Kragujevac Journal of Mathematics
Volume 45(3) (2021), Pages 477–489.

A STUDY OF CONFORMALLY FLAT QUASI-EINSTEIN SPACETIMES WITH APPLICATIONS IN GENERAL RELATIVITY

VENKATESHA[1] AND ARUNA KUMARA H[1]

Abstract. In this paper we consider conformally flat $(QE)_4$ spacetime and obtained several important results. We study application of conformally flat $(QE)_4$ spacetime in general relativity and Ricci soliton structure in a conformally flat $(QE)_4$ perfect fluid spacetime.

1. Introduction

An Einstein manifold is a Riemannian or pseudo-Riemannian manifold whose Ricci tensor S of type $(0,2)$ is non-zero and propotional to the metric tensor. Einstein manifolds form a natural subclass of various classes of Riemannian or pseudo-Riemannian manifolds by a curvature condition imposed on their Ricci tensor [4]. Also in Riemannian geometry as well as in general relativity theory, the Einstein manifold play an very important role.

The quasi-Einstein manifolds are generalization of Einstein manifolds. The notion of quasi-Einstein manifolds was introduced by Chaki and Maity [6] in 2000. According to them, a Riemannian manifold or pseudo-Riemannian manifold is said to be a quasi-Einstein manifold if its Ricci tensor S of type $(0,2)$ is non-zero and satisfies the condition

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y), \tag{1.1}$$

where α and β are real valued non-zero scalar functions and A is a non-zero 1-form equivalent to the vector field ω, i.e., $g(X,\omega) = A(X)$, $g(\omega,\omega) = 1$. Here A is called an associated 1-form and ω is called a generator. If $\beta = 0$, then the

Key words and phrases. Quasi-Einstein spacetime, perfect fluid spacetime, Einstein field equation, energy momentum tensor, Ricci solitons, conformal curvature tensor.

2010 *Mathematics Subject Classification.* Primary: 53C15. Secondary: 53C25.

Received: October 17, 2018.

Accepted: February 20, 2019.

manifold reduces to an Einstein manifold. This kind of n-dimensional manifold is denoted by $(QE)_n$. Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during the considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. For instance, the Robertson-Walker spacetimes and conformally flat almost pseudo-Ricci symmetric spacetimes are quasi-Einstein manifolds. Also, quasi-Einstein manifolds can be taken as a model of perfect fluid spacetime in general relativity. The importance of quasi-Einstein spacetimes lies in the fact that 4-dimensional pseudo-Riemannian manifolds are related to study of general relativistic fluid spacetime, where the generator vector field ω is taken as timelike velocity vector field, that is, $g(\omega, \omega) = -1$.

In the paper [5], Chaki and Ray studied spacetimes with covariant constant energy momentum tensor. In recent paper [11, 17], they studied the quasi-Einstein spacetime and generalized quasi-Einstein spacetime in general relativity. Additionally, there are many works related with spacetime in general relativity [1, 13, 14, 16, 19].

The authors De, Özgür and De showed that conformally flat almost pseudo-Ricci symmetric spacetime can be considered as a model of the perfect fluid spacetime in general relativity and also obeying Einstein equation without cosmological constant and having the vector as velocity vector is infinitesimally spatially isotropic relative to the unit timelike vector field [8]. In [9], they proved that conformally flat perfect fluid spacetime with semisymmetric energy momentum tensor is a spacetime of quasi constant curvature and such spacetime determines an equation of state in quintessence era, where the universe is in an accelerating phase. Therefore it is meaningful to study a conformally flat $(QE)_4$ spacetime in general relativity.

The present paper organized as follows. After preliminaries, in Section 3, we study conformally flat $(QE)_4$ spacetime. In Section 4, we prove that conformally flat Ricci pseudosymmetric $(QE)_4$ spacetime is an $N\left(\frac{2\alpha-5\beta}{6}\right)$ quasi-Einstien spacetime, provided $g(Y, Z)A(X) \neq g(X, Z)A(Y)$. In Section 5, we study conformally flat $(QE)_4$ perfect fluid spacetime and obtained some interesting results on conformally flat $(QE)_4$ spacetime in general reltivity. Finally, we study Ricci soliton structure of conformally flat $(QE)_4$ sapcetime in general relativity.

2. Preliminaries

Consider $(QE)_4$ spacetime with associated scalars α, β and associated 1-form A. Then by (1.1), we have

$$r = 4\alpha - \beta, \tag{2.1}$$

where r is a scalar curvature of the spacetime. If ω is orthogonal unit vector field, then $g(\omega, \omega) = -1$. Again from (1.1), we have

$$\begin{aligned} S(X, \omega) =& (\alpha - \beta)A(X), \\ S(\omega, \omega) =& \beta - \alpha. \end{aligned} \tag{2.2}$$

Let Q be the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor S. Then $g(QX, Y) = S(X, Y)$ for all X, Y.

3. Conformally Flat $(QE)_4$ Spacetime

A quasi-Einstein spacetime is said to be conformally flat, if the Weyl conformal curvature tensor C vanishes and is defined by [8, 22]

$$C(X,Y)Z = R(X,Y)Z - \frac{1}{2}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} + \frac{r}{6}\{g(Y,Z)X - g(X,Z)Y\}, \tag{3.1}$$

where Q is the Ricci operator defined by $g(QX, Y) = S(X, Y)$ and r is the scalar curvature.

Now, suppose that $(QE)_4$ spacetime is conformally flat. Then by (3.1), we get

$$R(X,Y)Z = \frac{1}{2}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} - \frac{r}{6}\{g(Y,Z)X - g(X,Z)Y\}. \tag{3.2}$$

From (1.1), we have

$$QX = \alpha X + \beta A(X)\omega. \tag{3.3}$$

Substituting (1.1) and (3.3) in (3.2), we obtain

$$\begin{aligned} R(X,Y,Z,W) = & \left(\frac{2\alpha+\beta}{6}\right)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ & + \frac{\beta}{2}\{g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z) \\ & + g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)\}, \end{aligned} \tag{3.4}$$

which leads to

$$R(X,Y,Z,W) = B(Y,Z)B(X,W) - B(X,Z)B(Y,W), \tag{3.5}$$

where

$$B(X,Y) = \sqrt{\frac{2\alpha+\beta}{6}}g(X,Y) + \frac{\beta\sqrt{3}}{\sqrt{4\alpha+2\beta}}A(X)A(Y). \tag{3.6}$$

It is known that an n-dimensional Riemannian or pseudo-Riemannian manifold whose curvature tensor R of type $(0, 4)$ satisfies the condition (3.5), where B is a symmetric tensor field of type $(0, 2)$, is called a special manifold with the associated symmetric tensor B and is denoted by the symbol $\psi(B)_n$. Recently, these type of manifolds are studied in [15, 18].

By virtue of (3.5) and (3.6), we have the following theorem.

Theorem 3.1. *A conformally flat $(QE)_4$ spacetime is $\psi(B)_4$ with associated symmetric tensor B given by* (3.6).

Chen and Yano [7] introduced the concept of manifold of a quasi-constant curvature. A spacetime is said to be of quasi-constant curvature if the curvature tensor R of type $(0,4)$ satisfies

$$\begin{aligned} R(X,Y,Z,W) =& a\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ &+ b\,\{g(X,W)\varpi(Y)\varpi(Z) - g(Y,W)\varpi(X)\varpi(Z) \\ &+ g(Y,Z)\varpi(X)\varpi(W) - g(X,Z)\varpi(Y)\varpi(W)\}\,, \end{aligned} \tag{3.7}$$

where a and b are scalars and there exists a unit vector field ν such that $g(X,\nu) = \varpi(X)$. If $b = 0$, then the spacetime is of constant curvature a. Comparing the equation (3.4) and (3.7), we have the following.

Theorem 3.2. *A conformally flat $(QE)_4$ spacetime is a spacetime of quasi-constant curvature.*

Let (M^4, g) be a conformally flat $(QE)_4$ spacetime. As $C = 0$, we have div$C = 0$, where div denotes the divergence. Hence, from (3.2) we have

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \frac{1}{6}\{g(Y,Z)dr(X) - g(X,Z)dr(Y)\}.$$

In view of (1.1), above relation takes the form

$$\begin{aligned} & d\alpha(X)g(Y,Z) - d\alpha(Y)g(X,Z) + d\beta(X)A(Y)A(Z) - d\beta(Y)A(X)A(Z) \\ & + \beta[(\nabla_X A)(Y)A(Z) + A(Y)(\nabla_X A)(Z) - (\nabla_Y A)(X)A(Z) \\ & - A(X)(\nabla_Y A)(Z)] \\ =& \frac{1}{6}\left[g(Y,Z)dr(X) - g(X,Z)dr(Y)\right]. \end{aligned} \tag{3.8}$$

Suppose the scalar curvature r is constant, then from (2.1) we have

$$4d\alpha(X) = d\beta(X). \tag{3.9}$$

Using above equation in (3.8), we get

$$\begin{aligned} & d\alpha(X)[g(Y,Z) + 4A(Y)A(Z)] - d\alpha(Y)[g(X,Z) + 4A(X)A(Z)] \\ & + \beta[(\nabla_X A)(Y)A(Z) + A(Y)(\nabla_X A)(Z) - (\nabla_Y A)(X)A(Z) - A(X)(\nabla_Y A)(Z)] = 0. \end{aligned} \tag{3.10}$$

Taking a frame field after contraction over Y and Z, we obtain from (3.10) that

$$d\alpha(X) + 4d\alpha(\omega)A(X) + \beta[(\nabla_\omega A)(X) + A(X)\sum_{i=1}^{4}\epsilon_i(\nabla_{e_i}A)(e_i)] = 0, \tag{3.11}$$

where $\epsilon_i = g(e_i, e_i) = \pm 1$. Plugging ω in place of Y and Z in (3.10), we get

$$3[d\alpha(X) + d\alpha(\omega)A(X)] + \beta(\nabla_\omega A)(X) = 0. \tag{3.12}$$

In view of (3.12) and (3.11), we obtain

$$-2d\alpha(X) + d\alpha(\omega)A(X) + \beta A(X)\sum_{i=1}^{4}\epsilon_i(\nabla_{e_i}A)(e_i) = 0. \tag{3.13}$$

Now, putting $X = \omega$ in the above equation, we get

$$\beta\sum_{i=1}^{4}\epsilon_i(\nabla_{e_i}A)(e_i) = -3d\alpha(\omega). \tag{3.14}$$

From (3.13) and (3.14), it follows that

$$d\alpha(X) = -d\alpha(\omega)A(X). \tag{3.15}$$

Setting $Z = \omega$ in (3.10) and then using (3.13) and $\beta \neq 0$, we get

$$(\nabla_X A)(Y) = (\nabla_Y A)(X). \tag{3.16}$$

Above equation shows that 1-form A is of Codazzi type, this means that generator ω is irrotational. By virtue of (3.15), (3.12) and $\beta \neq 0$, it follows that

$$(\nabla_\omega A)(X) = 0, \tag{3.17}$$

for all X, which implies that $\nabla_\omega \omega = 0$ and hence intergral curves of ω are geodesic. Again, setting $Y = \omega$ in (3.10) and then using (3.15) and (3.17), we get

$$(\nabla_X A)(Z) = -\frac{d\alpha(\omega)}{4\alpha}[A(X)A(Z) + g(X,Z)]. \tag{3.18}$$

Now, we consider non-vanishing scalar function $f = -\frac{d\alpha(\omega)}{4\alpha}$. Then, we have

$$\nabla_X f = \frac{d\alpha(\omega)}{4\alpha^2}d\alpha(X) - \frac{d^2\alpha(\omega,X)}{4\alpha}. \tag{3.19}$$

By virtue of (3.15), we get $d^2\alpha(X,Y) = -d^2\alpha(\omega,Y)A(X) - d\alpha(\omega)(\nabla_Y A)(X)$. In a Lorentzian manifold, the scalar function η satisfies the relation $d^2\eta(X,Y) = d^2\eta(Y,X)$, for all X, Y. In view of (3.16), the above relation becomes

$$d^2\alpha(\omega,X)A(Y) = d^2\alpha(\omega,Y)A(X).$$

Taking $Y = \omega$ in the above equation, we get

$$d^2\alpha(\omega,X) = -d^2\alpha(\omega,\omega)A(X) = -\psi A(X), \tag{3.20}$$

where $\psi = d^2\alpha(\omega,\omega)$ is a scalar function. Now in the consequence of (3.20) and (3.15), equation (3.19) takes the form

$$\nabla_X f = -\frac{1}{4\alpha^2}[\{d\alpha(\omega)\}^2 - \alpha\psi]A(X). \tag{3.21}$$

Now consider a 1-form h given by

$$h(X) = -\frac{d\alpha(\omega)}{4\alpha}A(X) = fA(X). \tag{3.22}$$

From (3.16), (3.21) and (3.22) we have $dh(X,Y) = 0$, i.e., the 1-form h is closed. Therefore (3.18) can be written as $(\nabla_X A)(Z) = h(X)A(Z) + fg(X,Z)$. This means that the generator ω corresponding to the 1-form A is a unit proper concircular vector field [20]. This leads to the following theorem.

Theorem 3.3. *In a conformally flat $(QE)_4$ spacetime with constant scalar curvature, the following properties hold:*

i. *the generator vector field ω is irrotational;*
ii. *the integral curves of ω are geodesic;*
iii. *the vector field ω corresponding to the 1-form A is a unit proper concircular vector field.*

Lemma 3.1. *In a conformally flat $(QE)_4$ spacetime, the curvature tensor R of type $(1,3)$ satisfies the following properties:*

(i) $R(X,Y)Z = \left(\frac{2\alpha+\beta}{6}\right)\{g(Y,Z)X - g(X,Z)Y\}$;
(ii) $R(X,\omega)Y = \left(\frac{\beta-\alpha}{3}\right)g(X,Y)\omega$;
(iii) $R(X,\omega)\omega = \left(\frac{\beta-\alpha}{3}\right)X$,

for all $X, Y, Z \in \omega^{\perp}$, the 3-dimensional distribution orthogonal to the generator ω.

Proof. In a conformally flat $(QE)_4$ spacetime, we have the relation (3.4). Since $\omega^{\perp}$ is a 3-dimensional distribution orthogonal to the generator ω, we have $g(X,\omega) = 0$ if and only if $X \in \omega^{\perp}$. Hence (3.4) yields the relation (i)-(iii) for all $X, Y, Z \in \omega^{\perp}$. This proves the lemma. $\square$

Let $X, Y, Z \in \omega^{\perp}$. Let K_1 be the sectional curvature of the plane determined by X and Y and K_2 be the sectional curvature of the plane determined by X and ω. Then

$$K_1 = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}, \quad K_2 = \frac{g(R(X,\omega)\omega,X)}{g(X,X)g(\omega,\omega) - g(X,\omega)^2}.$$

By virtue of (i) and (iii) in Lemma 3.1 , we have $K_1 = \frac{2\alpha+\beta}{6}$ and $K_2 = \frac{\alpha-\beta}{3}$. Hence, we state the following.

Lemma 3.2. *In a conformally flat $(QE)_4$ spacetime, the sectional curvature of all planes determined by $X, Y \in \omega^{\perp}$ is $\frac{2\alpha+\beta}{6}$ and the sectional curvature of all planes determined by X and ω, where $X \in \omega^{\perp}$ is $\frac{\alpha-\beta}{3}$.*

We note that K_1 and K_2 are constants if and only if α and β are constant. So the following corollary arises.

Corollary 3.1. *In a conformally flat $(QE)_4$ spacetime, the sectional curvature K_1 of all planes determined by X and Y as well as the sectional curvature K_2 of all planes determined by X and ω are constants if and only if α and β are constant.*

Remark 3.1. We know that, a pseudo-Riemannian manifold of constant sectional curvature is locally symmetric. Suppose α and β are constant, then from Corollary 3.1, we say that conformally flat $(QE)_4$ spacetime is locally symmetric if and only if $\frac{2\alpha+\beta}{6}$ is constant, provided that the vectors are orthogonal to the generator ω.

By virtue of (3.4), we obtain

$$\begin{aligned} R(X,Y)\omega =& \frac{\alpha-\beta}{3}\left\{A(Y)X - A(X)Y\right\}, \\ R(X,\omega)Y =& \frac{\alpha-\beta}{3}\left\{A(Y)X - g(X,Y)\omega\right\}. \end{aligned} \tag{3.23}$$

From the Theorem 3.2, we know that conformally flat $(QE)_4$ spacetime is of quasi-constant curvature and is said to be regular if $\alpha - \beta \neq 0$.

Bejan and Crasmareanu [3] proved that a parallel second order symmetric covariant tensor in a regular manifold of quasi-constant curvature is a constant multiple of the metric tensor. Hence we have the following.

Theorem 3.4. *A parallel and symmetric second order covariant tensor field in a conformally flat $(QE)_4$ spacetime with $\alpha \neq \beta$, is a constant multiple of the metric tensor, that is $h(X,Y) = h(\omega,\omega)g(X,Y)$, where h is a symmetric tensor field of type $(0,2)$.*

Let us consider a second order symmetric tensor $h = L_\omega g + 2S$, where L_ω is the Lie derivative with respect to ω. Then

$$h(\omega,\omega) = (L_\omega g)(\omega,\omega) + 2S(\omega,\omega). \tag{3.24}$$

Since $g(\omega,\omega) = -1$, it follows that

$$(\nabla_X A)(\omega) = g(\nabla_X \omega, \omega) = 0.$$

Therefore, $(L_\omega g)(\omega,\omega) = 2g(\nabla_\omega \omega, \omega) = 0$ (because $\nabla_\omega \omega \perp \omega$). In view of (2.2) and (3.24), we obtain

$$h(\omega,\omega) = 2(\beta - \alpha). \tag{3.25}$$

By virtue of Theorem 3.4 and (3.25), we have

$$h(X,Y) = 2(\beta - \alpha)g(X,Y). \tag{3.26}$$

Thus, we have $(L_\omega g)(X,Y) + 2S(X,Y) + 2(\alpha-\beta)g(X,Y) = 0$. This expression defines Ricci soliton on confomally flat $(QE)_4$ spacetime if $(\alpha - \beta)$ is constant. Hence, we conclude the following.

Theorem 3.5. *In a conformally flat $(QE)_4$ spacetime, the symmetric tensor field $h = L_\omega g + 2S$ of type $(0,2)$ is parallel with respect to Levi-Civita connection ∇ of g, then the relation (3.26) defines a Ricci soliton, provided that $\alpha - \beta$ is constant. In this case, Ricci soliton is called expanding or steady or shrinking according as $\alpha - \beta$ is positive or zero or negative, respectively.*

4. Conformally Flat Ricci pseudosymmetric $(QE)_4$ Spacetime

An n-dimensional pseudo-Riemannian manifold is said to be Ricci pseudosymmetric if the tensor $R \cdot S$ and Tachibana tensor $Q(g, S)$ are lineraly dependent, i.e.,

$$(R(X,Y) \cdot S(Z,W)) = L_S Q(g,S)(Z,W;X,Y) \tag{4.1}$$

holds on U_S, where $U_S = \left\{x \in M : S \neq \frac{r}{n} g \ at \ x\right\}$, L_S is a certain function on U_S and

$$(R(X,Y) \cdot S(Z,W)) = -S(R(X,Y)Z,W) - S(Z,R(X,Y)W), \tag{4.2}$$

$$L_S Q(g,S)(Z,W;X,Y) = -S((X\Lambda_g Y)Z,W) - S(Z,(X\Lambda_g Y)W), \tag{4.3}$$

$$(X\Lambda_g Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{4.4}$$

Suppose a conformally flat $(QE)_4$ spacetime is Ricci pseudosymmetric. Then making use of (4.2)–(4.4) in (4.1), we obtain

$$\begin{aligned} S(R(X,Y)Z,W) + S(Z,R(X,Y)W) =& L_S[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \\ & + g(Y,W)S(Z,X) - g(X,W)S(Y,Z)]. \end{aligned} \tag{4.5}$$

Substituting (1.1) in (4.5), we have

$$\begin{aligned} & A(R(X,Y)Z)A(W) + A(Z)A(R(X,Y)W) \\ =& L_S[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W) + g(Y,W)A(Z)A(X) \\ & - g(X,W)A(Y)A(Z)]. \end{aligned}$$

Plugging W by ω in previous equation and making use of the property $g(R(X,Y)\omega,\omega) = g(R(\omega,\omega)X,Y) = 0$, we get

$$A(R(X,Y)Z) = L_S[g(Y,Z)A(X) - g(X,Z)A(Y)]. \tag{4.6}$$

In view of (2.1) and (3.4), (4.6) yields

$$\left[L_S - \left(\frac{2\alpha - 5\beta}{6}\right)\right]\{g(Y,Z)A(X) - g(X,Z)A(Y)\} = 0,$$

which yields either $g(Y,Z)A(X) = g(X,Z)A(Y)$ or $\left[L_S - \left(\frac{2\alpha-5\beta}{6}\right)\right] = 0$.

Suppose $g(Y,Z)A(X) \neq g(X,Z)A(Y)$, then we have

$$L_S = \frac{2\alpha - 5\beta}{6}. \tag{4.7}$$

In view of (4.6) and (4.7), we have

$$R(X,Y)Z = \frac{2\alpha - 5\beta}{6}\{g(Y,Z)X - g(X,Z)Y\},$$

which means that the generator vector field ω belongs to $\frac{2\alpha-5\beta}{6}$-nullity distrbution. This leads to the following.

Theorem 4.1. *Every conformally flat Ricci pseudosymmetric $(QE)_4$ spacetime with $g(Y,Z)A(X) \neq g(X,Z)A(Y)$ is an $N\left(\frac{2\alpha-5\beta}{6}\right)$-quasi-Einstein spacetime.*

5. Conformally Flat $(QE)_4$ Spacetimes with Applications in General Relativity

Ricci tensor is a part of curvature of spacetime that determines the degree to which matter will tend to converge or diverge in time. It is related to the matter content of universe by means of the Einstein field equation

$$S(X,Y)+\left(\Lambda-\frac{r}{2}\right)g(X,Y)=\kappa T(X,Y),\quad \text{for all } X,Y, \tag{5.1}$$

where S is the Ricci tensor, r is the scalar curvature, Λ is the cosmological constant and κ is the gravitational constant. Einstein's field equation shows that the energy momentum tensor is symmetric of type $(0,2)$ with divergence zero.

For the perfect fluid matter distribution, the energy momentum tensor is given by

$$T(X,Y)=\rho g(X,Y)+(\sigma+\rho)A(X)A(Y), \tag{5.2}$$

where σ is energy density and ρ is the isotropic pressure of the fluid.

Here we consider a conformally flat $(QE)_4$ spacetime obeying Einstein's field equation with cosmological constant whose matter content is perfect fluid. Then, in view of (5.1) and (5.2), Ricci tensor takes the form

$$S(X,Y)=\left(\kappa\rho-\Lambda+\frac{r}{2}\right)g(X,Y)+\kappa(\sigma+\rho)A(X)A(Y). \tag{5.3}$$

Compare (5.3) with (1.1), we have

$$\alpha=\kappa\rho-\Lambda+\frac{r}{2},\quad \beta=\kappa(\sigma+\rho).$$

Contracting (5.3) and taking into account that $g(\omega,\omega)=-1$, we have

$$r=4\Lambda+\kappa(\sigma-3\rho). \tag{5.4}$$

By virtue of (5.4) and (5.3), it follows that

$$S(X,Y)=\left(\Lambda+\frac{\kappa(\sigma-\rho)}{2}\right)g(X,Y)+\kappa(\sigma+\rho)A(X)A(Y). \tag{5.5}$$

Now differentiating (5.5) covariantly, we get

$$\begin{aligned}(\nabla_X S)(Y,Z)=&\frac{\kappa}{2}X(\sigma-\rho)g(Y,Z)+\kappa X(\sigma+\rho)A(Y)A(Z)\\ &+\kappa(\sigma+\rho)[(\nabla_X A)(Y)A(Z)+A(Y)(\nabla_X A)(Z)].\end{aligned} \tag{5.6}$$

Let us suppose that conformally flat $(QE)_4$ perfect fluid spacetime is Ricci symmetric, i.e., $\nabla S=0$, then in view of (3.18) and (5.6), it follows that

$$\begin{aligned}0=&\frac{\kappa}{2}X(\sigma-\rho)g(Y,Z)+\kappa X(\sigma+\rho)A(Y)A(Z)\\ &+f\kappa(\sigma+\rho)[2A(X)A(Y)A(Z)+g(X,Y)A(Z)+g(X,Z)A(Y)].\end{aligned} \tag{5.7}$$

Taking contraction on (5.7) over Y and Z, we get

$$X(\sigma-3\rho)=0. \tag{5.8}$$

This shows that $\sigma - 3\rho$ is constant. Hence, we state the following.

Theorem 5.1. *If a conformally flat* $(QE)_4$ *perfect fluid spacetime obeying Einstein field equation with cosmological constant is Ricci symmetric, then* $\sigma - 3\rho$ *is constant.*

Remark 5.1. Let us take constant as zero in the equation (5.8). Then the isotropic pressure ρ is $\sigma/3$ which means that it characterizes radiation era. Therefore radiation has the equation of state $\upsilon = 1/3$ and it predicts that the resulting universe is isotropic and homogenous [10].

Let us consider the energy momentum tensor which is η-recurrent, i.e., $(\nabla_X T)(Y, Z) = \eta(X)T(Y, Z)$, where η is a nonzero 1-form. By Einstein field equation, this condition becomes

$$(\nabla_X S)(Y, Z) - \frac{dr(X)}{2} g(Y, Z) = \eta(X)S(Y, Z) + \eta(X)\left(\Lambda - \frac{r}{2}\right) g(Y, Z).$$

Recall that the scalar curvature r is constant. Replacing r from (5.4), S from (5.5) and ∇S from (5.6), we get

$$\begin{aligned}\kappa\rho\eta(X)g(Y, Z) =& \frac{\kappa}{2} X(\sigma - \rho)g(Y, Z) + \kappa X(\sigma + \rho)A(Y)A(Z)\\ &+ \kappa(\sigma + \rho)[(\nabla_X A)(Y)A(Z) + A(Y)(\nabla_X A)(Z) - \eta(X)A(Y)A(Z)].\end{aligned}$$

Plugging $Y = Z = \omega$ in the above equation, we have

$$X(\sigma + 3\rho) = 2\eta(X)(2\sigma + \rho). \tag{5.9}$$

Hence, we conclude the following.

Theorem 5.2. *If the energy momentum tensor* T *of conformally flat* $(QE)_4$ *perfect fluid spacetime is* η*-recurrent, then energy density and isotropic pressure satisfies the relation* (5.9).

Remark 5.2. For an η-recurrent energy-momentum tensor, if energy density and isotropic pressure are constants, then $\sigma = -1/2\rho$. For a perfect fluid, T is given in (5.2) which takes the form $T(X, Y) = \rho\left[g(X, Y) + \frac{1}{2}A(X)A(Y)\right]$.

In this case we observe that the equation of state υ is -2 which is less than -1, showimg that the existence of phantom energy. We know that phantom energy is a hypothetical form of dark energy with $\upsilon < -1$ [2]. The existence of phantom energy could cause the expansion of the universe to accelerate so quickly that a scenario known as the **Big Rip**, a possible end to the universe occurs and violates weak energy condition.

6. Ricci Soliton Structure in a Conformally Flat $(QE)_4$ Perfect Fluid Spacetime

The present authors recently studied the Ricci soliton structure in perfect fluid spacetime with torse-forming vector field in [19]. In this section, we consder a Ricci soliton structure in a conformally flat $(QE)_4$ perfect fluid spacetime.

The idea of Ricci solitons was introduced by Hamilton[12]. Ricci solitons also correspond to selfsimilar solutions of Hamilton's Ricci flow. They are natural generalizations of Einstein metrics and is defined by

$$(L_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \tag{6.1}$$

for some constant λ and a vector field V. The Ricci soliton is said to be shrinking, steady, and expanding according as λ is negative, zero, and positive respectively.

In view of (5.5), Ricci soliton equation (6.1) takes the form

$$(L_V g)(Y,Z) = -2\left(\Lambda + \lambda + \frac{\kappa(\sigma-\rho)}{2}\right) g(Y,Z) - 2\kappa(\sigma+\rho)A(Y)A(Z). \tag{6.2}$$

In this case we assume that the energy density σ and isotropic pressure ρ are constants. Now differentiating (6.2) covariantly along an arbitrary vector field X provides

$$(\nabla_X L_V g)(Y,Z) = -2\kappa(\sigma+\rho)\left[(\nabla_X A)(Y)A(Z) + A(Y)(\nabla_X A)(Z)\right]. \tag{6.3}$$

Suppose the vector field ω is concurrent, i.e., $\nabla_X \omega = \xi X$, where ξ is a nonzero constant, then $(\nabla_X A)(Y) = \xi g(X,Y)$. Therefore, (6.3) becomes

$$(\nabla_X L_V g)(Y,Z) = -2\xi\kappa(\sigma+\rho)\left[g(X,Y)A(Z) + g(X,Z)A(Y)\right]. \tag{6.4}$$

The identity

$$(\nabla_X L_V g)(Y,Z) = g((L_V\nabla)(X,Y),Z) + g((L_V\nabla)(X,Z),Y), \tag{6.5}$$

can be found from the commutation formula [21]

$$\left(L_V\nabla_X g - \nabla_X L_V g - \nabla_{[V,X]}g\right)(Y,Z) = -g((L_V\nabla)(X,Y),Z) - g((L_V\nabla)(X,Z),Y).$$

Using (6.4) in (6.5) and a straightforward combinatorial computation shows that

$$(L_V\nabla)(Y,Z) = -2\xi\kappa(\sigma+\rho)A(Z)Y. \tag{6.6}$$

Now, substituting $Y = Z = \omega$ in the well known formula [21], we have

$$(L_V\nabla)(X,Y) = \nabla_X\nabla_Y V - \nabla_{\nabla_X Y}V + R(V,X)Y,$$

and then making use of (6.6) we obtain $\nabla_\omega\nabla_\omega V + R(V,\omega)\omega = 2\xi\kappa(\sigma+\rho)\omega$.

If $\sigma+\rho = 0$, then $\nabla_\omega\nabla_\omega V + R(V,\omega)\omega = 0$, i.e., V is Jacobi along ω.
Next, differentiating the (6.6) along an arbitrary vector field X we have

$$(\nabla_X L_V\nabla)(Y,Z) = -2\xi^2\kappa(\sigma+\rho)g(X,Z)Y. \tag{6.7}$$

According to Yano [21], we have the following commutation formula:

$$(L_V R)(X,Y)Z = (\nabla_X L_V\nabla)(Y,Z) - (\nabla_Y L_V\nabla)(X,Z).$$

In view of (6.7), we obtain

$$(L_V R)(X,Y)Z = 2\xi^2\kappa(\sigma+\rho)[g(Y,Z)X - g(X,Z)Y]. \tag{6.8}$$

Substituting $Y = Z = \omega$ in (6.8), we obtain

$$(L_V R)(X,\omega)\omega = 2\xi^2\kappa(\sigma+\rho)[-X - A(X)\omega]. \tag{6.9}$$

Taking $Y=\omega$ in (3.23), then Lie differentiate along V and making use of (6.2) and (6.9), we find that

$$\begin{aligned} &2\xi^2\kappa(\sigma+\rho)[-X-A(X)\omega]+R(X,L_V\omega)\omega+R(X,\omega)L_V\omega \\ &=\frac{\alpha-\beta}{3}\left[-A(X)L_V\omega+2\left(\Lambda+\lambda-\frac{\kappa(\sigma+3\rho)}{2}\right)A(X)\omega-g(X,L_V\omega)\omega\right]. \end{aligned} \tag{6.10}$$

Plugging $Y=Z=\omega$ in (6.2), we get

$$g(L_V\omega,\omega)=\left[\frac{\kappa(\sigma+3\rho)}{2}-\Lambda-\lambda\right]. \tag{6.11}$$

Contracting (6.10) over X, then making use of (5.5) and (6.11) gives

$$\left[\Lambda-\frac{\kappa(\sigma+3\rho)}{2}\right]\cdot\left[\frac{\kappa(\sigma+3\rho)}{2}-\Lambda-\lambda\right]=3\xi^2(\sigma+\rho). \tag{6.12}$$

If $\sigma+\rho=0$, then (6.12) gives a relation

$$\lambda=\kappa\rho-\Lambda.$$

This shows that Ricci soliton is expanding if $\kappa\rho>\Lambda$, steady if $\kappa\rho=\Lambda$ and shrinking if $\kappa\rho<\Lambda$. Hence, we can state the following theorem.

Theorem 6.1. *Let M^4 be a conformally flat $(QE)_4$ perfect fluid spacetime whose energy density and isotropic pressure are constants. If M^4 admits a non-trivial (non-Einstein) Ricci soliton with velocity vector of the fluid is concurrent and $\sigma+\rho=0$, i.e., the spacetime represents inflation, then*

(i) *V is Jacobi along the geodesic determined by ω;*
(ii) *the Ricci soliton is expanding, steady and shrinking according as $\kappa\rho>\Lambda$, $\kappa\rho=\Lambda$ and $\kappa\rho<\Lambda$, respectively.*

Acknowledgements. The authors would like to express their deep thanks to the referee for his/her careful reading and many valuable suggestions towards the improvement of the paper.

References

[1] Z. Ahsan and S. A. Siddiqui, *Concircular curvature tensor and fluid spacetimes,* Internat. J. Theoret. Phys. **48** (2009), 3202–3212.
[2] L. Amendola and S. Tsujikawa, *Dark Energy: Theory and Observations*, Cambridge University Press, Cambridge, 2010.
[3] C. L. Bejan and M. Crasmareanu, *Ricci solitons in manifolds with quasi-constant curvature,* Publ. Math. Debrecen **78**(1) (2011), 235–243.
[4] A. L. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin, 1987.
[5] M. C. Chaki and S. Ray, *Spacetimes with covariant constant energy momentum tensor*, Internat. J. Theoret. Phys. **35**(5) (1996), 1027–1032.
[6] M. C. Chaki and R. K. Maity, *On quasi Einstein manifolds,* Publ. Math. Debrecen **57** (2000), 297–306.

[7] B. Y. Chen and K. Yano, *Hypersurfaces of a conformally flat space,* Tensor (N.S.) **26** (1972), 315–321.
[8] A. De, C. Özgür and U. C. De, *On conformally flat almost pseudo-Ricci symmetric spacetimes,* Internat. J. Theoret. Phys. **51** (2012), 2878–2887.
[9] U. C. De and L. Velimirović, *Spacetimes with semisymmetric energy momentum tensor,* Internat. J. Theoret. Phys. **54** (2015), 1779–1783.
[10] G. F. R. Ellis, *Relativistic Cosmology*, in: R. K. Sachs (Ed.) *General Relativity and Cosmology*, Academic Press, London, 1971.
[11] S. Güler and S. A. Demirbağ, *A study of generalized quasi Einstein spacetime with application in general relativity,* International Journal of Theoretical Physics **55** (2016), 548–562.
[12] R. S. Hamilton, *The Ricci flow on surfaces,* Contemp. Math. **71** (1988), 237–261.
[13] S. Mallick and U. C. De, *Spacetimes with pseudosymmetric energy momentum tensor,* Communications Physics **26**(2) (2016), 121–128.
[14] S. Mallick, Y. J. Suh and U. C. De, *A spacetime with pseudo-projective curvature tensor*, J. Math. Phys. **57** (2016), Paper ID 062501.
[15] D. G. Prakasha and B. S. Hadimani, *η-Ricci soliton on para-Sasakian manifolds,* J. Geom. **108** (2017), 383–392.
[16] S. Ray and Guha, *On perfect fluid pseudo Ricci symmetric spacetime,* Tensor (N.S.) **67** (2006), 101–107.
[17] A. A. Shaikh, D. W. Yoon and S. K. Hui, *On quasi Eienstien spacetime,* Tsuukiba J. Math. **33**(2) (2009), 305–326.
[18] Venkatesha and D. M. Naik, *Certain results on K-para conatct and para Sasakian manifold,* J. Geom. **108** (2017), 939–952.
[19] Venkatesha and H. A. Kumara, *Ricci soliton and geometrical structure in a perfect fluid spacetime with torse-forming vector field*, Afr. Mat. (2019), DOI 10.1007/s13370-019-00679-y.
[20] K. Yano, *Concircular geometry I,* Proc. Imp. Acad. Tokyo. **16** (1940), 195–200.
[21] K. Yano, *Integral Formulas in Riemannian Geometry,* Marcel Dekker, New York, 1970.
[22] K. Yano and M. Kon, *Structure on Manifold,* Series in Pure Mathematics, World Scientific Publishing, Singapore, 1984.

Department of Mathematics,
Kuvempu University,
Shankaraghatta - 577 451, Shivamoga, Karnataka, INDIA
Email address: vensmath@gmail.com
Email address: arunmathsku@gmail.com