x) = \mathbf{I}_c^{\mu-\gamma;\psi}\mathrm{D}_c^{\gamma;\psi}\mathbf{f}(x) \quad \text{and} \quad \mathrm{D}_d^{\gamma,\beta;\psi}\mathbf{f}(x) = \mathbf{I}_d^{\mu-\gamma;\psi}\mathrm{D}_d^{\gamma;\psi}\mathbf{f}(x),$$

with $\mu = \gamma + \beta(n-\gamma)$ and $\mathbf{I}_c^{\mu-\gamma;\psi}$, $\mathbf{I}_d^{\mu-\gamma;\psi}$, $\mathrm{D}_c^{\gamma;\psi}$ and $\mathrm{D}_d^{\gamma;\psi}$ as defined in (2.1), (2.2), (2.3) and (2.4).

In this paper we take $\Omega = A_1 \times \cdots \times A_N = [c_1, d_1] \times \cdots \times [c_N, d_N]$ where $0 < c_i < d_i$ for all $i \in \mathbb{N}$, $0 < \gamma_1, \ldots, \gamma_N < 1$. Consider also $\psi(\cdot)$ be an increasing and positive monotone function on $(c_1, d_1), \ldots, (c_N, d_N)$, having a continuous derivative $\psi'(\cdot)$ on $(c_1, d_1], \ldots, (c_N, d_N]$.

• The ψ-Riemann-Liouville fractional partial integral of order γ of N-variables $\mathbf{f} = (\mathbf{f}_1, \ldots, \mathbf{f}_N)$ is defined by

$$\mathbf{I}_{c,x}^{\gamma;\psi}\mathbf{f}(x) = \frac{1}{\Gamma(\gamma)} \int_{A_1}\int_{A_2} \cdots \int_{A_N} \psi'(y)(\psi(x) - \psi(y))^{\gamma-1}\mathbf{f}(y)dy,$$

with $\psi'(y)(\psi(x)-\psi(y))^{\gamma-1} = \psi'(y_1)(\psi(x_1)-\psi(y_1))^{\gamma_1-1}\cdots\psi'(y_N)(\psi(x_N)-\psi(y_N))^{\gamma_N-1}$ and $\Gamma(\gamma)=\Gamma(\gamma_1)\Gamma(\gamma_2)\cdots\Gamma(\gamma_N)$, $x_i = x_1x_2\cdots x_N$ and $dy_i = dy_1dy_2\cdots dN$, for all $i\in\{1,2,\ldots,N\}$.

• The ψ-Hilfer fractional partial derivative of N-variables of order γ and type β, $0\le\beta\le 1$, is defined by

$$\mathrm{D}^{\gamma,\beta;\psi}_{c,x_i}\mathbf{f}(x_i) = \mathbf{I}^{\beta(n-\gamma);\psi}_{c,x_i}\left(\frac{1}{\psi'(x_i)}\cdot\frac{\partial^N}{\partial x_i}\right)\mathbf{I}^{(1-\beta)(n-\gamma);\psi}_{c,x_i}\mathbf{f}(x_i),$$

with $\partial x_i = \partial x_1, \partial x_2, \ldots, \partial x_N$ and $\psi'(x_i) = \psi'(x_1)\psi'(x_2)\cdots\psi'(x_N)$ for all $i\in\{1,2,\ldots,N\}$. Analogously, it is defined $\mathrm{D}^{\gamma,\beta;\psi}_{d,x_i}(\cdot)$.

Now that we have all the necessary tools, we are ready to commence our study. To facilitate this, we define the working space $\mathbb{H}^{\gamma,\beta,\psi}_{p(x)}(\Omega)$ as follow

$$\mathbb{H}^{\gamma,\beta,\psi}_{p(x)}(\Omega) = \mathbb{H} := \left\{u\in L^{p(x)}(\Omega) : |\mathrm{D}^{\gamma,\beta;\psi}_{0^+}u| \in L^{p(x)}(\Omega)\right\},$$

equipped with the norm $\|u\|_{\mathbb{H}} = ||u||_{L^{p(x)}} + ||\mathrm{D}^{\gamma,\beta,\psi}_{0^+}u||_{L^{p(x)}}$.

Proposition 2.2 ([33])**.** *Let $0<\gamma\le 1$, $0\le\beta\le 1$ and $1<p(x)$. The ψ-Hilfer fractional derivative space $\mathbb{H}^{\gamma,\beta,\psi}_{p(x)}(\Omega)$ is a reflexive and separable Banach space.*

Remark 2.2. We can define $\mathbb{H}(\Omega) := \mathbb{H}^{\gamma,\beta,\psi}_{p(x),0}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $\mathbb{H}^{\gamma,\beta,\psi}_{p(x)}(\Omega)$ which can be renormed by the equivalent norm $\|u\| := \||\mathrm{D}^{\gamma,\beta,\psi}_{0^+}u|\|_{L^{p(x)}}$. This space is a separable and reflexive Banach space [37].

Proposition 2.3 ([37])**.** *Let Ω a Lipschitz bounded domain in Ω. Let $p\in C^0(\overline{\Omega})$. If $r:\overline{\Omega}\to(1,+\infty)$ such that $1\le r(x)<p^\star(x)$. Then, the embedding $\mathbb{H}(\Omega)\hookrightarrow L^{r(x)}(\Omega)$ is compact.*

2.3. **The** (S_+) **property.** In this subsection, we prove the (S_+) property of the operator

$$\mathcal{L} := \mathrm{D}^{\gamma,\beta;\psi}_{T}\left(|\mathrm{D}^{\gamma,\beta;\psi}_{0^+}u|^{p(x)-2} + \frac{|\mathrm{D}^{\gamma,\beta;\psi}_{0^+}u|^{2p(x)-2}}{\sqrt{1+|\mathrm{D}^{\gamma,\beta;\psi}_{0^+}u|^{2p(x)}}}\right).$$

For this, let consider the following functional:

$$\mathcal{A}(u) = \int_\Omega \frac{1}{p(x)}\left(|\mathrm{D}^{\gamma,\beta;\psi}_{0^+}u|^{p(x)} + \sqrt{1+|\ \mathrm{D}^{\gamma,\beta;\psi}_{0^+}u|^{2p(x)}}\right)dx, \quad \text{for all } u\in\mathbb{H}(\Omega).$$

Note that $\mathcal{A}\in C^1(\mathbb{H}(\Omega),\mathbb{R})$ and the derivative operator of $\mathcal{A}$ in weak sense $\mathcal{A}'$: $\mathbb{H}(\Omega)\to(\mathbb{H}(\Omega))^*$ is such that

$$\langle\mathcal{A}'(u),v\rangle = \int_\Omega\left(|\mathrm{D}^{\gamma,\beta;\psi}_{0^+}u|^{p(x)-2} + \frac{|\ \mathrm{D}^{\gamma,\beta;\psi}_{0^+}u|^{2p(x)-2}}{\sqrt{1+|\mathrm{D}^{\gamma,\beta;\psi}_{0^+}u|^{2p(x)}}}\right)\ \mathrm{D}^{\gamma,\beta;\psi}_{0^+}u\mathrm{D}^{\gamma,\beta;\psi}_{0^+}v dx,$$

for all $u,v\in\mathbb{H}(\Omega)$.

Proposition 2.4. *The mapping $\mathcal{A}':\mathbb{H}(\Omega)\to(\mathbb{H}(\Omega))^*$ is a convex, bounded homeomorphism and strictly monotone operator, and is a mapping of type (S_+).*

Proof. It is clear that $\mathcal{A}$ is a continuous, bounded, and strictly monotone operator. Considering that $\mathcal{A}'$ is a continuous, bounded and strictly monotone operator, if $u_n \rightharpoonup u$ and $\overline{\lim}_{n\to+\infty}(\mathcal{A}'(u_n)-\mathcal{A}'(u),u_n-u)\leq 0$, then $\lim\limits_{n\to+\infty}(\mathcal{A}'(u_n)-\mathcal{A}'(u),u_n-u)=0$. According to Fatou lemma, we have

$$\lim_{n\to+\infty}\int_\Omega \frac{1}{p(x)}\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)}dx\geq \int_\Omega \frac{1}{p(x)}|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{p(x)}dx,$$

and note that $\sqrt{1+|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{2p(x)}}\geq |\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{p(x)}$ for all $u\in\mathbb{H}(\Omega)$, then we get

$$\begin{aligned}&\underline{\lim}_{n\to+\infty}\int_\Omega \frac{1}{p(x)}\left(\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)}+\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)}}\right)dx\\ &\geq\int_\Omega \frac{1}{p(x)}\left(\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u\right|^{p(x)}+\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u\right|^{2p(x)}}\right)dx.\end{aligned}\tag{2.5}$$

With $u_n\rightharpoonup u$, we have

$$\lim_{n\to+\infty}(\mathcal{A}'(u_n),u_n-u)=\lim_{n\to+\infty}(\mathcal{A}'(u_n)-\mathcal{A}'(u),u_n-u)=0.\tag{2.6}$$

Moreover, we also have

$$\begin{aligned}&(\mathcal{A}'(u_n),u_n-u)\\ =&\int_\Omega\left(\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)-2}+\frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)-2}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)}}}\right)\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\mathrm{D}_{0^+}^{\gamma,\beta;\psi}(u_n-u)dx\\ =&\int_\Omega\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)}dx-\int_\Omega\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)-2}\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\mathrm{D}_{0^+}^{\gamma,\beta;\psi}udx\\ &+\int_\Omega\frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)}}}dx-\int_\Omega\frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)-2}\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\ \mathrm{D}_{0^+}^{\gamma,\beta;\psi}u}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)}}}dx\\ \geq&\int_\Omega\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)}dx-\int_\Omega\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)-1}\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u\right|dx\\ &+\int_\Omega\frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)}}}dx-\int_\Omega\frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)-1}|\ \mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{2p(x)}}}dx:=I_1+I_2-I_3.\end{aligned}$$

Note that

$$\begin{aligned}I_1=&\int_\Omega\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)}dx-\int_\Omega\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)-1}|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|dx\\ \geq&\int_\Omega\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)}dx-\int_\Omega\left(\frac{p(x)-1}{p(x)}\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)}+\frac{1}{p(x)}\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u\right|\right)dx\\ \geq&\int_\Omega\frac{1}{p(x)}\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u_n\right|^{p(x)}dx-\int_\Omega\frac{1}{p(x)}\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u\right|dx,\end{aligned}$$

$$
\begin{aligned}
I_2 &= \int_\Omega \frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}} dx = \int_\Omega \frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}} \cdot \frac{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}} dx \\
&= \int_\Omega \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} \frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} dx \\
&= \int_\Omega \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} \Bigg(1 - \frac{1}{1+\underbrace{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}_{\geq c_1 > 0}}\Bigg) dx \\
&\geq \int_\Omega \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} \left(1 - \frac{1}{1+c_1}\right) dx \geq c_2 \int_\Omega \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} dx
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= -\int_\Omega \frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)-1} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}} dx \\
&\geq -\int_\Omega \left(\frac{p(x)-1}{p(x)} \cdot \frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)-1}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}} + \frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right| \right) dx.
\end{aligned}
$$

Then,

$$
\begin{aligned}
I_2 + I_3 \geq& \int_\Omega \frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}} dx - \int_\Omega \frac{p(x)-1}{p(x)} \cdot \frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}} dx \\
&- \int_\Omega \frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right| dx \\
\geq& \int_\Omega \frac{1}{p(x)} \cdot \frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)-1}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} dx} - \int_\Omega \frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right| dx \\
\geq& c_2 \int_\Omega \frac{1}{p(x)} \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} dx - \int_\Omega \frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right| dx \\
\geq& c_2 \int_\Omega \frac{1}{p(x)} \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} dx - \int_\Omega \frac{1}{p(x)} \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|^{2p(x)}} dx \\
\geq& c_3 \left(\int_\Omega \frac{1}{p(x)} \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} dx - \int_\Omega \frac{1}{p(x)} \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|^{2p(x)}} dx \right).
\end{aligned}
$$

Therefore,

$$
(\mathcal{A}'(u_n), u_n - u)
$$

$$\begin{aligned}
&\geq \int_\Omega \frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{p(x)} dx - \int_\Omega \frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right| dx \\
&+ c_3 \left(\int_\Omega \frac{1}{p(x)} \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} dx - \int_\Omega \frac{1}{p(x)} \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|^{2p(x)}} dx \right). \tag{2.7}
\end{aligned}$$

Based on (2.5)–(2.7), we deduce

$$\begin{aligned}
&\lim_{n\to+\infty} \int_\Omega \left(\frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{p(x)} + c_3 \frac{1}{p(x)} \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} \right) dx \\
&= \int_\Omega \left(\frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|^{p(x)} + c_3 \frac{1}{p(x)} \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|^{2p(x)}} \right) dx. \tag{2.8}
\end{aligned}$$

Following (2.8), it can be inferred that the integrals of the family of functions

$$\left\{ \left(\frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{p(x)} + c_3 \frac{1}{p(x)} \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} \right) \right\}$$

posses absolutely equicontinuity on Ω. Given that

$$\begin{aligned}
&\frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n - \mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|^{p(x)} + c_3 \frac{1}{p(x)} \left| \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|} - \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|} \right|^{2p(x)} \\
\leq & L_1 \left(\frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{p(x)} + \frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|^{p(x)} \right) \\
&+ L_2 \left(c_3 \frac{1}{p(x)} \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} + c_3 \frac{1}{p(x)} \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|^{2p(x)}} \right).
\end{aligned}$$

The integrals of family

$$\left\{ \frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n - \mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|^{p(x)} + c_3 \frac{1}{p(x)} \left| \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|} - \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|} \right|^{2p(x)} \right\}$$

are also absolutely equicontinuous on Ω. Hence,

$$\begin{aligned}
&\lim_{n\to+\infty} \int_\Omega \left(\frac{1}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_{\mathrm{n}}(x) - \mathrm{D}_{0^+}^{\gamma,\beta;\psi} u(x)\right|^{p(x)} \right. \\
&\left. + c_3 \frac{1}{p(x)} \left| \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_{\mathrm{n}}(x)\right|} - \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u(x)\right|} \right|^{2p(x)} \right) dx = 0. \tag{2.9}
\end{aligned}$$

According to (2.9) we have

$$\lim_{n\to+\infty} \int_\Omega \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n(x) - \mathrm{D}_{0^+}^{\gamma,\beta;\psi} u(x)\right|^{p(x)} dx = 0 \tag{2.10}$$

and

$$\lim_{n\to+\infty} \int_\Omega \left| \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n(x)\right|} - \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u(x)\right|} \right|^{2p(z)} dx = 0. \tag{2.11}$$

By Proposition 2.1, along with (2.10) and (2.11), we have $u_n \to u$, i.e., $\mathcal{L}$ is of type (S_+). By the strictly monotonicity, $\mathcal{L}$ is an injection. On the other hand, since

$$\lim_{\|u\|\to+\infty} \frac{1}{\|u\|} \left(\mathcal{A}'(u), u\right) = \lim_{\|u\|\to+\infty} \frac{1}{\|u\|} \int_\Omega \left(|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u|^{p(x)} + \frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u\right|^{2p(u)}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} x\right|^{2p(x)}}} \right) dx$$
$$= +\infty,$$

$\mathcal{A}'$ is coercive. Therefore, $\mathcal{A}'$ is a surjection. Hence, $\mathcal{A}'$ has an inverse mapping $(\mathcal{A}')^{-1}$: $(\mathbb{H}(\Omega))^* \to \mathbb{H}(\Omega)$. Subsequently the continuity of $(\mathcal{A}')^{-1}$ is sufficient to ensure $\mathcal{A}'$ to be a bomeomorphism.

If $g_n, g \in (\mathbb{H}(\Omega))^*$, $g_n \to g$, let $u_n = (\mathcal{A}')^{-1}(g_n)$ and $u = (\mathcal{A}')^{-1}(g)$. Then $\mathcal{A}'(u_n) = g_n$, $\mathcal{A}'(u) = g$. Hence, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{H}(\Omega)$. Without loss of generality, we can assume that $u_n \rightharpoonup u_0$. Since $g_n \to g$, then

$$\lim_{n\to+\infty} \left(\mathcal{A}'(u_n) - \mathcal{A}'(u_0), u_n - u_0\right) = \lim_{n\to+\infty} (g_n, u_n - u_0) = 0.$$

Given that $\mathcal{A}'$ is of type (S_+) and $u_n \to u_0$, we deduce that $u_n \to u$, and therefore, $\mathcal{A}'$ is continuous. $\square$

Definition 2.1. We say that $u \in \mathbb{H}(\Omega)$ is a weak solution of (1.3), if for every $\mu \in \mathbb{H}(\Omega)$, the following holds:

$$\int_\Omega \left(|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u|^{p(x)-2}\, \mathrm{D}_{0^+}^{\gamma,\beta;\psi} u + \frac{|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u|^{2p(x)-2}\, \mathrm{D}_{0^+}^{\gamma,\beta;\psi} u}{\sqrt{1+|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u|^{2p(x)}}} \right) \mathrm{D}_{0^+}^{\gamma,\beta;\psi} \mu(x) dx$$
$$= \xi \int_\Omega g(x,u)\mu(x) dx.$$

3. Main Results

The primary outcome established in this paper is formulated as follows.

Theorem 3.1. *Assume that (g_1)-(g_4) are satisfied. Then, the problem* (1.3) *has at least one nontrivial weak solution for all $\xi > 0$.*

Let us introduce the energy functional $\mathbb{E}_\xi : \mathbb{H}(\Omega) \to \mathbb{R}$ associated to problem (1.3), which is defined as follow

$$\mathbb{E}_\xi(u) = \int_\Omega \frac{1}{p(x)} \left(|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u|^{p(x)} + \sqrt{1+|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u|^{2p(x)}} \right) dx - \xi \int_\Omega G(x,u) dx. \tag{3.1}$$

Keep in mind that $\mathbb{E}_\xi \in C^1\big(\mathbb{H}(\Omega), \mathbb{R}\big)$ and it is noteworthy that the critical points of $\mathbb{E}_\xi$ correspond to weak solutions of (1.3) and its Gateaux derivative is

$$\langle \mathbb{E}_\xi'(u), u\rangle = \int_\Omega \left(|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u|^{p(x)} + \frac{|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u|^{2p(x)}}{\sqrt{1+|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u|^{2p(x)}}} \right) dx - \xi \int_\Omega u g(x,u) dx. \tag{3.2}$$

Mountain-pass geometry.

Next, we illustrate that the energy function follows the mountain pass geometry.

Lemma 3.1. *Given that the conditions (g_1)-(g_3) hold true. Then we have the following assertions.*

(i) *There exists $\omega \in \mathbb{H}(\Omega)$, $\omega > 0$ such that $\mathbb{E}_\xi(t\omega) \to -\infty$ as $t \to +\infty$.*

(ii) *There exist $e > 0$ and $\eta > 0$ such that $\mathbb{E}_\xi(u) \geq \eta$ for any $u \in \mathbb{H}(\Omega)$ with $\|u\| = e$.*

Proof. (i) By applying (g_2), it can be inferred that for all $K > 0$, there exist $C_K > 0$, such that

$$G(x,t) \geq K|t|^{p^+} - C_K, \quad \text{for all } x \in \Omega, \text{ for all } t \in \mathbb{R}.$$

Let $\omega \in \mathbb{H}(\Omega)$ with $\omega > 0$, then from last inequality, one has

$$\begin{aligned}
\mathbb{E}_\xi(t\omega) &= \int_\Omega \frac{1}{p(x)}\Big(t^{p(x)}|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}\omega|^{p(x)} + \sqrt{1 + t^{2p(x)}|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}\omega|^{2p(x)}}\Big)dx - \xi\int_\Omega G(x,t\omega)dx \\
&\leq \int_\Omega \frac{1}{p(x)}\left(t^{p(x)}|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}\omega|^{p(x)} + 1 + t^{p(x)}|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}\omega|^{p(x)}\right)dx - \xi\int_\Omega G(x,t\omega)dx \\
&\leq \int_\Omega \frac{1}{p(x)}\left(t^{p(x)}|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}\omega|^{p(x)} + t^{p(x)} + t^{p(x)}|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}\omega|^{p(x)}\right)dx - \xi\int_\Omega G(x,t\omega)dx
\end{aligned}$$

$$\leq t^{p^+}\left[\int_\Omega \frac{1}{p(x)}\left(2|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}\omega|^{p(x)} + 1\right)dx - \xi K\int_\Omega \omega^{p^+}dx\right] + \xi C_K|\Omega|, \tag{3.3}$$

where $t > 1$ and $|\Omega|$ denotes the Lebesgue measure of Ω. Then, from (3.3), if K is large enough such that

$$\int_\Omega \frac{1}{p(x)}\left(2\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}\omega\right|^{p(x)} + 1\right)dx - \xi K\int_\Omega \omega^{p^+}dx < 0,$$

thus we get

$$\mathbb{E}_\xi(t\omega) \to -\infty, \quad \text{as } t \to +\infty.$$

This concludes the proof of (i).

(ii) According to Proposition 2.3, there exists constant $c_3 > 0$ such that

$$\|u\|_{L^{p^+}} \leq c_3\|u\| \quad \text{and} \quad \|u\|_{L^{r(x)}} \leq c_3\|u\|. \tag{3.4}$$

It follows from (g_1) and (g_3), that for all given $\varepsilon > 0$ there exists $C_\varepsilon > 0$, such that

$$G(x,t) \leq \frac{\varepsilon}{p^+}|t|^{p^+} + C_\varepsilon|t|^{r(x)}, \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}. \tag{3.5}$$

Therefore, for $u \in \mathbb{H}(\Omega)$ with $\|u\| < 1$, we have from (3.4) and (3.5)

$$\begin{aligned}
\mathbb{E}_\xi(u) &= \int_\Omega \frac{1}{p(x)}\Big(|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{p(x)} + \sqrt{1 + |\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{2p(x)}}\Big)dx - \xi\int_\Omega G(x,u)dx \\
&\geq \int_\Omega \frac{1}{p(x)}|\mathrm{D}_{0^+}^{\gamma,\beta;\psi}u|^{p(x)}dx - \frac{\xi}{p^+}\varepsilon\int_\Omega |u|^{p^+}dx - \xi C_\varepsilon\int_\Omega |u|^{r(x)}dx
\end{aligned}$$

$$\geq \frac{1}{p^+}\|u\|^{p^+} - \frac{\xi}{p^+}\varepsilon c_3^{p^+}\|u\|^{p^+} - \xi C_\varepsilon c_3^{r^-}\|u\|^{r^-}$$

$$\geq \frac{1}{p^+}\left(1 - \xi\varepsilon c_3^{p^+}\right)\|u\|^{p^+} - \xi C_\varepsilon c_3^{r^-}\|u\|^{r^-}. \tag{3.6}$$

This implies the existence of $e > 0$ and $\eta > 0$ such that $\mathbb{E}_\xi(u) \geq \eta > 0$ for every $u \in \mathbb{H}(\Omega)$ and $\|u\| = e$. This completes the proof of (ii). □

Lemma 3.2. *Assume that* (g_1)-(g_3) *hold and* $0 < \xi_0 < \mu_0$, *then* $\mathbb{E}_\xi$ *possesses uniform mountain-pass geometric structure around* $u = 0$ *for* $\xi \in [\xi_0, \mu_0]$, *i.e., there is an* $\varrho \in \mathbb{H}(\Omega)$ *such that* $\mathbb{E}_{\xi_0}(\varrho) < 0$ *for any* $\xi \in [\xi_0, \mu_0]$, *and there are* $\rho > 0, v > 0$ *such that* $\mathbb{E}_\xi(u) \geq v$ *for any* $\xi \in [\xi_0, \mu_0]$ *and* $u \in \mathbb{H}(\Omega)$ *with* $\|u\| = \rho$.

Proof. Fix $\varepsilon > 0$ small enough such that $(1 - \mu_0\varepsilon c_3^{p^+}) \geq \frac{1}{2}$, then by (3.6), one has

$$\mathbb{E}_\xi(u) \geq \frac{1}{2p^+}\|u\|^{p^+} - \mu_0 C_\varepsilon c_3^{\alpha^-}\|u\|^{r^-}, \quad \text{when } \|u\| < 1,\ \xi_0 \leq \xi \leq \mu_0.$$

Hence, there exist $\rho = \rho\,(\mu_0, \varepsilon) > 0$ and $v = v\,(\mu_0, \varepsilon) > 0$ such that

$$\mathbb{E}_\xi(u) \geq v, \quad \|u\| = \rho, \quad \text{for all } \xi \in [\xi_0, \mu_0]\,.$$

Furthermore, from Lemma 3.1 (i), we can choose $\varrho = t_0\omega \in \mathbb{H}(\Omega)$ with t_0 large enough such that $\mathbb{E}_{\xi_0}(\varrho) < 0$. Then for any $0 < \xi_0 < \xi$, we have

$$\mathbb{E}_\xi(\varrho) < \mathbb{E}_{\xi_0}(\varrho) < 0, \quad \xi_0 \leq \xi \leq \mu_0.$$

This implies that

$$\mathbb{E}_\xi(\varrho) < 0, \quad \xi_0 \leq \xi \leq \mu_0.$$ □

The boundedness of Cerami sequence.

Lemma 3.3. *Assume that* (g_1)-(g_4) *are satisfied. Then the functional* $\mathbb{E}_\xi$ *satisfies the* (C_c) *condition for any* $c > 0$.

Proof. Suppose that $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{H}(\Omega)$ is a (C_c) sequence for $\mathbb{E}_\xi$, that is,

$$\mathbb{E}_\xi\,(u_n) \to c > 0, \quad (1 + \|u_n\|)\,\mathbb{E}'_\xi(u_n) \to 0, \quad \text{as } n \to +\infty.$$

This indicates that

$$c = \mathbb{E}_\xi\,(u_n) + o(1), \quad \langle \mathbb{E}'_\xi\,(u_n)\,, u_n\rangle = o(1), \tag{3.7}$$

where $o(1) \to 0$ as $n \to +\infty$.

Claim. The sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{H}(\Omega)$. Let's assume the contrary. By considering a subsequence if needed, we can assume that $\|u_n\| \to +\infty$ as $n \to +\infty$. Let define

$$w_n = \frac{u_n}{\|u_n\|}, \quad n \geq 1.$$

We can assume that

$$\left\{\begin{array}{l} w_n \rightarrow w \ \text{ strongly in } \ L^{r(x)}(\Omega), \\ w_n \rightarrow w \ \text{ strongly in } L^{p^+}(\Omega), \\ w_n(x) \rightarrow w(x) \ \text{ a.e in } \Omega, \\ w_n \rightharpoonup w \ \text{ weakly in } \ \mathbb{H}(\Omega). \end{array}\right. \tag{3.8}$$

Let $\tilde{\Omega} := \{x \in \Omega : w(x) \neq 0\}$. Then, in $\tilde{\Omega}$, one has

$$\lim_{n\to+\infty} w_n(x) = \lim_{n\to+\infty} \frac{u_n(x)}{\|u_n\|} = w(x) \neq 0,$$

this implies that

$$|u_n(x)| \rightarrow +\infty, \qquad \text{a.e. in } \ \tilde{\Omega}. \tag{3.9}$$

Furthermore, based on (g_2), we obtain

$$\lim_{n\to+\infty} \frac{G\left(x, u_n(x)\right)}{|u_n(x)|^{p^+}} = +\infty, \quad x \in \tilde{\Omega}.$$

This implies that

$$\lim_{n\to+\infty} \frac{G\left(x, u_n(x)\right)}{|u_n(x)|^{p^+}} \left|w_n(x)\right|^{p^+} = +\infty, \quad x \in \tilde{\Omega}. \tag{3.10}$$

Due to the assumptions (g_2), there exists a positive constant K such that

$$\frac{G(x,t)}{|t|^{p^+}} > 1, \tag{3.11}$$

for any $x \in \Omega$ and $t \in \mathbb{R}$ with $|t| \geq K$. Since $G(x,t)$ is continuous on $\overline{\Omega} \times [-K, K]$, there exists a positive constant c_4 such that

$$|G(x,t)| \leq c_4, \tag{3.12}$$

for all $(x,t) \in \overline{\Omega} \times [-K, K]$. Therefore, from (3.11) and (3.12), we can see that there is a constant $c_5 > 0$ such that for any $(x,t) \in \overline{\Omega} \times \mathbb{R}$, we have

$$G(x,t) \geq c_5,$$

which signifies that

$$\frac{G\left(x, u_n(x)\right) - c_5}{\|u_n\|^{p^+}} \geq 0.$$

This implies that

$$\frac{G\left(x, u_n(x)\right)}{|u_n(x)|^{p^+}} \left|w_n(x)\right|^{p^+} - \frac{c_5}{\|u_n\|^{p^+}} \geq 0. \tag{3.13}$$

Using (3.7), we derive that

$$
\begin{aligned}
c &= \mathbb{E}_{\xi}\left(u_n\right)+o(1), \\
&= \int_{\Omega} \frac{1}{p(x)}\left(\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{p(x)}+\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}\right) dx-\xi \int_{\Omega} G\left(x, u_n\right) dx+o(1), \\
&\geq \frac{2}{p^+}\left\|u_n\right\|^{p^-}-\xi \int_{\Omega} G\left(x, u_n\right) dx+o(1).
\end{aligned}
$$

Hence, we can see that

$$
\int_{\Omega} G\left(x, u_n\right) dx \geq \frac{2}{\xi p^+}\left\|u_n\right\|^{p^-}-\frac{c}{\xi}+o(1) \rightarrow+\infty, \quad \text{as } n \rightarrow+\infty. \tag{3.14}
$$

Similarly, from (3.7), we also have

$$
\begin{aligned}
c &= \mathbb{E}_{\xi}\left(u_n\right)+o(1), \\
&= \int_{\Omega} \frac{1}{p(x)}\left(\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{p(x)}+\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}\right) dx-\xi \int_{\Omega} G\left(x, u_n\right) dx+o(1), \\
&\leq \int_{\Omega} \frac{1}{p(x)}\left(\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{p(x)}+1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{p(x)}\right) dx-\xi \int_{\Omega} G\left(x, u_n\right) dx+o(1), \\
&\leq \frac{2}{p^-}\left\|u_n\right\|^{p^+}+\frac{1}{p^-}|\Omega|-\xi \int_{\Omega} G\left(x, u_n\right) dx+o(1).
\end{aligned}
$$

Therefore, it follows from (3.14) that

$$
\left\|u_n\right\|^{p^+} \geq \frac{p^-}{2} c-\frac{|\Omega|}{2}+\frac{p^-}{2} \xi \int_{\Omega} G\left(x, u_n\right) dx-o(1)>0, \tag{3.15}
$$

for n large enough.

• Let prove that $\left|\tilde{\Omega}\right|=0$. Indeed, if $\left|\tilde{\Omega}\right| \neq 0$, then according to (3.10), (3.13), (3.15) and Fatou's Lemma, one has

$$
\begin{aligned}
+\infty &= \int_{\tilde{\Omega}} \liminf_{n \rightarrow+\infty} \frac{G\left(x, u_n(x)\right)}{\left|u_n(x)\right|^{p^+}}\left|w_n(x)\right|^{p^+} dx-\int_{\tilde{\Omega}} \limsup_{n \rightarrow+\infty} \frac{c_5}{\left\|u_n\right\|^{p^+}} dx \\
&= \int_{\tilde{\Omega}} \liminf_{n \rightarrow+\infty}\left[\frac{G\left(x, u_n(x)\right)}{\left|u_n(x)\right|^{p^+}}\left|w_n(x)\right|^{p^+}-\frac{c_5}{\left\|u_n\right\|^{p^+}}\right] dx \\
&\leq \liminf_{n \rightarrow+\infty} \int_{\tilde{\Omega}}\left[\frac{G\left(x, u_n(x)\right)}{\left|u_n(x)\right|^{p^+}}\left|w_n(x)\right|^{p^+}-\frac{c_5}{\left\|u_n\right\|^{p^+}}\right] dx \\
&\leq \liminf_{n \rightarrow+\infty} \int_{\Omega}\left[\frac{G\left(x, u_n(x)\right)}{\left|u_n(x)\right|^{p^+}}\left|w_n(x)\right|^{p^+}-\frac{c_5}{\left\|u_n\right\|^{p^+}}\right] dx \\
&= \liminf_{n \rightarrow+\infty} \int_{\Omega} \frac{G\left(x, u_n(x)\right)}{\left|u_n(x)\right|^{p^+}}\left|w_n(x)\right|^{p^+} dx-\limsup_{n \rightarrow+\infty} \int_{\Omega} \frac{c_5}{\left\|u_n\right\|^{p^+}} dx \\
&= \liminf_{n \rightarrow+\infty} \int_{\Omega} \frac{G\left(x, u_n(x)\right)}{\left\|u_n\right\|^{p^+}} dx
\end{aligned}
$$

$$\leq \liminf_{n\to+\infty} \frac{1}{\frac{p^-c}{2} - \frac{|\Omega|}{2} + \frac{\xi p^-}{2}\int_\Omega G(x,u_n)\,dx - o(1)} \int_\Omega G(x,u_n(x))\,dx. \tag{3.16}$$

Hence, combining (3.14) and (3.16), we conclude that

$$+\infty \leq \frac{2}{\xi p^-}.$$

This leads to a contradiction. This demonstrates that $\left|\tilde{\Omega}\right| = 0$ and then $w(x) = 0$ a.e. in Ω. Since $\mathbb{E}_\xi(tu_n)$ is continuous in $t \in [0,1]$, for each n there exists $t_n \in [0,1]$, $n = 1,2,\ldots$, such that

$$\mathbb{E}_\xi(t_n u_n) := \max_{t\in[0,1]} \mathbb{E}_\xi(tu_n). \tag{3.17}$$

We can see that, for $t_n > 0$ we have

$$\mathbb{E}_\xi(t_n u_n) \geq c_\xi > 0 = \mathbb{E}_\xi(0).$$

- If $t_n < 1$, then $\frac{d}{dt}\mathbb{E}_\xi(tu_n)\Big|_{t=t_n} = 0$, which implies $\langle \mathbb{E}'_\xi(t_n u_n), t_n u_n\rangle = 0$.
- If $t_n = 1$, then, according to (3.7), we have $\langle \mathbb{E}'_\xi(u_n), u_n\rangle = o(1)$.

Hence,

$$\langle \mathbb{E}'_\xi(t_n u_n), t_n u_n\rangle = o(1). \tag{3.18}$$

Let $\{s_k\}_{k\in\mathbb{N}}$ be a positive sequence of real numbers such that $s_k > 1$ for any k and $\lim_{k\to+\infty} s_k = +\infty$. Then,

$$\|s_k w_n\| = \left\| s_k \frac{u_n}{\|u_n\|}\right\| = s_k > 1, \quad \text{for any } k \text{ and } n.$$

Let fix k. Since $w_n \to 0$ in $L^{r(x)}(\Omega)$, and $w_n(x) \to 0$ a.e. $x \in \Omega$ as $n \to +\infty$. Then, from (g_1) and the Lebesgue dominated convergence theorem we can deduce that

$$\lim_{n\to+\infty} \int_\Omega G(x, s_k w_n)\,dx = 0. \tag{3.19}$$

From the fact that $\|u_n\| \to +\infty$ as $n \to +\infty$. Therefore, we have

$$\|u_n\| > s_k \quad \text{or} \quad 0 < \frac{s_k}{\|u_n\|} < 1, \quad \text{for } n \text{ large enough.}$$

Thus, by (3.19), we can infer that

$$\begin{aligned}
\mathbb{E}_\xi(t_n u_n) \geq& \mathbb{E}_\xi\left(\frac{s_k}{\|u_n\|}u_n\right) = \mathbb{E}_\xi(s_k w_n)\\
&= \int_\Omega \frac{1}{p(x)}\left(\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} s_k w_n\right|^{p(x)} + \sqrt{1 + \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} s_k w_n\right|^{2p(x)}}\right) dx\\
&\quad - \xi \int_\Omega G(x, s_k w_n)\,dx\\
&\geq \int_\Omega \frac{2}{p(x)} \left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} s_k w_n\right|^{p(x)} dx - \xi \int_\Omega G(x, s_k w_n)\,dx
\end{aligned}$$

$$\geq \frac{2s_k^{p^-}}{p^+}\|w_n\|^{p^-} - \xi \int_\Omega G(x, s_k w_n)\,dx = \frac{2s_k^{p^-}}{p^+}. \tag{3.20}$$

From (3.20), as we let $n, k \to +\infty$, we obtain

$$\lim_{n\to+\infty} \mathbb{E}_\xi(t_n u_n) = +\infty. \tag{3.21}$$

Due to (g_4) and (3.7), for sufficiently large n, one has

$$\begin{aligned}
\mathbb{E}_\xi(t_n u_n) =& \mathbb{E}_\xi(t_n u_n) - \frac{1}{2p^+}\langle \mathbb{E}'_\xi(t_n u_n), t_n u_n\rangle + o(1) \\
=& \int_\Omega \frac{1}{p(x)}\left(\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} t_n u_n\right|^{p(x)} + \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} t_n u_n\right|^{2p(x)}}\right)dx \\
& - \frac{1}{2p^+}\int_\Omega \left(\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} t_n u_n\right|^{p(x)} + \frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} t_n u_n\right|^{2p(x)}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} t_n u_n\right|^{2p(x)}}}\right)dx \\
& - \xi\int_\Omega G(x, t_n u_n)\,dx + \frac{\xi}{2p^+}\int_\Omega g(x, t_n u_n)\, t_n u_n dx + o(1) \\
=& \int_\Omega \left[\frac{1}{p(x)}\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} t_n u_n\right|^{2p(x)}} - \frac{1}{2p^+}\frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} t_n u_n\right|^{2p(x)}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} t_n u_n\right|^{2p(x)}}}\right]dx \\
& + \frac{\xi}{2p^+}\int_\Omega \Big[g(x, t_n u_n)\, t_n u_n - 2p^+ G(x, t_n u_n)\Big]dx \\
& + \int_\Omega \left[\frac{1}{p(x)} - \frac{1}{2p^+}\right]\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} t_n u_n\right|^{p(x)} dx + o(1) \\
\leq& \int_\Omega \left[\frac{1}{p(x)}\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}} - \frac{1}{2p^+}\frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}}\right]dx \\
& + \frac{\xi}{2p^+}\int_\Omega \Big[g(x, u_n)\, u_n - 2p^+ G(x, u_n) + c_3\Big]dx \\
& + \int_\Omega \left[\frac{1}{p(x)} - \frac{1}{2p^+}\right]\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{p(x)} dx + o(1) \\
=& \frac{1}{2p^+}\int_\Omega \left[\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{p(x)} + \frac{\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}{\sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}}\right]dx \\
& + \int_\Omega \frac{1}{p(x)}\left(\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{p(x)} + \sqrt{1+\left|\mathrm{D}_{0^+}^{\gamma,\beta;\psi} u_n\right|^{2p(x)}}\right)dx \\
& - \frac{\xi}{2p^+}\int_\Omega g(x, u_n)\, u_n dx - \xi\int_\Omega G(x, u_n)\,dx + \frac{\xi}{2p^+}c_3|\Omega| + o(1)
\end{aligned}$$

$$=\mathbb{E}_{\xi}\left(u_{n}\right)+\frac{1}{2 p^{+}}\left\langle\mathbb{E}_{\xi}^{\prime}\left(u_{n}\right), u_{n}\right\rangle+\frac{\xi}{2 p^{+}} c_{3}|\Omega|+o(1) \rightarrow c+\frac{\xi}{2 p^{+}} c_{3}|\Omega|, \tag{3.22}$$

as $n \rightarrow +\infty$. By combining (3.21) and (3.22), we arrive at a contradiction. This establishes that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{H}(\Omega)$. Thus, from (3.8) and Hölder inequality, we have

$$\begin{aligned}\left|\int_{\Omega} g\left(x, u_{n}\right)\left(u_{n}-u\right) d x\right| \leq \int_{\Omega}\left|g\left(x, u_{n}\right)\right|\left|u_{n}-u\right| d x \leq C \int_{\Omega}\left|1+\left|u_{n}\right|^{r(x)-1}\right|\left|u_{n}-u\right| d x \\ \leq 2 C\left\|u_{n}-u\right\|_{L^{r(x)}}\left\|1+\left|u_{n}\right|^{r(x)-1}\right\|_{L^{r^{\prime}(x)}} \rightarrow 0, \quad \text{as } n \rightarrow+\infty,\end{aligned}$$

which implies that

$$\lim_{n \rightarrow+\infty} \int_{\Omega} g\left(x, u_{n}\right)\left(u_{n}-u\right) d x=0. \tag{3.23}$$

However, according to (3.7), we have

$$\left\langle\mathbb{E}_{\xi}^{\prime}\left(u_{n}\right), u_{n}-u\right\rangle \rightarrow 0, \quad \text{as } n \rightarrow+\infty. \tag{3.24}$$

By combining (3.23) and (3.24), we obtain

$$\left\langle\mathcal{A}\left(u_{n}\right), u_{n}-u\right\rangle=\xi \int_{\Omega} g\left(x, u_{n}\right)\left(u_{n}-u\right) d x+\left\langle\mathbb{E}_{\xi}^{\prime}\left(u_{n}\right), u_{n}-u\right\rangle \longrightarrow 0, \quad \text{as } n \rightarrow+\infty.$$

According to Proposition 2.4, we conclude that $u_n \rightarrow u$ in $\mathbb{H}(\Omega)$. This establishes that $\mathbb{E}_{\xi}(u)$ satisfies the (C_c) condition on $\mathbb{H}(\Omega)$. □

Conclusion of the proof of Theorem 3.1**.** Indeed, based on Lemmas 3.2 and 3.3, the functional $\mathbb{E}_{\xi}$ satisfies all conditions of the mountain pass Theorem 1.1. Therefore, the functional $\mathbb{E}_{\xi}$ has a critical value $c \geq v > 0$. Thus problem (1.3) has at least one nontrivial weak solution in $\mathbb{H}(\Omega)$.

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[1]Laboratory of Applied Mathematics and Scientic Computing,
Faculty of Sciences and Technology
Sultan Moulay Slimane University,
PO Box 523, 23000 Beni mellal, Morocco.

*Corresponding Author
Email address: arhrrabi.elhoussain@gmail.com
ORCID iD: https://orcid.org/0000-0002-9239-4603
Email address: h.elhouari94@gmail.com
ORCID iD: https://orcid.org/0000-0003-4240-1286