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BI-UNIVALENT FUNCTION SUBCLASSES WITH (p, q)-DERIVATIVE OPERATOR LINKED TO HORADAM POLYNOMIALS

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Abstract. In the open unit disk $\{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$, two subclasses of bi-univalent functions related to Horadam polynomials are presented and examined in this paper. For functions belonging to the recently established classes, we obtain the estimates of the first two coefficients. Furthermore, an estimate of the Fekete-Szegö problem is provided for functions in these classes. We also provide some observations and draw relevant connections to earlier research.

1. Preliminaries

The quantum calculus is very important because it is applied in numerous subfields within computer science, mathematics, physics, and other relevant disciplines. The significance of the q-derivative operator is demonstrated by its ability to be applied to various subclasses of holomorphic functions. Jackson [27] first investigated the q-analogue of the derivative and integral operator in 1908, along with some of its uses. Later, in [26], the concept of q-extension of the family of starlike functions was introduced. Subsequently, q-calculus was studied in the context of theory of univalent functions by Srivastava [38] and several mathematicians: the q-analogue of the Ruscheweyh operator was presented in [29]; authors looked at some of its applications for multivalent functions in [7, 8]; the convolution concept was used in [56] to establish the q-starlike functions linked to the generalized conic domain. Many authors have recently published several articles on sbclasses of q-starlike functions

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and other related classes (see [2, 17, 19, 31, 42–44, 47]). Scholars investigating these topics might find Srivastava's expository review paper [39] to be beneficial.

Furthermore, the extension of the q-calculus to the (p, q)-calculus, was taken into consideration by the researchers. The (p, q)-calculus, which includes the (p, q)-number, is first examined around the same time (1991) and subsequently on its own by [6, 11, 12, 54]. In [12], the (p, q)-number was explored as a way to unify or generalize various forms of q-oscillator algebras. The investigation of the (p, q)-number in [11] allows for the construction of a (p, q)-Harmonic oscillator. The (p, q)-numbers are investigated in [54] in order to calculate the (p, q)-Stirling numbers. Fibonacci oscillators were studied with the presentation of the (p, q)-number in [6]. Consequently, many mathematical, physical, and chemical problems require knowledge of (p, q)-calculus. Expanding upon the previously mentioned papers, numerous scientist have studied the (p, q)-calculus in a variety of research fields since 1991. A syntax for embedding the q-series into a (p, q)-series was given by the results in [28]. Additionally, they looked into (p, q)-hypergeometric series and discovered some outcomes that matched (p, q)-extensions of the well-known q-identities. The q-identities are extended correspondingly to yield the (p, q)-series (see, e.g., [5]). We provide some basic definitions of the (p, q)-calculus concepts used in this paper. The (p, q)-bracket number is given by $[j]_{p,q} = p^{j-1} + p^{j-2}q + p^{j-3}q^2 + \cdots + pq^{j-2} + q^{j-1} = \frac{p^j - q^j}{p-q}$, $p \neq q$, which is an extesion of q-number (see [27]), that is $[j]_q = \frac{1-q^j}{1-q}$, $q \neq 1$. Note that $[j]_{p,q}$ is symmetric and if $p = 1$, then $[j]_{p,q}=[j]_q$.

Let $\{\zeta \in \mathbb{C} : |\zeta| < 1\} = \mathfrak{D}$ be the open unit disk, and let the set of complex numbers be $\mathbb{C}$. Let $\mathbb{N}$ represent the natural number set and $\mathbb{R}$ the real number set.

Definition 1.1 ([53]). Let $0 < q < p \leq 1$ and g be a function defined on $\mathbb{C}$. Then the (p, q)-derivative of g is defined by

$$D_{p,q}g(\zeta) = \frac{g(p\zeta) - g(q\zeta)}{(p-q)\zeta}, \quad \zeta \neq 0,$$

and $D_{p,q}g(0) = g'(0)$, provided $g'(0)$ exists.

We note that $D_{p,q}\zeta^j = [j]_{p,q}\zeta^{j-1}$ and $D_{p,q}\ln(\zeta) = \frac{\ln(p/q)}{(p-q)\zeta}$. Also, we observe that $[j]_{p,q} \to j$ if $p = 1$ and $q \to 1^-$. Therefore, $D_{p,q}g(\zeta) \to g'(\zeta)$ as $p = 1$ and $q \to 1^-$. Any function's (p, q)-derivative is a linear operator. More accurately $D_{p,q}(ag(\zeta) + bh(\zeta)) = aD_{p,q}g(\zeta) + bD_{p,q}h(\zeta)$, for any constants a and b. The product rule and quotient rule are satisfied by the (p, q)-derivative (see [35]). The exponential functions are used to define the (p, q)-analogues of many functions, including sine, cosine, and tangent, in the same way as their well-known Euler expressions. In addition, the (p, q)-derivatives of these functions have been examined by Duran et al. [15].

The set of functions g that are regular in $\mathfrak{D}$ and have the following form is represented by $\mathcal{A}$:

$$g(\zeta) = \zeta + \sum_{j=2}^{\infty} d_j \zeta^j, \quad \zeta \in \mathfrak{D}, \tag{1.1}$$

with $g'(0) - 1 = 0 = g(0)$. $\mathcal{S}$ represents a sub-set of $\mathcal{A}$ that consists of univalent functions in $\mathfrak{D}$. If $g \in \mathcal{A}$ is of the form (1.1), then

$$D_{p,q} g(\zeta) = 1 + \sum_{j=2}^{\infty} [j]_{p,q} d_j \zeta^{j-1}, \quad \zeta \in \mathfrak{D}. \tag{1.2}$$

We say that $\mathfrak{g}_1$ is subordinate to $\mathfrak{g}_2$ for $\mathfrak{g}_1, \mathfrak{g}_2 \in \mathcal{A}$ regular in $\mathfrak{D}$, if there is a Schwarz function $\psi(\zeta)$ that is regular in $\mathfrak{D}$ with $|\psi(\zeta)| < 1$, $\psi(0) = 0$, such that $\mathfrak{g}_1(\zeta) = \mathfrak{g}_2(\psi(\zeta))$ (see [16]). The notation $\mathfrak{g}_1 \prec \mathfrak{g}_2$ or $\mathfrak{g}_1(\zeta) \prec \mathfrak{g}_2(\zeta)$, $\zeta \in \mathfrak{D}$, denotes this subordination. Specifically, when $\mathfrak{g}_2 \in \mathcal{S}$, we have

$$\mathfrak{g}_1(\zeta) \prec \mathfrak{g}_2(\zeta) \Leftrightarrow \mathfrak{g}_1(0) = \mathfrak{g}_2(0) \text{ and } \mathfrak{g}_1(\mathfrak{D}) \subset \mathfrak{g}_2(\mathfrak{D}).$$

The Koebe theorem (see [16]) states that the inverse of each function g in $\mathcal{S}$ is given by

$$g^{-1}(w) = w - d_2 w^2 + (2d_2^2 - d_3)w^3 - (5d_2^3 - 5d_2 d_3 + d_4)w^4 + \cdots = f(w) \tag{1.3}$$

satisfying $\zeta = g^{-1}(g(\zeta))$, $g(g^{-1}(w)) = w$, $|w| < r_0(g)$ and $1/4 \leq r_0(g)$, $\zeta, w \in \mathfrak{D}$. In $\mathfrak{D}$, a member g of $\mathcal{A}$ given by (1.1) is called bi-univalent if $g \in \mathcal{S}$ and $g^{-1} \in \mathcal{S}$. The set of such functions in $\mathfrak{D}$ is represented by σ. $\frac{1}{2}\log\left(\frac{1+\zeta}{1-\zeta}\right)$, $-\log(1-\zeta)$ and $\frac{\zeta}{1-\zeta}$ are some of the functions in the σ family. Nevertheless, despite being in $\mathcal{S}$, $\zeta - \frac{\zeta^2}{2}$, $\frac{\zeta}{1-\zeta^2}$, and the Koebe function do not belong to σ. For a concise analysis and to discover some of the remarkable characteristics of the family σ, see [9, 10, 30, 51] and the citation provided in these papers. Similar to the well-known subclasses of the family $\mathcal{S}$, Srivastava et al. [45] have introduced a number of subclasses of the family σ. In reality, many writers have since investigated a variety of alternative subfamilies of σ as follow-ups to the aforementioned subfamilies (see, for example [14, 20, 21, 41, 52]). The majority of these publications focus on the analysis of the Fekete-Szegó problem of functions in distinct σ subclasses.

The (p,q)-calculus was used previously to study several subclasses of the class $\mathcal{S}$ and the class σ. In [46], the subordination principle is used to define the classes of (p,q)-starlike and (p,q)-convex functions. Novel subclasses of the class σ associated with (p,q)-differential operators have also been presented and examined in a number of studies (refer to [3, 4, 13, 23, 32, 33]).

The Horadam polynomials $\mathcal{H}_j(\varkappa, u, v; r, s)$ (or $\mathcal{H}_j(\varkappa)$) have recently been studied and quantified using the recurrence relation given below by Horzum and Koçer [24] (see also [25]):

$$\mathcal{H}_j(\varkappa) = r\varkappa\mathcal{H}_{j-1}(\varkappa) + s\mathcal{H}_{j-2}(\varkappa), \tag{1.4}$$

with $\mathcal{H}_1(\varkappa) = u$, $\mathcal{H}_2(\varkappa) = v\varkappa$, where $\varkappa$, $r, s, u, v \in \mathbb{R}$, and $j \in \mathbb{N}\backslash\{1, 2\}$. It is evident from (1.4) that

$$\mathcal{H}_3(\varkappa) = rv\varkappa^2 + su = Z. \tag{1.5}$$

For $j \in \mathbb{N}$, the sequence $\mathcal{H}_j(\varkappa)$, has the following generating function (see [24]):

$$\mathsf{H}(\varkappa, \zeta) := \sum_{j=1}^{\infty} \mathcal{H}_j(\varkappa)\zeta^{j-1} = \frac{(v - ur)\varkappa\zeta + u}{1 - r\varkappa\varsigma - s\zeta^2}, \tag{1.6}$$

where $\varkappa \in \mathbb{R}$, $\zeta \in \mathbb{C}$ with $\varkappa \neq \mathfrak{R}(\zeta)$.

For specific selections of u, v, r and s, Horadam polynomials $\mathcal{H}_j(\varkappa, u, v, ;, r, s)$ leads to various known polynomials (see [40]). Interesting findings regarding coefficient estmates and Fekete-Szegö functional [18] have been found in [22, 34, 36, 48, 49, 55] for members of certain subclasses of σ associated with Horadam polynomials.

We present two new subclasses of σ subordinate to polynomials $\mathcal{H}_j(\varkappa)$ as in (1.4) and its generating function (1.6). The Fekete-Szegö functional on specific subclasses of σ and the patterns in the citations discussed above on coefficient-related problems serve as the inspiration for these families. $\mathsf{H}(\varkappa, \varsigma)$ is as in (1.6), $g^{-1}(w) = f(w)$ as in (1.3), $\varkappa \in \mathbb{R}$, $\zeta \in \mathfrak{D}$ and $w \in \mathfrak{D}$ are assumed throughout this paper, unless otherwise mentioned.

Definition 1.2. A function g in σ that possesses the series (1.1) is said to be a part of the class $\mathfrak{Y}^{\tau,\delta}_{\sigma,p,q}(\nu, \varkappa)$, $\nu \geq 1$, $0 < \delta \leq 1$ and $\tau \geq 1$, if

$$\frac{1}{2}\left\{\frac{\nu[D_{p,q}(\zeta D_{p,q}g(\zeta))]^\tau + (1-\nu)}{D_{p,q}g(\zeta)} + \left(\frac{\nu[D_{p,q}(\zeta D_{p,q}g(\zeta))]^\tau + (1-\nu)}{D_{p,q}g(\zeta)}\right)^{\frac{1}{\delta}}\right\}$$
$$\prec 1 - u + \mathsf{H}(\varkappa, \zeta)$$

and

$$\frac{1}{2}\left\{\frac{\nu[D_{p,q}(wD_{p,q}f(w))]^\tau + (1-\nu)}{D_{p,q}f(w)} + \left(\frac{\nu[D_{p,q}(wD_{p,q}f(w))]^\tau + (1-\nu)}{D_{p,q}f(w)}\right)^{\frac{1}{\delta}}\right\}$$
$$\prec 1 - u + \mathsf{H}(\varkappa, w).$$

For particular selections of p, q, ν and τ, the family $\mathfrak{Y}^{\tau,\delta}_{\sigma,p,q}(\nu, \varkappa)$ includes many new and existing subfamilies of σ. This is shown below.

1. $\mathfrak{H}^{\delta}_{\sigma,p,q}(\nu, \varkappa) \equiv \mathfrak{Y}^{1,\delta}_{\sigma,p,q}(\nu, \varkappa)$, $\nu \geq 1$, $0 < \delta \leq 1$, is the set of members g of σ that satisfy

$$\frac{1}{2}\left\{\frac{\nu D_{p,q}(\zeta D_{p,q}g(\zeta)) + (1-\nu)}{D_{p,q}g(\zeta)} + \left(\frac{\nu D_{p,q}(\zeta D_{p,q}g(\zeta)) + (1-\nu)}{D_{p,q}g(\zeta)}\right)^{\frac{1}{\delta}}\right\}$$
$$\prec 1 - u + \mathsf{H}(\varkappa, \zeta)$$

and

$$\frac{1}{2}\left\{\frac{\nu D_{p,q}(wD_{p,q}f(w))+(1-\nu)}{D_{p,q}f(w)}+\left(\frac{\nu D_{p,q}(wD_{p,q}f(w))+(1-\nu)}{D_{p,q}f(w)}\right)^{\frac{1}{\delta}}\right\}$$
$$\prec 1-u+\mathsf{H}(\varkappa,w).$$

2. $\mathfrak{I}^{\tau,\delta}_{\sigma,p,q}(\varkappa)\equiv\mathfrak{Y}^{\tau,\delta}_{\sigma,p,q}(1,\varkappa)$, $0<\delta\leq 1$ and $\tau\geq 1$, is the set of elements g of σ that satisfy

$$\frac{1}{2}\left\{\frac{[D_{p,q}(\zeta D_{p,q}g(\zeta))]^{\tau}}{D_{p,q}g(\zeta)}+\left(\frac{[D_{p,q}(\zeta D_{p,q}g(\zeta))]^{\tau}}{D_{p,q}g(\zeta)}\right)^{\frac{1}{\delta}}\right\}\prec 1-u+\mathsf{H}(\varkappa,\zeta)$$

and

$$\frac{1}{2}\left\{\frac{[D_{p,q}(wD_{p,q}f(w))]^{\tau}}{D_{p,q}f(w)}+\left(\frac{[D_{p,q}(wD_{p,q}f(w))]^{\tau}}{D_{p,q}f(w)}\right)^{\frac{1}{\delta}}\right\}\prec 1-u+\mathsf{H}(\varkappa,w).$$

3. If $p=1$ and $q\to 1^{-}$ in the set $\mathfrak{Y}^{\tau,\delta}_{\sigma,p,q}(\nu,\varkappa)$, then we obtain a subset $\Upsilon^{\tau,\delta}_{\sigma}(\nu,\varkappa)$, $\tau\geq 1$, $0<\delta\leq 1$, $\nu\geq 1$, which is the collection of members g of σ that satisfy

$$\frac{1}{2}\left\{\frac{\nu[(\zeta g'(\zeta))']^{\tau}+(1-\nu)}{g'(\zeta)}+\left(\frac{\nu[(\zeta\ g'(\zeta))']^{\tau}+(1-\nu)}{g'(\zeta)}\right)^{\frac{1}{\delta}}\right\}\prec 1-u+\mathsf{H}(\varkappa,\zeta)$$

and

$$\frac{1}{2}\left\{\frac{\nu[(wf'(w))']^{\tau}+(1-\nu)}{f'(w)}+\left(\frac{\nu[(wf'(w))']^{\tau}+(1-\nu)}{f'(w)}\right)^{\frac{1}{\delta}}\right\}\prec 1-u+\mathsf{H}(\varkappa,w).$$

4. $\mathfrak{Z}^{\tau}_{\sigma,p,q}(\nu,\varkappa)\equiv\mathfrak{Y}^{\tau,1}_{\sigma,p,q}(\nu,\varkappa)$, $\nu\geq 1$ and $\tau\geq 1$ is the set of elements g of σ that satisfy

$$\left\{\frac{\nu[D_{p,q}(\zeta D_{p,q}g(\zeta))]^{\tau}+(1-\nu)}{D_{p,q}g(\zeta)}\right\}\prec 1-u+\mathsf{H}(\varkappa,\zeta)$$

and

$$\left\{\frac{\nu[D_{p,q}(wD_{p,q}f(w))]^{\tau}+(1-\nu)}{D_{p,q}f(w)}\right\}\prec 1-u+\mathsf{H}(\varkappa,w).$$

Remark 1.1. i) $\mathfrak{H}^{\delta}_{\sigma,p,q}(1,\varkappa)\equiv\mathfrak{I}^{1,\delta}_{\sigma,p,q}(\varkappa)$;
ii) $\mathfrak{H}^{1}_{\sigma,p,q}(\nu,\varkappa)\equiv\mathfrak{Z}^{1}_{\sigma,p,q}(\nu,\varkappa)$;
iii) $\mathfrak{Z}^{\tau}_{\sigma,p,q}(1,\varkappa)\equiv\mathfrak{I}^{\tau,1}_{\sigma,p,q}(\varkappa)$.

Definition 1.3. A function g in σ that possesses the series (1.1) is said to be a part of the class $\mathfrak{T}^{\tau,\delta}_{\sigma,p,q}(\gamma,\varkappa)$, $0\leq\gamma\leq 1$, $0<\delta\leq 1$ and $\tau\geq 1$, if

$$\frac{1}{2}\left\{\frac{\zeta(D_{p,q}g(\zeta))^{\tau}}{\gamma g(\zeta)+(1-\gamma)\zeta}+\left(\frac{\zeta(D_{p,q}g(\zeta))^{\tau}}{\gamma g(\zeta)+(1-\gamma)\zeta}\right)^{\frac{1}{\delta}}\right\}\prec 1-u+\mathsf{H}(\varkappa,\zeta)$$

and

$$\frac{1}{2}\left\{\frac{w(D_{p,q}f(w))^{\tau}}{\gamma f(w)+(1-\gamma)w}+\left(\frac{w(D_{p,q}f(w))^{\tau}}{\gamma f(w)+(1-\gamma)w}\right)^{\frac{1}{\delta}}\right\}\prec 1-u+\mathsf{H}(\varkappa,w).$$

For particular selections of γ, τ, and δ, the family $\mathfrak{T}^{\tau,\delta}_{\sigma,p,q}(\gamma,\varkappa)$ includes many new and preexisting subfamilies of σ, as shown below.

1. $\mathfrak{C}^{\tau,\delta}_{\sigma,p,q}(\varkappa)\equiv\mathfrak{T}^{\tau,\delta}_{\sigma,p,q}(0,\varkappa)$, $0<\delta\leq 1$ and $\tau\geq 1$, is the set of members g of σ that satisfy

$$\frac{1}{2}\left((D_{p,q}g(\zeta))^{\tau}+(D_{p,q}g(\zeta))^{\frac{\tau}{\delta}}\right)\prec 1-u+\mathsf{H}(\varkappa,\zeta)$$

and

$$\frac{1}{2}\left((D_{p,q}f(w))^{\tau}+D_{p,q}f(w)^{\frac{\tau}{\delta}}\right)\prec 1-u+\mathsf{H}(\varkappa,w).$$

2. $\mathfrak{D}^{\tau,\delta}_{\sigma,p,q}(x)\equiv\mathfrak{T}^{\tau,\delta}_{\Sigma,p,q}(1,\varkappa)$, $0<\delta\leq 1$ and $\tau\geq 1$, is the family of elements g of σ that satisfy

$$\frac{1}{2}\left\{\frac{\zeta(D_{p,q}g(\zeta))^{\tau}}{g(\zeta)}+\left(\frac{\zeta(D_{p,q}g(\zeta))^{\tau}}{g(\zeta)}\right)^{\frac{1}{\delta}}\right\}\prec 1-u+\mathsf{H}(\varkappa,\zeta)$$

and

$$\frac{1}{2}\left\{\frac{w(D_{p,q}f(w))^{\tau}}{f(w)}+\left(\frac{w(D_{p,q}f(w))^{\tau}}{f(w)}\right)^{\frac{1}{\delta}}\right\}\prec 1-u+\mathsf{H}(\varkappa,w).$$

3. If $p=1$ and $q\to 1^{-}$ in the class $\mathfrak{T}^{\tau,\delta}_{\sigma,p,q}(\nu,\varkappa)$, then we have a subset $\Gamma^{\tau,\delta}_{\sigma}(\nu,\varkappa)$, $\tau\geq 1$, $0<\delta\leq 1$, $0\leq\gamma\leq 1$ of $g\in\sigma$ that satisfy

$$\frac{1}{2}\left\{\frac{\zeta(g'(\zeta))^{\tau}}{\gamma g(\zeta)+(1-\gamma)\zeta}+\left(\frac{\zeta(g'(\zeta))^{\tau}}{\gamma g(\zeta)+(1-\gamma)\zeta}\right)^{\frac{1}{\delta}}\right\}\prec 1-u+\mathsf{H}(\varkappa,\zeta)$$

and

$$\frac{1}{2}\left\{\frac{w(f'(w))^{\tau}}{\gamma f(w)+(1-\gamma)w}+\left(\frac{w(f'(w))^{\tau}}{\gamma f(w)+(1-\gamma)w}\right)^{\frac{1}{\delta}}\right\}\prec 1-u+\mathsf{H}(\varkappa,w).$$

4. $\mathfrak{O}^{\tau}_{\sigma,p,q}(\gamma,\varkappa)\equiv\mathfrak{T}^{\tau,1}_{\sigma,p,q}(\gamma,\varkappa)$, $0\leq\gamma\leq 1$ and $\tau\geq 1$, is the group of $g\in\sigma$ that satisfy

$$\frac{\zeta(D_{p,q}g(\zeta))^{\tau}}{\gamma g(\zeta)+(1-\gamma)\zeta}\prec 1-u+\mathsf{H}(\varkappa,\zeta)$$

and

$$\frac{w(D_{p,q}f(w))^{\tau}}{\gamma f(w)+(1-\gamma)w}\prec 1-u+\mathsf{H}(\varkappa,w).$$

Remark 1.2. i) $\mathfrak{D}^{\tau,1}_{\sigma,p,q}(x)\equiv\mathfrak{O}^{\tau}_{\sigma,p,q}(1,\varkappa)$;
ii) $\Gamma^{\tau,1}_{\sigma}(\nu,\varkappa)\equiv\mathfrak{O}^{\tau}_{\sigma,p=1,q\to 1^{-}}(\gamma,\varkappa)$.

In Section 2, we find estimates for $|d_2|$, $|d_3|$ and $|d_3-\mu d_2^2|$, $\mu\in\mathbb{R}$, for functions in $\mathfrak{Y}^{\tau,\delta}_{\sigma,p,q}(\nu,\varkappa)$. In Section 3, we find estimates for $|d_2|$, $|d_3|$ and $|d_3-\mu d_2^2|$, $\mu\in\mathbb{R}$, for functions in $\mathfrak{T}^{\tau,\delta}_{\sigma,p,q}(\gamma,\varkappa)$. Interesting results are also presented along with relevant connections to the published research.

2. Results for the Class $\mathfrak{Y}^{\tau,\delta}_{\sigma,p,q}(\nu,\varkappa)$

First, we determine the coefficient estimates for any function $g\in\mathfrak{Y}^{\tau,\delta}_{\sigma,p,q}(\nu,\varkappa)$, the class as defined in Definition 1.2.

Theorem 2.1. *Let $0<\delta\le 1$, $\nu\ge 1$ and $\tau\ge 1$. If $g\in\mathfrak{Y}^{\tau,\delta}_{\sigma,p,q}(\nu,\varkappa)$, then*

$$|d_2|\le\frac{2\delta|v\varkappa|\sqrt{|v\varkappa|}}{\sqrt{|(2\delta(\delta+1)X+(1-\delta)W^2[2]^2_{p,q})(v\varkappa)^2-(\delta+1)^2W^2[2]^2_{p,q}Z|}},\tag{2.1}$$

$$|d_3|\le\frac{4\delta^2(v\varkappa)^2}{(\delta+1)^2W^2[2]^2_{p,q}}+\frac{2\delta|v\varkappa|}{(\delta+1)U[3]_{p,q}},\tag{2.2}$$

and for $\mu\in\mathbb{R}$

$$|d_3-\mu d_2^2|\le\begin{cases}\frac{2\delta|v\varkappa|}{(\delta+1)U[3]_{p,q}}, & |1-\mu|\le J,\\ \frac{4\delta^2|v\varkappa|^3\,|1-\mu|}{|(2\delta(\delta+1)X+(1-\delta)^2W^2[2]^2_{p,q})(v\varkappa)^2-(\delta+1)^2W^2[2]^2_{p,q}Z|}, & |1-\mu|\ge J,\end{cases}\tag{2.3}$$

where

$$J=\left|\frac{(2\delta(\delta+1)X+(1-\delta)W^2[2]^2_{p,q})v^2\varkappa^2-(\delta+1)^2W^2[2]^2Z}{2\delta(1+\delta)U[3]_{p,q}v^2\varkappa^2}\right|,\tag{2.4}$$

$$X=U[3]_{p,q}+V[2]^2_{p,q},\tag{2.5}$$

$$U=\nu\tau[3]_{p,q}-1,\tag{2.6}$$

$$V=1-\nu\tau[2]_{p,q}+\frac{\nu\tau(\tau-1)[2]^2_{p,q}}{2},\tag{2.7}$$

$$W=\nu\tau[2]_{p,q}-1,\tag{2.8}$$

and Z is as in (1.5).

Proof. Let $g\in\mathfrak{Y}^{\tau,\delta}_{\sigma,p,q}(\gamma,\varkappa)$. Then, because of Definition 1.3, we obtain

$$\frac{1}{2}\left\{\frac{\nu[D_{p,q}(\zeta D_{p,q}g(\zeta))]^\tau+(1-\nu)}{D_{p,q}g(\zeta)}+\left(\frac{\nu[D_{p,q}(\zeta D_{p,q}g(\zeta))]^\tau+(1-\nu)}{D_{p,q}g(\zeta)}\right)^{\frac{1}{\delta}}\right\}$$
$$=1-u+\mathsf{H}(\varkappa,\mathfrak{m}(\zeta))\tag{2.9}$$

and

$$\frac{1}{2}\left\{\frac{\nu[D_{p,q}(wD_{p,q}f(w))]^\tau+(1-\nu)}{D_{p,q}f(w)}+\left(\frac{\nu[D_{p,q}(wD_{p,q}f(w))]^\tau+(1-\nu)}{D_{p,q}f(w)}\right)^{\frac{1}{\delta}}\right\}$$
$$=1-u+\mathsf{H}(\varkappa,\mathfrak{n}(w)),\tag{2.10}$$

where

$$\mathfrak{m}(\zeta) = m_1\zeta + m_2\zeta^2 + m_3\zeta^3 + \cdots \quad \text{and} \quad \mathfrak{n}(w) = n_1 w + n_2 w^2 + n_3 w^3 + \cdots, \tag{2.11}$$

are some functions holomorphic in $\mathfrak{D}$ with $|\mathfrak{m}(\zeta)| < 1$, $|\mathfrak{n}(w)| < 1$ and is known that

$$|\mathfrak{m}_i| \leq 1 \quad \text{and} \quad |\mathfrak{n}_i| \leq 1, \quad i \in \mathbb{N}. \tag{2.12}$$

From (2.9)–(2.11), it follows that

$$\begin{aligned} &\frac{1}{2}\left\{\frac{\nu[D_{p,q}(\zeta D_{p,q}g(\zeta))]^\tau + (1-\nu)}{D_{p,q}g(\zeta)} + \left(\frac{\nu[D_{p,q}(\zeta D_{p,q}g(\zeta))]^\tau + (1-\nu)}{D_{p,q}g(\zeta)}\right)^{\frac{1}{\delta}}\right\} \\ &= 1 - u + \mathcal{H}_1(\varkappa) + \mathcal{H}_2(\varkappa)\mathfrak{m}(\zeta) + \mathcal{H}_3(\varkappa)\mathfrak{m}^2(\zeta) + \cdots \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} &\frac{1}{2}\left\{\frac{\nu[D_{p,q}(wD_{p,q}f(w))]^\tau + (1-\nu)}{D_{p,q}f(w)} + \left(\frac{\nu[D_{p,q}(wD_{p,q}f(w))]^\tau + (1-\nu)}{D_{p,q}f(w)}\right)^{\frac{1}{\delta}}\right\} \\ &= 1 - u + \mathcal{H}_1(\varkappa) + \mathcal{H}_2(\varkappa)\mathfrak{n}(w) + \mathcal{H}_3(\varkappa)\mathfrak{n}^2(w) + \cdots. \end{aligned} \tag{2.14}$$

In the light of (1.4), we determine from (2.13) and (2.14) that

$$\begin{aligned} &\frac{1}{2}\left\{\frac{\nu[D_{p,q}(\zeta D_{p,q}g(\zeta))]^\tau + (1-\nu)}{D_{p,q}g(\zeta)} + \left(\frac{\nu[D_{p,q}(\zeta D_{p,q}g(\zeta))]^\tau + (1-\nu)}{D_{p,q}g(\zeta)}\right)^{\frac{1}{\delta}}\right\} \\ &= 1 + \mathcal{H}_2(\varkappa)\mathfrak{m}_1\zeta + [\mathcal{H}_2(\varkappa)\mathfrak{m}_2 + \mathcal{H}_3(\varkappa)\mathfrak{m}_1^2]\zeta^2 + \cdots \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} &\frac{1}{2}\left\{\frac{\nu[D_{p,q}(wD_{p,q}f(w))]^\tau + (1-\nu)}{D_{p,q}f(w)} + \left(\frac{\nu[D_{p,q}(wD_{p,q}f(w))]^\tau + (1-\nu)}{D_{p,q}f(w)}\right)^{\frac{1}{\delta}}\right\} \\ &= 1 + \mathcal{H}_2(\varkappa)\mathfrak{n}_1 w + [\mathcal{H}_2(\varkappa)\mathfrak{n}_2 + \mathcal{H}_3(\varkappa)\mathfrak{n}_1^2]\mathrm{w}^2 + \cdots. \end{aligned} \tag{2.16}$$

Using (1.2) and comparing (2.15) and (2.16), we have

$$\frac{(\delta+1)W[2]_{p,q}}{2\delta}\, d_2 = \mathcal{H}_2(\varkappa)\mathfrak{m}_1, \tag{2.17}$$

$$\left(\frac{\delta+1}{2\delta}\right)\left(U[3]_{p,q}d_3 + V[2]_{p,q}^2 d_2^2\right) + \left(\frac{1-\delta}{4\delta^2}\right) W^2[2]_{p,q}^2 d_2^2 = \mathcal{H}_2(\varkappa)\mathfrak{m}_2 + \mathcal{H}_3(\varkappa)\mathfrak{m}_1^2, \tag{2.18}$$

$$-\frac{(\delta+1)W[2]_{p,q}}{2\delta}\, d_2 = \mathcal{H}_2(\varkappa)\mathfrak{n}_1 \tag{2.19}$$

and

$$\left(\frac{\delta+1}{2\delta}\right)\left(U[3]_{p,q}(2d_2^2 - d_3) + V[2]_{p,q}^2 d_2^2\right) + \left(\frac{1-\delta}{4\delta^2}\right) W^2[2]_{p,q}^2 d_2^2 = \mathcal{H}_2(\varkappa)\mathfrak{n}_2 + \mathcal{H}_3(\varkappa)\mathfrak{n}_1^2, \tag{2.20}$$

where U, V and W are as mentioned in (2.6), (2.7) and (2.8), respectively. From (2.17) and (2.19), we easily obtain

$$\mathfrak{m}_1 = -\mathfrak{n}_1, \tag{2.21}$$

and also

$$\frac{(\delta+1)^2 W^2 [2]_{p,q}^2}{2\delta^2}\, d_2^2 = (\mathfrak{m}_1^2 + \mathfrak{n}_1^2)(\mathcal{H}_2(\varkappa))^2. \tag{2.22}$$

The bound on $|d_2|$ is obtained by adding (2.18) and (2.20)

$$\left[\left(\frac{\delta+1}{\delta}\right) X + \left(\frac{1-\delta}{2\delta^2}\right) W^2[2]_{p,q}^2\right] d_2^2 = \mathcal{H}_2(\varkappa)(\mathfrak{m}_2+\mathfrak{n}_2) + \mathcal{H}_3(\varkappa)(\mathfrak{m}_1^2+\mathfrak{n}_1^2), \tag{2.23}$$

where X is as in (2.5). The value of $\mathfrak{m}_1^2 + \mathfrak{n}_1^2$ from (2.22) is substituted in (2.23), yielding

$$d_2^2 = \frac{2\delta^2 \mathcal{H}_2^3(\varkappa)(\mathfrak{m}_2+\mathfrak{n}_2)}{(2\delta(\delta+1)X + (1-\delta)W^2[2]_{p,q}^2)\mathcal{H}_2^2(\varkappa) - (\delta+1)^2 W^2 [2]_{p,q}^2 \mathcal{H}_3(\varkappa)}. \tag{2.24}$$

Using (1.5) and applying (2.12) to the coefficients $\mathfrak{m}_2$ and $\mathfrak{n}_2$ yields (2.1).

We deduct (2.20) from (2.18) to get the bound on $|d_3|$:

$$d_3 = d_2^2 + \frac{\mathcal{H}_2(\varkappa)(\mathfrak{m}_2-\mathfrak{n}_2)}{\left(\frac{\delta+1}{\delta}\right) U[3]_{p,q}}. \tag{2.25}$$

Then in view of (2.21) and (2.22), (2.25) becomes

$$d_3 = \frac{2\delta^2 \mathcal{H}_2^2(\varkappa)(\mathfrak{m}_1^2+\mathfrak{n}_1^2)}{(\delta+1)^2 W^2[2]_{p,q}^2} + \frac{\delta \mathcal{H}_2(\varkappa)(\mathfrak{m}_2-\mathfrak{n}_2)}{(\delta+1)U[3]_{p,q}},$$

and applying (2.12) for the coefficients $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{n}_1$ and $\mathfrak{n}_2$ we get (2.2).

From (2.24) and (2.25), for $\mu \in \mathbb{R}$, we get in view of (1.4) that

$$\begin{aligned} &|d_3 - \mu d_2^2| \\ =&|\mathcal{H}_2(\varkappa)| \left| \left(\mathfrak{B}_2(\mu,\varkappa) + \frac{\delta}{(\delta+1)U[3]_{p,q}}\right)\mathfrak{m}_2 + \left(\mathfrak{B}_2(\mu,\varkappa) - \frac{\delta}{(\delta+1)U[3]_{p,q}}\right)\mathfrak{n}_2\right|, \end{aligned}$$

where

$$\mathfrak{B}_2(\mu,\varkappa) = \frac{2\delta^2(1-\mu)\mathcal{H}_2^2(\varkappa)}{(2\delta(\delta+1)X + (1-\delta)W^2[2]_{p,q}^2)\mathcal{H}_2^2(\varkappa) - (\delta+1)^2 W^2[2]_{p,q}^2 \mathcal{H}_3(\varkappa)}.$$

Clearly

$$|d_3 - \mu d_2^2| \le \begin{cases} \frac{2\delta|\mathcal{H}_2(\varkappa)|}{(\delta+1)U[3]_{p,q}}, & 0 \le |\mathfrak{B}_2(\mu,\varkappa)| \le \frac{\delta}{(\delta+1)U[3]_{p,q}}, \\ 2|\mathcal{H}_2(\varkappa)||\mathfrak{B}_2(\mu,\varkappa)|, & \mathfrak{B}_2(\mu,\varkappa)| \ge \frac{\delta}{(\delta+1)U[3]_{p,q}}, \end{cases}$$

which results in (2.3) with J as in (2.4) and Z as in (1.5). □

Corollary 2.1. *Let us assume that* $\tau = 1$ *in Theorem* 2.1. *Then, for any function* $g \in \mathfrak{H}^{\delta}_{\sigma,p,q}(\nu, \varkappa)$ *the upper bounds of* $|d_2|$, $|d_3|$ *and* $|d_3 - \mu d_2^2|$, $\mu \in \Re$, *are given by* (2.1), (2.2) *and* (2.3), *respectively, with* $U = U_1 = \nu[3]_{p,q} - 1$, $V = V_1 = 1 - \nu[2]_{p,q}$, $W = W_1 = -V_1$ *and* $X = X_1 = U_1[3]_{p,q} + V_1[2]^2_{p,q}$. *To change* U_1, V_1, W_1 *and* X_1 *for* U, V, W *and* X, *respectively, for* J *in* (2.4) *and* Z, *as stated in* (1.5).

Corollary 2.2. *Let us assume that* $\nu = 1$ *in Theorem* 2.1. *Then for any function* $g \in \mathfrak{I}^{\tau,\delta}_{\sigma,p,q}(\varkappa)$ *the upper bounds of* $|d_2|$, $|d_3|$ *and* $|d_3 - \mu d_2^2|$, $\mu \in \Re$, *are given by* (2.1), (2.2) *and* (2.3), *respectively, with* $U = U_2 = \tau[3]_{p,q} - 1$, $V = V_2 = 1 - \tau[2]_{p,q} + \frac{\tau(\tau-1)[2]^2_{p,q}}{2}$, $W = W_2 = \tau[2]_{p,q} - 1$ *and* $X = X_2 = U_2[3]_{p,q} + V_2[2]^2_{p,q}$. *For* J *in* (2.4), U_2, V_2, W_2 *and* X_2 *should be used in place of* U, V, W *and* X, *respectively, and* Z, *as stated in* (1.5).

Taking $p = 1$ and $q \to 1^-$ in the Theorem 2.1, we get the following.

Corollary 2.3. *Let* $0 < \delta \leq 1$, $\tau \geq 1$ *and* $\nu \geq 1$. *If* $g \in \Upsilon^{\tau,\delta}_{\sigma}(\nu, \varkappa)$, *then*

$$|d_2| \leq \frac{\delta|\upsilon\varkappa|\sqrt{2|\upsilon\varkappa|}}{\sqrt{|(\delta(\delta+1)Y + 2(1-\delta)(2\nu\tau-1)^2)(\upsilon\varkappa)^2 - 2(\delta+1)^2(2\nu\tau-1)^2 Z|}},$$

$$|d_3| \leq \left(\frac{\delta\upsilon\varkappa}{(\delta+1)(2\nu\tau-1)}\right)^2 + \frac{2\delta|\upsilon\varkappa|}{3(\delta+1)(3\nu\tau-1)},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{2\delta|\upsilon\varkappa|}{3(\delta+1)(3\nu\tau-1)}, & |1-\mu| \leq J_1, \\ \frac{2\delta^2|\upsilon\varkappa|^3\,|1-\mu|}{|(\delta(\delta+1)Y+2(1-\delta)(2\nu\tau-1)^2)(\upsilon\varkappa)^2 - 2(\delta+1)^2(2\nu\tau-1)^2 Z|}, & |1-\mu| \geq J_1, \end{cases}$$

where

$$J_1 = \left|\frac{(\delta(\delta+1)Y + 2(1-\delta)(2\nu\tau-1)^2)(\upsilon\varkappa)^2 - 2(\delta+1)^2(2\nu\tau-1)^2 Z}{3\delta(\delta+1)(3\nu\tau-1)\upsilon^2\varkappa^2}\right|,$$

$Y = 8\nu\tau^2 - 7\nu\tau + 1$ *and* Z *is as in* (1.5).

Remark 2.1. i) In Corollary 2.3, $\delta = 1$ yields Theorem 2.2 of Swamy and Sailaja [50]. Moreover, we obtain Corollaries 2.3 and 2.4 of Swamy and Sailaja [50], for $\tau = 1$ and $\nu = 1$, respectively.

ii) Using $\delta = \tau = \nu = 1$ in Corollary 2.3 we derive Corollary 1 of Horhan et al. [34].

Taking $\delta = 1$ in the above theorem, we get the following.

Corollary 2.4. *If* $g \in \mathfrak{Z}^{\tau}_{\sigma,p,q}(\nu, \varkappa)$, $\nu \geq 1$ *and* $\tau \geq 1$, *then*

$$|d_2| \leq \frac{|\upsilon\varkappa|\sqrt{|\upsilon\varkappa|}}{\sqrt{|X(\upsilon\varkappa)^2 - W^2[2]^2_{p,q}Z|}}, \quad |d_3| \leq \frac{(\upsilon\varkappa)^2}{W^2[2]^2_{p,q}} + \frac{|\upsilon\varkappa|}{U[3]_{p,q}},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \le \begin{cases} \frac{|v\varkappa|}{U[3]_{p,q}}, & |1-\mu| \le \left| \frac{Xv^2\varkappa^2 - W^2[2]^2 Z}{U[3]_{p,q}v^2\varkappa^2} \right|, \\ \frac{|v\varkappa|^3\,|1-\mu|}{|X(v\varkappa)^2 - W^2[2]_{p,q}^2 Z|}, & |1-\mu| \ge \left| \frac{Xv^2\varkappa^2 - W^2[2]^2 Z}{U[3]_{p,q}v^2\varkappa^2} \right|, \end{cases}$$

where X, U, V *and* W *are as detailed in* (2.5), (2.6), (2.7) *and* (2.8), *respectively.*

Remark 2.2. i) In Corollary 2.4, $q \to 1^-$ and $p = 1$ yields Theorem 2.2 of Swamy and Sailaja [50]. Moreover, we obtain Corollaries 2.3 and 2.4 of Swamy and Sailaja [50], for $\tau = 1$ and $\nu = 1$, respectively.

ii) Using $\nu = \tau = 1$, $q \to 1^-$ and $p = 1$ in Corollary 2.4 we derive Corollary 1 of Horhan et al. [34].

3. Results for the Class $\mathfrak{T}_{\sigma,p,q}^{\tau,\delta}(\gamma,\varkappa)$

First, we calculate the coefficient estimates for any function $g \in \mathfrak{T}_{\sigma,p,q}^{\tau,\delta}(\gamma,\varkappa)$, the class as defined in Definition 1.3.

Theorem 3.1. *Let* $0 < \delta \le 1$, $\tau \ge 1$ *and* $0 \le \nu \le 1$. *If* $g \in \mathfrak{T}_{\sigma,p,q}^{\tau,\delta}(\gamma,\varkappa)$, *then*

$$|d_2| \le \frac{2\delta|v\varkappa|\sqrt{|v\varkappa|}}{\sqrt{|(2\delta(\delta+1)(A+S)+(1-\delta)B^2)(v\varkappa)^2 - (\delta+1)^2B^2Z|}}, \tag{3.1}$$

$$|d_3| \le \frac{4\delta^2(v\varkappa)^2}{(\delta+1)^2B^2} + \frac{2\delta|v\varkappa|}{(\delta+1)A}, \tag{3.2}$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \le \begin{cases} \frac{2\delta|v\varkappa|}{(\delta+1)A}, & |1-\mu| \le \mathbb{Q}, \\ \frac{4\delta^2|v\varkappa|^3\,|1-\mu|}{|(2\delta(\delta+1)(A+S)+(1-\delta)B^2)(v\varkappa)^2-(\delta+1)^2B^2Z|}, & |1-\mu| \ge \mathbb{Q}, \end{cases} \tag{3.3}$$

where

$$\mathbb{Q} = \left| \frac{(2\delta(\delta+1)(A+S)+(1-\delta)B)^2)v^2\varkappa^2 - (\delta+1)^2B)^2Z}{2\delta(1+\delta)Av^2\varkappa^2} \right|, \tag{3.4}$$

$$A = \tau[3]_{p,q} - \gamma, \tag{3.5}$$

$$S = \frac{\tau(\tau-1)[2]_{p,q}^2}{2} - \gamma\tau[2]_{p,q} + \gamma^2, \tag{3.6}$$

$$B = \tau[2]_{p,q} - \gamma, \tag{3.7}$$

and Z *is as in* (1.5).

Proof. Let $g \in \mathfrak{T}_{\sigma,p,q}^{\tau,\delta}(\gamma,\varkappa)$. Following that, due to Definition 1.3, we obtain

$$\frac{1}{2}\left\{ \frac{\zeta(D_{p,q}g(\zeta))^\tau}{\gamma g(\zeta)+(1-\gamma)\zeta} + \left(\frac{\zeta(D_{p,q}g(\zeta))^\tau}{\gamma g(\zeta)+(1-\gamma)\zeta} \right)^{\frac{1}{\delta}} \right\} = 1 - u + \mathsf{H}(\varkappa,\mathfrak{m}(\zeta)) \tag{3.8}$$

and

$$\frac{1}{2}\left\{\frac{w(D_{p,q}f(w))^{\tau}}{\gamma f(w)+(1-\gamma)w}+\left(\frac{w(D_{p,q}f(w))^{\tau}}{\gamma f(w)+(1-\gamma)w}\right)^{\frac{1}{\delta}}\right\}=1-u+\mathsf{H}(\varkappa,\mathfrak{n}(w)), \tag{3.9}$$

where $\mathfrak{m}(\zeta)$ and $\mathfrak{n}(w)$ are regular functions as given in (2.11) satisfying (2.12).

By going through the steps in the proof of Theorem 2.1 to obtain (2.13), (2.14), (2.15), and (2.16), one can easily get the following in view (3.8) and (3.9).

$$\frac{(\delta+1)B}{2\delta}d_2 =\mathcal{H}_2(\varkappa)\mathfrak{m}_1, \tag{3.10}$$

$$\left(\frac{\delta+1}{2\delta}\right)\left(Ad_3+Sd_2^2\right)+\left(\frac{1-\delta}{4\delta^2}\right)B^2d_2^2 =\mathcal{H}_2(\varkappa)\mathfrak{m}_2+\mathcal{H}_3(\varkappa)\mathfrak{m}_1^2, \tag{3.11}$$

$$-\frac{(\delta+1)B}{2\delta}d_2 =\mathcal{H}_2(\varkappa)\mathfrak{n}_1 \tag{3.12}$$

and

$$\left(\frac{\delta+1}{2\delta}\right)\left(A(2d_2^2-d_3)+Sd_2^2\right)+\left(\frac{1-\delta}{4\delta^2}\right)B^2d_2^2 = \mathcal{H}_2(\varkappa)\mathfrak{n}_2+\mathcal{H}_3(\varkappa)\mathfrak{n}_1^2, \tag{3.13}$$

where A, S and B are as mentioned in (3.5), (3.6) and (3.7), respectively. From (3.10) and (3.12), we easily obtain

$$\mathfrak{m}_1 = -\mathfrak{n}_1, \tag{3.14}$$

and also

$$\frac{(\delta+1)^2B^2}{2\delta^2}\,d_2^2 = (\mathfrak{m}_1^2+\mathfrak{n}_1^2)(\mathcal{H}_2(\varkappa))^2. \tag{3.15}$$

We add (3.11) and (3.13) to obtain the bound on $|d_2|$:

$$\left(\left(\frac{\delta+1}{\delta}\right)(A+S)+\left(\frac{1-\delta}{2\delta^2}\right)B^2\right)d_2^2 = \mathcal{H}_2(\varkappa)(\mathfrak{m}_2+\mathfrak{n}_2)+\mathcal{H}_3(\varkappa)(\mathfrak{m}_1^2+\mathfrak{n}_1^2). \tag{3.16}$$

The value of $\mathfrak{m}_1^2+\mathfrak{n}_1^2$ from (3.15) is substituted in (3.16) to obtain

$$d_2^2 = \frac{2\delta^2\mathcal{H}_2^3(\varkappa)(\mathfrak{m}_2+\mathfrak{n}_2)}{(2\delta(\delta+1)(A+S)+(1-\delta)B^2)\mathcal{H}_2^2(\varkappa)-(\delta+1)^2B^2\mathcal{H}_3(\varkappa)}. \tag{3.17}$$

Using (1.5) and applying (2.12) for the coefficients $\mathfrak{m}_2$ and $\mathfrak{n}_2$, we obtain (3.1).

We subtract (3.13) from (3.11) to get the bound on $|d_3|$:

$$d_3 = d_2^2+\frac{\mathcal{H}_2(\varkappa)(\mathfrak{m}_2-\mathfrak{n}_2)}{\left(\frac{\delta+1}{\delta}\right)A}. \tag{3.18}$$

Then in view of (3.14) and (3.15), (3.18) becomes

$$d_3 = \frac{2\delta^2\mathcal{H}_2^2(\varkappa)(\mathfrak{m}_1^2+\mathfrak{n}_1^2)}{(\delta+1)^2B^2}+\frac{\delta\mathcal{H}_2(\varkappa)(\mathfrak{m}_2-\mathfrak{n}_2)}{(\delta+1)A},$$

and applying (2.12) for the coefficients $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{n}_1$ and $\mathfrak{n}_2$ we get (3.2).

From (3.17) and (3.18), for $\mu \in \mathbb{R}$, we get in view of (1.4) that

$$|d_3 - \mu d_2^2| = \frac{|\mathcal{H}_2(\varkappa)|}{2} \left| \left(\mathfrak{B}_1(\mu,\varkappa) + \frac{\delta}{(\delta+1)A}\right)\mathfrak{m}_2 + \left(\mathfrak{B}_1(\mu,\varkappa) - \frac{\delta}{(\delta+1)A}\right)\mathfrak{n}_2\right|,$$

where

$$\mathfrak{B}_1(\mu,\varkappa) = \frac{2\delta^2(1-\mu)\mathcal{H}_2^2(\varkappa)}{(2\delta(\delta+1)(A+S)+(1-\delta)B^2)\mathcal{H}_2^2(\varkappa) - (\delta+1)^2B^2\mathcal{H}_3(\varkappa)}.$$

Clearly,

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{|\mathcal{H}_2(\varkappa)|}{(\delta+1)A}, & 0 \leq |\mathfrak{B}_1(\mu,\varkappa)| \leq \frac{\delta}{(\delta+1)A}, \\ |\mathcal{H}_2(\varkappa)| \cdot |\mathfrak{B}_1(\mu,\varkappa)|, & |\mathfrak{B}_1(\mu,\varkappa)| \geq \frac{\delta}{(\delta+1)A}, \end{cases}$$

from which we conclude (3.3) with $\mathbb{Q}$ as in (3.4) and Z as in (1.5). □

Corollary 3.1. *Let $\gamma = 0$ in the above theorem. Then, the upper bounds of $|d_2|$, $|d_3|$, and $|d_3 - \mu d_2^2|$, $\mu \in \Re$, for any function $g \in \mathfrak{H}^{\delta}_{\sigma,p,q}(\varkappa)$ are given by* (3.1), (3.2) *and* (3.3), *respectively, with $A = A_1 = \tau[3]_{p,q}$, $S = S_1 = \frac{\tau(\tau-1)[2]^2_{p,q}}{2}$ and $B = B_1 = \tau[2]_{p,q}$. A_1, S_1 and B_1 should be used in place of A, S and B, respectively, for $\mathbb{Q}$ in* (3.4), *and Z is as stated in* (1.5).

Corollary 3.2. *Let $\gamma = 1$ in the above theorem. Then, the upper bounds of $|d_2|$, $|d_3|$ and $|d_3 - \mu d_2^2|$, $\mu \in \Re$, for any function $g \in \mathfrak{I}^{\tau,\delta}_{\sigma,p,q}(\varkappa)$ are given by* (3.1), (3.2) *and* (3.3), *respectively, with $A = A_2 = \tau[3]_{p,q} - 1$, $S = S_2 = 1 + \frac{\tau(\tau-1)[2]^2_{p,q}}{2} - \tau[2]_{p,q}$ and $B = B_2 = \tau[2]_{p,q} - 1$. A_2, S_2 and B_2 should be used in place of A, S and B, respectively, for $\mathbb{Q}$ in* (3.4), *and Z is as stated in* (1.5).

Taking $p = 1$ and $q \to 1^-$ in the Theorem 3.1, we get the following.

Corollary 3.3. *Let $0 < \delta \leq 1$, $\tau \geq 1$ and $0 \leq \gamma \leq 1$. If $g \in \Gamma^{\tau,\delta}_{\sigma}(\gamma,\varkappa)$, then*

$$|d_2| \leq \frac{2\delta|v\varkappa|\sqrt{2|v\varkappa|}}{\sqrt{|2\delta(\delta+1)F(v\varkappa)^2 - (\delta+1)^2(2\tau-\gamma)^2Z|}},$$

$$|d_3| \leq \left(\frac{2\delta v\varkappa}{(\delta+1)(2\tau-\gamma)}\right)^2 + \frac{2\delta|v\varkappa|}{(\delta+1)(3\tau-\gamma)},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{2\delta|v\varkappa|}{(\delta+1)(3\tau-\gamma)}, & |1-\mu| \leq \mathbb{Q}_1, \\ \frac{2\delta^2|v\varkappa|^3\,|1-\mu|}{|2\delta(\delta+1)F(v\varkappa)^2-(\delta+1)^2(2\tau-\gamma)^2Z|}, & |1-\mu| \geq \mathbb{Q}_1, \end{cases}$$

where

$$\mathbb{Q}_1 = \left|\frac{2\delta(\delta+1)F(v\varkappa)^2 - (\delta+1)^2(2\tau-\gamma)^2Z}{\delta(\delta+1)(3\tau-\gamma)v^2\varkappa^2}\right|,$$

$$F = (2\tau+1)(\tau-\gamma) + \gamma^2 + (1-\delta)(2\tau-\gamma)^2,$$

and Z is as given in (1.5).

Remark 3.1. i) In Corollary 3.3, $\gamma = 1$ and $\tau = 1$ yield Theorem 2 of Srivastava et al. [40].

ii) Corollary 3.3, where $\delta = 1$, yields Corollary 3.3 in [49]. Furthermore, Theorem 2.1 of [1] is obtained when $\gamma = 1$.

iii) In Corollary 3.3, $\delta = 1, \tau = 1$ and $\gamma = 0$ yield Corollary 2 of Horan et al. [34].

Corollary 3.4. *If* $g \in \mathfrak{O}^{\tau}_{\sigma,p,q}(\gamma, \varkappa)$, $0 \leq \gamma \leq 1$ *and* $\tau \geq 1$, *then*

$$|d_2| \leq \frac{|v\varkappa|\sqrt{|v\varkappa|}}{\sqrt{|(A+S)(v\varkappa)^2 - B^2 Z|}}, \quad |d_3| \leq \frac{(v\varkappa)^2}{B^2} + \frac{|v\varkappa|}{A},$$

and for $\mu \in \mathbb{R}$

$$|d_3 - \mu d_2^2| \leq \begin{cases} \frac{|v\varkappa|}{A}, & |1-\mu| \leq \left|\frac{(A+S)v^2\varkappa^2 - B^2 Z}{Av^2\varkappa^2}\right|, \\ \frac{|v\varkappa|^3\,|1-\mu|}{|(A+S)(v\varkappa)^2 - B^2 Z|}, & |1-\mu| \geq \left|\frac{(A+S)v^2\varkappa^2 - B^2 Z}{Av^2\varkappa^2}\right|, \end{cases}$$

where Z, A, S *and* B *are as mentioned in* (1.5), (3.5), (3.6) *and* (3.7), *respectively.*

Remark 3.2. i) From Corollary 3.4, letting $p = 1$ and $q \to 1^-$, we obtain Corllary 3.3, which is demonstrated in [49]. Moreover, Theorem 2.1 of [1] is obtained when $\gamma = 1$.

ii) We obtain Corollary 2 of Horan et al. [34], if we take $p = 1$, $q \to 1^-$, $\gamma = 0$ and $\tau = 1$ in Corollary 3.4.

4. Conclusion

This study establishes upper bounds on $|d_2|$ and $|d_3|$ for funcions in two subfamilies of σ related to Horadam polynomials. Moreover, the Fekete-Szegö functional $|d_3 - \mu d_2^2|$, $\mu \in \mathbb{R}$ has been identified for functions in these subfamilies. By varying the parameters in Theorem 2.1 and Theorem 3.1, we have been able to highlight a number of implications. Additionally, pertinent links to the ongoing research are found. Nevertheless, this paper does not address all of the significant subclasses of σ that exist in the literature. For example, authors [32, 37] have examined various subclasses involving operators introduced in (p, q)-calculus. It is recommended that the interested reader review these papers and the associated references.

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