Kragujevac Journal of Mathematics
Volume 42(2) (2018), Pages 287–297.

SOME MONOTONICITY PROPERTIES AND INEQUALITIES FOR THE (p,k)-GAMMA FUNCTION

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Abstract. In this paper, the authors present some complete monotonicity properties and some inequalities involving the (p,k)-analogue of the Gamma function. The established results provide the (p,k)-generalizations for some results known in the literature.

1. Introduction

In a recent paper [12], the authors introduced a (p,k)-analogue of the Gamma function defined for $p \in \mathbb{N}$, $k > 0$ and $x \in \mathbb{R}^+$ as

$$\begin{aligned}\Gamma_{p,k}(x) &= \int_0^p t^{x-1}\left(1-\frac{t^k}{pk}\right)^p dt \\ &= \frac{(p+1)!k^{p+1}(pk)^{\frac{x}{k}-1}}{x(x+k)(x+2k)\dots(x+pk)}, \qquad (1.1)\end{aligned}$$

and satisfying the basic properties:

$$\begin{aligned}\Gamma_{p,k}(x+k) &= \frac{pkx}{x+pk+k}\Gamma_{p,k}(x), \qquad (1.2)\\ \Gamma_{p,k}(ak) &= \frac{p+1}{p}k^{a-1}\Gamma_p(a), \quad a \in \mathbb{R}^+, \\ \Gamma_{p,k}(k) &= 1.\end{aligned}$$

Key words and phrases. Gamma function, (p,k)-analogue, completely monotonic function, logarithmically completely monotonic function, inequality.
2010 *Mathematics Subject Classification.* Primary: 33B15. Secondary: 26A48, 33E50.
Received: August 06, 2016.
Accepted: March 07, 2017.

The (p,k)-analogue of the Digamma function is defined for $x>0$ as

$$\begin{aligned}(1.3)\qquad \psi_{p,k}(x) &= \frac{d}{dx}\ln\Gamma_{p,k}(x) = \frac{1}{k}\ln(pk) - \sum_{n=0}^{p}\frac{1}{nk+x}\\ &= \frac{1}{k}\ln(pk) - \int_0^\infty \frac{1-e^{-k(p+1)t}}{1-e^{-kt}}e^{-xt}\,dt.\end{aligned}$$

Also, the (p,k)-analogue of the Polygamma functions are defined as

$$\begin{aligned}(1.4)\qquad \psi_{p,k}^{(m)}(x) &= \frac{d^m}{dx^m}\psi_{p,k}(x) = \sum_{n=0}^{p}\frac{(-1)^{m+1}m!}{(nk+x)^{m+1}}\\ &= (-1)^{m+1}\int_0^\infty \left(\frac{1-e^{-k(p+1)t}}{1-e^{-kt}}\right)t^m e^{-xt}\,dt,\end{aligned}$$

where $m\in\mathbb{N}$ and $\psi_{p,k}^{(0)}(x)\equiv\psi_{p,k}(x)$.

The functions $\Gamma_{p,k}(x)$ and $\psi_{p,k}(x)$ satisfy the following commutative diagrams.

$$\begin{array}{ccc} \Gamma_{p,k}(x) & \xrightarrow{p\to\infty} & \Gamma_k(x)\\ {\scriptstyle k\to1}\downarrow & & \downarrow{\scriptstyle k\to1}\\ \Gamma_p(x) & \xrightarrow[p\to\infty]{} & \Gamma(x), \end{array} \qquad\qquad \begin{array}{ccc} \psi_{p,k}(x) & \xrightarrow{p\to\infty} & \psi_k(x)\\ {\scriptstyle k\to1}\downarrow & & \downarrow{\scriptstyle k\to1}\\ \psi_p(x) & \xrightarrow[p\to\infty]{} & \psi(x). \end{array}$$

From the identity (1.2), the following relations are established:

$$(1.5)\qquad \psi_{p,k}(x+k)-\psi_{p,k}(x) = \frac{1}{x}-\frac{1}{x+pk+k},$$

$$(1.6)\qquad \psi_{p,k}^{(m)}(x+k)-\psi_{p,k}^{(m)}(x) = \frac{(-1)^m m!}{x^{m+1}}-\frac{(-1)^m m!}{(x+pk+k)^{m+1}},\quad m\in\mathbb{N}.$$

It follows easily from (1.3) and (1.4) that for $x>0$,

(i) $\psi_{p,k}(x)$ is increasing;
(ii) $\psi_{p,k}^{(m)}(x)$ is positive and decreasing if m is odd;
(iii) $\psi_{p,k}^{(m)}(x)$ is negative and increasing if m is even.

Next, we recall the following definitions and lemmas which will be used in the paper.

A function f is said to be *completely monotonic* on an interval I, if f has derivatives of all order and satisfies

$$(1.7)\qquad (-1)^m f^{(m)}(x)\geq 0,\quad \text{for}\quad x\in I, m\in\mathbb{N}_0.$$

If the inequality (1.7) is strict, then f is said to be *strictly completely monotonic* on I.

A positive function f is said to be *logarithmically completely monotonic* on an interval I, if f satisfies

$$(1.8)\qquad (-1)^m[\ln f(x)]^{(m)}\geq 0,\quad \text{for}\quad x\in I, m\in\mathbb{N}_0.$$

If the inequality (1.8) is strict, then f is said to be *strictly logarithmically completely monotonic* on I.

Lemma 1.1 ([1])**.** *If h is completely monotonic on $(0,\infty)$, then $\exp(-h)$ is also completely monotonic on $(0,\infty)$.*

Lemma 1.2 ([1])**.** *Let a_i and b_i, $i = 1, 2, \ldots, n$ be real numbers such that $0 < a_1 \leq a_2 \leq \cdots \leq a_n$, $0 < b_1 \leq b_2 \leq \cdots \leq b_n$ and $\sum_{i=1}^{\lambda} a_i \leq \sum_{i=1}^{\lambda} b_i$ for $\lambda \in \mathbb{N}$. If f is a decreasing and convex function on $\mathbb{R}$, then*

$$\sum_{i=1}^{n} f(b_i) \leq \sum_{i=1}^{n} f(a_i).$$

Lemma 1.3 ([4])**.** *Let $f''(x)$ be completely monotonic on $(0,\infty)$. Then for $0 \leq s \leq 1$, the functions*

$$\mu(x) = \exp\left(-\left(f(x+1) - f(x+s) - (1-s)f'\left(x + \frac{1+s}{2}\right)\right)\right),$$
$$\eta(x) = \exp\left(f(x+1) - f(x+s) - \frac{(1-s)}{2}\left(f'(x+1) + f'(x+s)\right)\right),$$

are logarithmically completely monotonic on $(0,\infty)$.

In this paper, our goal is to establish some complete monotonicity properties and some inequalities involving the (p,k)-analogue of the Gamma function. For additional information on results of this nature, one could refer to [3], [8] and the related references therein.

2. Main Results

We now present our findings in this section.

Theorem 2.1. *Let $p \in \mathbb{N}$, $k > 0$ and $m \in \mathbb{N}_0$. Then the function $\psi'_{p,k}(x)$ is strictly completely monotonic on $(0,\infty)$.*

Proof. It follows directly from (1.4) that

$$\begin{aligned}(-1)^m \left(\psi'_{p,k}(x)\right)^{(m)} &= (-1)^m \psi_{p,k}^{(m+1)}(x) \\ &= (-1)^m \sum_{n=0}^{p} \frac{(-1)^{m+2}(m+1)!}{(nk+x)^{m+2}} \\ &= (-1)^{2m+2} \sum_{n=0}^{p} \frac{(m+1)!}{(nk+x)^{m+2}} > 0,\end{aligned}$$

which concludes the proof. □

Remark 2.1. It follows from Lemma 1.1 that $\exp(-\psi_{p,k}(x))$ is also completely monotonic.

Theorem 2.2. *Let $p \in \mathbb{N}$, $k > 0$ and $a \in (0,1)$. Then the function*

$$Q(x) = \psi_{p,k}(x+a) - \psi_{p,k}(x),$$

is strictly completely monotonic on $(0,\infty)$. In particular, Q is decreasing and convex.

Proof. By direct computation, we obtain

$$\begin{aligned}(-1)^m \left(Q(x)\right)^{(m)} &= (-1)^m \left[\psi_{p,k}^{(m)}(x+a) - \psi_{p,k}^{(m)}(x)\right] \\ &= (-1)^m \left[\sum_{n=0}^{p} \frac{(-1)^{m+1}m!}{(nk+x+a)^{m+1}} - \sum_{n=0}^{p} \frac{(-1)^{m+1}m!}{(nk+x)^{m+1}}\right] \\ &= (-1)^{2m+1}m! \sum_{n=0}^{p} \left[\frac{1}{(nk+x+a)^{m+1}} - \frac{1}{(nk+x)^{m+1}}\right] \\ &> 0,\end{aligned}$$

which establishes the result. In particular, $Q'(x) = \psi'_{p,k}(x+a) - \psi'_{p,k}(x) \leq 0$ since $\psi'_{p,k}(x)$ is decreasing. Hence Q is decreasing. Furthermore, $Q''(x) = \psi''_{p,k}(x+a) - \psi''_{p,k}(x) \geq 0$ implying that Q is convex. □

Remark 2.2. Theorem 2.2 generalizes the the previous result [10, Theorem 1].

In the following theorem, we prove a generalization of the results of Mortici [11].

Theorem 2.3. *Let $p \in \mathbb{N}$, $k > 0$ and $\alpha \in (0,1)$. Then the function*

$$T(x) = \psi_{p,k}(x+\alpha) - \psi_{p,k}(x) - \frac{\alpha}{x},$$

is strictly completely monotonic on $(0,\infty)$. Particularly, T is decreasing and convex.

Proof. Similarly, by direct computation, we obtain

$$\begin{aligned}&(-1)^m \left(T(x)\right)^{(m)} \\ =&(-1)^m \left[\psi_{p,k}^{(m)}(x+\alpha) - \psi_{p,k}^{(m)}(x) - (-1)^{m+1}(m-1)!\frac{\alpha}{x^{m+1}}\right] \\ =&(-1)^m \left[\sum_{n=0}^{p} \frac{(-1)^{m+1}m!}{(nk+x+\alpha)^{m+1}} - \sum_{n=0}^{p} \frac{(-1)^{m+1}m!}{(nk+x)^{m+1}} + \frac{(-1)^{m+2}\alpha(m-1)!}{x^{m+1}}\right] \\ =&(-1)^{2m+1}m! \sum_{n=0}^{p} \left[\frac{1}{(nk+x+\alpha)^{m+1}} - \frac{1}{(nk+x)^{m+1}}\right] + (-1)^{2m+2}\frac{\alpha(m-1)!}{x^{m+1}} \\ >&0.\end{aligned}$$

Hence T is strictly completely monotonic on $(0,\infty)$. In particular,

$$\begin{aligned}T'(x) &= \psi'_{p,k}(x+\alpha) - \psi'_{p,k}(x) + \frac{\alpha}{x^2} \\ &= -\frac{1}{x^2} + \frac{1}{(x+p\alpha+\alpha)^2} + \frac{\alpha}{x^2} \\ &= -\frac{1-\alpha}{x^2} + \frac{1}{(x+p\alpha+\alpha)^2} \leq 0,\end{aligned}$$

as a result of (1.6). Thus T is decreasing. Next,

$$\begin{aligned} T''(x) &= \psi''_{p,k}(x+\alpha) - \psi''_{p,k}(x) - \frac{2\alpha}{x^3} \\ &= \frac{2}{x^3} - \frac{2}{(x+p\alpha+\alpha)^3} - \frac{2\alpha}{x^3} \\ &= 2\left(\frac{1-\alpha}{x^3} - \frac{1}{(x+p\alpha+\alpha)^3}\right) \geq 0. \end{aligned}$$

Hence T is convex. □

Remark 2.3. By letting $p \to \infty$ and $k = 1$ in Theorem 2.3, we obtain the main result of [11].

Theorem 2.4. *Let $p \in \mathbb{N}$, $k > 0$, $m \in \mathbb{N}_0$, a_i and b_i, $i = 1, 2, \dots, n$, be such that $0 < a_1 \leq a_2 \leq \cdots \leq a_n$, $0 < b_1 \leq b_2 \leq \cdots \leq b_n$ and $\sum_{i=1}^{\lambda} a_i \leq \sum_{i=1}^{\lambda} b_i$ for $\lambda \in \mathbb{N}$. Then the function*

$$H(x) = \prod_{i=1}^{n} \frac{\Gamma_{p,k}(x+a_i)}{\Gamma_{p,k}(x+b_i)},$$

is completely monotonic on $(0, \infty)$.

Proof. Let h be defined by $h(x) = \sum_{i=1}^{n} \left[\ln \Gamma_{p,k}(x+b_i) - \ln \Gamma_{p,k}(x+a_i)\right]$. Then for $m \in \mathbb{N}_0$, we have

$$\begin{aligned} (-1)^m \left(h'(x)\right)^{(m)} =& (-1)^m \sum_{i=1}^{n} \left[\psi_{p,k}^{(m)}(x+b_i) - \psi_{p,k}^{(m)}(x+a_i)\right] \\ =& (-1)^m \sum_{i=1}^{n} \Bigg[(-1)^m \sum_{s=0}^{p} \frac{m!}{(sk+x+b_i)^{m+1}} \\ & - (-1)^m \sum_{s=0}^{p} \frac{m!}{(sk+x+a_i)^{m+1}}\Bigg] \\ =& (-1)^{2m+1} m! \sum_{i=1}^{n}\sum_{s=0}^{p} \left[\frac{1}{(sk+x+b_i)^{m+1}} - \frac{1}{(sk+x+a_i)^{m+1}}\right]. \end{aligned}$$

Since $\frac{1}{x^m}$ is decreasing and convex on $\mathbb{R}$ for $m \in \mathbb{N}_0$, then by Lemma 1.2 we obtain

$$\sum_{i=1}^{n} \left[\frac{1}{(sk+x+b_i)^{m+1}} - \frac{1}{(sk+x+a_i)^{m+1}}\right] \leq 0.$$

Thus, $(-1)^m \left(h'(x)\right)^{(m)} \geq 0$ for $m \in \mathbb{N}_0$. Hence $h'(x)$ is completely monotonic on $(0, \infty)$. Then by Lemma 1.1,

$$\exp(-h(x)) = \prod_{i=1}^{n} \frac{\Gamma_{p,k}(x+a_i)}{\Gamma_{p,k}(x+b_i)} = H(x),$$

is completely monotonic on $(0, \infty)$. □

Remark 2.4. By letting $p \to \infty$ in Theorem 2.4, we obtain the result of [6, Theorem 2.6].

Remark 2.5. By letting $k = 1$ in Theorem 2.4, we obtain the result of [7, Theorem 13].

Remark 2.6. By letting $p \to \infty$ and $k = 1$ in Theorem 2.4, we obtain the result of [1, Theorem 10].

Theorem 2.5. *Let $p \in \mathbb{N}$, $k > 0$ and $a \in (0,1)$. Then the inequality*

$$0 < \psi_{p,k}(x+a) - \psi_{p,k}(x) \leq \frac{a(p+1)}{1+a(p+1)},$$

is satisfied for $x \in [1, \infty)$.

Proof. Let Q be defined as in Theorem 2.2. Since Q is decreasing, then for $x \in [1, \infty)$, we obtain

$$0 = \lim_{x\to\infty} Q(x) < Q(x) \leq Q(1) = \psi_{p,k}(a+1) - \psi_{p,k}(1),$$

which by (1.5) yields the desired result. □

Theorem 2.6. *Let $p \in \mathbb{N}$ and $k > 0$. Then the inequality*

$$\frac{1}{k}\ln\left(\frac{pkx}{x+pk+k}\right) - \frac{1}{x} + \frac{1}{x+pk+k} \leq \psi_{p,k}(x) \leq \frac{1}{k}\ln\left(\frac{pkx}{x+pk+k}\right), \tag{2.1}$$

holds for $x > 0$.

Proof. It follows from (1.2) that $\ln\Gamma_{p,k}(x+k) - \ln\Gamma_{p,k}(x) = \ln\left(\frac{pkx}{x+pk+k}\right)$. Let $g(x) = \ln\Gamma_{p,k}(x)$. Then by the classical mean value theorem, the exits a $\lambda \in (x, x+k)$ such that

$$\frac{g(x+k) - g(x)}{k} = \frac{\ln\Gamma_{p,k}(x+k) - \ln\Gamma_{p,k}(x)}{k} = \psi_{p,k}(\lambda).$$

Since $\psi_{p,k}(x)$ is increasing, then for $\lambda \in (x, x+k)$, we have

$$\psi_{p,k}(x) \leq \psi_{p,k}(\lambda) \leq \psi_{p,k}(x+k),$$

which implies

$$\psi_{p,k}(x) \leq \frac{1}{k}\ln\left(\frac{pkx}{x+pk+k}\right) \leq \psi_{p,k}(x+k).$$

Then by (1.5) we obtain

$$\psi_{p,k}(x) \leq \frac{1}{k}\ln\left(\frac{pkx}{x+pk+k}\right) \leq \psi_{p,k}(x) + \frac{1}{x} - \frac{1}{x+pk+k},$$

yielding the result (2.1). □

Remark 2.7. Let $p \to \infty$ and $k = 1$ in (2.1). Then we obtain the result

$$\ln x - \frac{1}{x} \leq \psi(x) \leq \ln x, \tag{2.2}$$

for the classical digamma function, $\psi(x)$.

Theorem 2.7. *Let $p \in \mathbb{N}$ and $k > 0$. Then the inequality*

$$\frac{1}{k}\left(\frac{1}{x} - \frac{1}{x+pk+k}\right) \leq \psi'_{p,k}(x) \leq \frac{1}{k}\left(\frac{1}{x} - \frac{1}{x+pk+k}\right) + \frac{1}{x^2} - \frac{1}{(x+pk+k)^2}, \tag{2.3}$$

holds for $x > 0$.

Proof. Consider the function $\psi_{p,k}(x)$ on the interval $(x, x+k)$. By the mean value theorem, the exits a $c \in (x, x+k)$ such that

$$\frac{1}{k}\left(\frac{1}{x} - \frac{1}{x+pk+k}\right) = \frac{\psi_{p,k}(x+k) - \psi_{p,k}(x)}{k} = \psi'_{p,k}(c).$$

Since $\psi'_{p,k}(x)$ is decreasing, then for $c \in (x, x+k)$, we have

$$\psi'_{p,k}(x+k) \leq \psi'_{p,k}(c) \leq \psi'_{p,k}(x),$$

which implies

$$\psi'_{p,k}(x+k) \leq \frac{1}{k}\left(\frac{1}{x} - \frac{1}{x+pk+k}\right) \leq \psi'_{p,k}(x).$$

Then by (1.6), we obtain

$$\psi'_{p,k}(x) - \frac{1}{x^2} + \frac{1}{(x+pk+k)^2} \leq \frac{1}{k}\left(\frac{1}{x} - \frac{1}{x+pk+k}\right) \leq \psi'_{p,k}(x),$$

which results to (2.3). □

Remark 2.8. Let $p \to \infty$ and $k = 1$ in (2.3). Then we obtain the result

$$\frac{1}{x} \leq \psi'(x) \leq \frac{1}{x} + \frac{1}{x^2}, \tag{2.4}$$

for the trigamma function, $\psi'(x)$.

Remark 2.9. The right side of (2.2) and the left side of (2.4) are however weaker than the results obtained in [5, Theorem 3].

Theorem 2.8. *Let $p \in \mathbb{N}$, $k > 0$ and $0 \leq s \leq 1$. Then the functions*

$$u(x) = \frac{\Gamma_{p,k}(x+s)}{\Gamma_{p,k}(x+1)} \exp\left((1-s)\psi_{p,k}\left(x + \frac{1-s}{2}\right)\right),$$
$$w(x) = \frac{\Gamma_{p,k}(x+1)}{\Gamma_{p,k}(x+s)} \exp\left(-\frac{1-s}{2}\left(\psi_{p,k}(x+1) + \psi_{p,k}(x+s)\right)\right),$$

are logarithmically completely monotonic on $(0, \infty)$.

Proof. Let $f(x) = \ln \Gamma_{p,k}(x)$ and recall that $f''(x) = \psi'_{p,k}(x)$ is completely monotonic on $(0, \infty)$ (See Theorem 2.1). Then the results follow directly from Lemma 1.3. □

Theorem 2.9. *Let $p \in \mathbb{N}$, $k > 0$ and $0 \leq s \leq 1$. Then the inequality*

$$\exp\left(\frac{1-s}{2}\left(\psi_{p,k}(x+s)+\psi_{p,k}(x+1)\right)\right) \leq \frac{\Gamma_{p,k}(x+1)}{\Gamma_{p,k}(x+s)} \leq \exp\left((1-s)\psi_{p,k}\left(x+\frac{1+s}{2}\right)\right), \tag{2.5}$$

is satisfied for $x > 0$.

Proof. We employ the Hermite-Hadamard's inequality which states that: if $f(x)$ is convex on $[a, b]$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

Let $f(x) = -\psi_{p,k}(x)$, $a = x + s$ and $b = x + 1$. Then we have

$$-\psi_{p,k}\left(x+\frac{1+s}{2}\right) \leq -\frac{1}{1-s}\int_{x+s}^{x+1} \psi_{p,k}(t)\,dt \leq -\frac{\psi_{p,k}(x+s)+\psi_{p,k}(x+1)}{2},$$

which implies

$$\frac{\psi_{p,k}(x+s)+\psi_{p,k}(x+1)}{2} \leq \frac{1}{1-s}\ln\frac{\Gamma_{p,k}(x+1)}{\Gamma_{p,k}(x+s)} \leq \psi_{p,k}\left(x+\frac{1+s}{2}\right).$$

Then by taking exponents, we obtain the desired result. □

Remark 2.10. By letting $p \to \infty$ in Theorems 2.8 and 2.9, we respectively obtain the results of Theorems 2.1 and 2.3 of [6].

Remark 2.11. By letting $k = 1$ in Theorems 2.8 and 2.9, we respectively obtain the results of Theorems 2.3 and 2.4 of [9].

Remark 2.12. The q-analogue of these results can also be found in [4].

The following theorem is a (p, k)-generalization of Lemma 2.1 of [2]. We derive our results by using similar techniques.

Theorem 2.10. *Let $p \in \mathbb{N}$ and $k > 0$. Then the function*

$$f(x) = \frac{1}{[\Gamma_{p,k}(x+k)]^{\frac{1}{x}}},$$

is logarithmically completely monotonic on $(0, \infty)$.

Proof. We employ the Leibniz's rule for n-times differentiable functions u and v, which is given by

$$[u(x)v(x)]^{(n)} = \sum_{s=0}^{n}\binom{n}{s}u^{(s)}(x)v^{(n-s)}(x).$$

That is,

$$\begin{aligned}(\ln f(x))^{(n)} &= \left[\left(\frac{1}{x}\right)(-\ln\Gamma_{p,k}(x+k))\right]^{(n)} \\ &= \sum_{s=0}^{n}\binom{n}{s}\left(\frac{1}{x}\right)^{(s)}(-\ln\Gamma_{p,k}(x+k))^{(n-s)} \\ &= -\frac{1}{x^{n+1}}\sum_{s=0}^{n}\binom{n}{s}(-1)^s s!x^{n-s}\psi_{p,k}^{(n-s-1)}(x+k) \\ &\triangleq -\frac{1}{x^{n+1}}\phi(x).\end{aligned}$$

This implies

$$\begin{aligned}\phi'(x) =&\sum_{s=0}^{n}\binom{n}{s}(-1)^s s!(n-s)x^{n-s-1}\psi_{p,k}^{(n-s-1)}(x+k) \\ &+\sum_{s=0}^{n}\binom{n}{s}(-1)^s s!x^{n-s}\psi_{p,k}^{(n-s)}(x+k) \\ =&\sum_{s=0}^{n-1}\binom{n}{s}(-1)^s s!(n-s)x^{n-s-1}\psi_{p,k}^{(n-s-1)}(x+k) \\ &+x^n\psi_{p,k}^{(n)}(x+k)+\sum_{s=1}^{n}\binom{n}{s}(-1)^s s!x^{n-s}\psi_{p,k}^{(n-s)}(x+k) \\ =&\sum_{s=0}^{n-1}\binom{n}{s}(-1)^s s!(n-s)x^{n-s-1}\psi_{p,k}^{(n-s-1)}(x+k) \\ &+x^n\psi_{p,k}^{(n)}(x+k)+\sum_{s=0}^{n-1}\binom{n}{s+1}(-1)^{s+1}(s+1)!x^{n-s-1}\psi_{p,k}^{(n-s-1)}(x+k) \\ =&\sum_{s=0}^{n-1}\left[\binom{n}{s}(n-s)-\binom{n}{s+1}(s+1)\right](-1)^s s!(n-s)x^{n-s-1}\psi_{p,k}^{(n-s-1)}(x+k) \\ &+x^n\psi_{p,k}^{(n)}(x+k) \\ =&x^n\psi_{p,k}^{(n)}(x+k) \\ =&x^n(-1)^{n+1}\sum_{s=0}^{p}\frac{n!}{(k(s+1)+x)^{n+1}}.\end{aligned}$$

Suppose that n is odd. Then,

$$\phi'(x)>0 \implies \phi(x)>\phi(0)=0 \implies (\ln f(x))^{(n)}<0.$$

Thus $(-1)^n(\ln f(x))^{(n)}>0$. Also, suppose that n is even. Then

$$\phi'(x)<0 \implies \phi(x)<\phi(0)=0 \implies (\ln f(x))^{(n)}>0,$$

yielding $(-1)^n (\ln f(x))^{(n)} > 0$. Therefore, for every $n \in \mathbb{N}$, we have

$$(-1)^n (\ln f(x))^{(n)} > 0,$$

which concludes the proof. □

Remark 2.13. By letting $p \to \infty$ in Theorem 2.10, we recover the results of Theorem 2.8 of [6].

Remark 2.14. By letting $k = 1$ in Theorem 2.10, we recover the results of Theorem 2.1 of [9].

3. Conclusion

In the study, the authors established some complete monotonicity properties and some inequalities involving the (p, k)-analogue of the Gamma function which was recently introduced in [12]. The established results provide the (p, k)-generalizations for some results known in the literature.

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