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RIEMANN SOLITONS ON KANTOWSKI-SACHS SPACETIMES

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ABSTRACT. We consider Riemann soliton vector fields on Kantowski-Sachs spacetimes. Then we classify all Riemann soliton vector fields on Kantowski-Sachs spacetimes and show which are gradient type and Killing vector field.

1. INTRODUCTION

The concept of spacetime can be viewed as a Lorentzian manifold, which involves the study of the manifold's vectors and offers a suitable framework for investigating cosmological models. Kantowski Sachs spacetime represents a cosmological model characterized by both homogeneity and anisotropy, serving a dual purpose. Evolution and describes the interior of a Schwarzschild black hole in a vacuum scenario. This spacetime typically exhibits future and past singularities, which can manifest as anisotropic structures such as a pancake, a cigar, or a barrel, or as an isotropic point-like structure, relating to the primary conditions related to matter and anisotropic shear. At these classical singularities, the geodesic evolution ceases, as indicated by the divergences in the expansion and shear scalars, as well as the energy density in the presence of matter. The existence of singularities suggests that general relativity is approaching the limits of its applicability, necessitating a quantum gravitational approach to spacetime.

The Kantowski-Sachs spacetimes can be locally represented as those that have an isometry group acting on space-like two-dimensional orbits of positive curvature, making them spherically symmetric. Additionally, they possess an extra space-like Killing field that does not reside within these orbits. Some authors assume the presence of a perfect fluid source, with a corresponding time-like congruence orthogonal to the

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space-like hypersurfaces generated by the four aforementioned space-like Killing vector fields. However, this additional condition is not universally accepted.

Kantowski-Sachs spacetime plays a significant role in cosmological investigations and is considered a potential nominee for a premature era of the universe. Its astrophysical significance is undeniable. Reddy [21] explored a Bianchi-type V inflationary universe within the framework of general relativity and examined the Kantowski–Sachs cosmological model featuring a massless scalar field with a flat potential. Mishra [19] employed the gauge function method to develop a cosmological model of the universe within the Kantowski-Sachs spacetime, utilizing a perfect fluid as the matter field. Hussain et al. [17] classified Kantowski-Sachs spacetime metrics via conformal Ricci collineations. Additionally, Shaikh and Chakraborty [22] studied the curvature properties of the Kantowski-Sachs metric. Also, see [18, 24].

In contrast, the metric of symmetry is often simplified to classify the solutions of Einstein's field equations, with soliton representing a crucial symmetry linked to the geometric flow of spacetime geometry. Hamilton's introduction of the Ricci flow and the Yamabe flow, along with their associated solitons, has facilitated the comprehension of kinematics [14]. The connection between Ricci solitons and Perelman's Ricci flow has opened up new avenues for researching spacetime geometry under systematic flows. This connection not only sheds light on fundamental principles governing the structure of the cosmos but also enhances our ability to model and understand cosmological phenomena.

Recent years have witnessed a broader exploration of Ricci solitons across various geometric contexts. These investigations aim to elucidate how the Ricci soliton condition interacts with extra geometric structures and conditions, thereby enriching our knowledge of both Ricci solitons and the manifolds they inhabit. For further details on the interaction between spacetimes and solitons, relevant citations include [2, 6, 9, 11–13, 15, 20, 23, 27]. Ali and Khan [1] have classified all Ricci soliton vector fields on Kantowski-Sachs spacetime.

The Riemann flow on a pseudo-Riemannian manifold (M^n, g) with the Riemann curvature tensor R [25] is defined by

$$\frac{\partial}{\partial t}G(t) = -2R(g(t)),$$

where $G = \frac{1}{2}g \odot g$ and for two $(0,2)$-tensors ω and θ the product $\odot$ determined by

$$\begin{aligned}(\omega \odot \theta)(U_1, U_2, U_3, U_4) &= \omega(U_1, U_4)\theta(U_2, U_3) + \omega(U_2, U_3)\theta(U_1, U_4) \\ &\quad - \omega(U_1, U_3)\theta(U_2, U_4) - \omega(U_2, U_4)\theta(U_1, U_3),\end{aligned}$$

for all vector fields U_1, U_2, U_3, U_4. Manifold (M^n, g) is called a Riemann soliton [16] and denoted by (M^n, g, μ, Y) if

$$2R + \mu g \odot g + g \odot \mathcal{L}_Y g = 0, \tag{1.1}$$

for some vector field Y and constant μ where $\mathcal{L}_Y$ denotes the Lie derivative along Y. If μ is positive or negative or zero, then the Riemann soliton is called expanding or shrinking or steady, respectively. The Riemann soliton is called gradient Riemann soliton, if $Y = \mathrm{grad} f$ for some smooth function f, and (1.1) becomes

$$2R + \mu g \odot g + 2g \odot \nabla^2 f = 0.$$

The Riemann soliton is associated with the Riemann flow as a fixed point.

Numerous researchers have conducted investigations on Riemann solitons within various manifold contexts, including contact geometry [10,26], almost co-Kahler manifolds [4], and para-Sasakian manifolds [7]. Furthermore, please refer to [3,5,8]. In this paper, we classify all Riemann soliton vector fields on Kantowski-Sachs spacetimes and determine which of them are of gradient type. Additionally, we establish which of these vector fields is a Killing vector field.

The paper is structured as follows. In Section 2, we review important concepts and fundamental formulas related to Kantowski-Sachs spacetimes that will be utilized throughout the paper. In Section 3, we classify Riemann solitons on Kantowski-Sachs spacetimes.

2. Preliminaries

In spherical coordinate system (t, r, x, y) at a neighborhood of optional point, Kantowski-Sachs metric is given by

$$g = dt^2 - e^{2A(t)}dr^2 - e^{2B(t)}dx^2 - e^{2B(t)}\sin^2 x dy^2, \tag{2.1}$$

where A and B are functions of t.

Let ∇ be the Levi-Civita connection of the metric g. The curvature tensor R of g is determined by

$$R(V, W) = \nabla_{[V,W]} - [\nabla_V, \nabla_W]$$

and the Ricci tensor S of g is described by $S(V, W) = tr(Z \to R(V, Z)W)$. Let

$$\partial_1 = \frac{\partial}{\partial t}, \quad \partial_2 = \frac{\partial}{\partial r}, \quad \partial_3 = \frac{\partial}{\partial x}, \quad \partial_4 = \frac{\partial}{\partial y}.$$

The components of ∇ are given by

$$\nabla = \begin{pmatrix} 0 & A'\partial_2 & B'\partial_3 & B'\partial_4 \\ A'\partial_2 & A'e^{2A}\partial_1 & 0 & 0 \\ B'\partial_3 & 0 & B'e^{2B}\partial_1 & \cot x\, \partial_4 \\ B'\partial_4 & 0 & \cot x\, \partial_4 & B'e^{2B}\sin^2 x\, \partial_1 - \sin x \cos x\, \partial_3 \end{pmatrix}$$

and the only non-zero components of the Riemannian curvature tensor associated to (2.1) are given as follows

$$\begin{aligned} R_{1212} &= e^{2A}(A'^2 + A''), \\ R_{1414} &= e^{2B}(B'^2 + B'') \sin^2 x, \\ R_{2323} &= -A'B'e^{2A+2B}, \\ R_{2424} &= -A'B'e^{2A+2B} \sin^2 x, \\ R_{1313} &= e^{2B}(B'^2 + B''), \\ R_{3434} &= -(e^{2B} + e^{4B}B'^2) \sin^2 x. \end{aligned} \tag{2.2}$$

The Ricci tensor of Kantowski-Sachs metric is determined by

$$S = \begin{pmatrix} S_1 & 0 & 0 & 0 \\ 0 & S_2 & 0 & 0 \\ 0 & 0 & S_3 & 0 \\ 0 & 0 & 0 & S_3 \sin^2 x \end{pmatrix},$$

with respect to the basis $\{\partial_1, \partial_2, \partial_3, \partial_4\}$, where

$$\begin{aligned} S_1 &= -(2B'^2 + 2B'' + A'^2 + A''), \\ S_2 &= e^{2A}(2A'B' + A'^2 + A''), \\ S_3 &= 1 + e^{2B}(2B'^2 + A'B' + B''). \end{aligned}$$

We assume an optional vector field $Y = Y^i\partial_i$ on Kantowski-Sachs spacetimes, where $Y^i = Y_i(t, r, x, y),\ i = 1, 2, 3, 4$ are smooth maps. We obtain

$$\begin{aligned} (\mathcal{L}_Y g)_{11} &= 2\partial_1 Y^1, \\ (\mathcal{L}_Y g)_{12} &= -e^{2A}\partial_1 Y^2 + \partial_2 Y^1, \\ (\mathcal{L}_Y g)_{13} &= -e^{2B}\partial_1 Y^3 + \partial_3 Y^1, \\ (\mathcal{L}_Y g)_{14} &= -e^{2B}(\sin^2 x)\partial_1 Y^4 + \partial_4 Y^1, \\ (\mathcal{L}_Y g)_{22} &= -2e^{2A}(\partial_2 Y^2 + A'Y^1), \\ (\mathcal{L}_Y g)_{23} &= -e^{2B}\partial_2 Y^3 - e^{2A}\partial_3 Y^2, \\ (\mathcal{L}_Y g)_{24} &= -e^{2B}(\sin^2 x)\partial_2 Y^4 - e^{2A}\partial_4 Y^2, \\ (\mathcal{L}_Y g)_{33} &= -2e^{2B}(\partial_3 Y^3 + B'Y^1), \\ (\mathcal{L}_Y g)_{34} &= -e^{2B}\left((\sin^2 x)\partial_3 Y^4 + \partial_4 Y^3\right), \\ (\mathcal{L}_Y g)_{44} &= -2(\sin x)e^{2B}\left((\sin x)\partial_4 Y^4 + (\sin x)B'Y^1 + (\cos x)Y^3\right), \end{aligned} \tag{2.3}$$

where $(\mathcal{L}_Y g)_{ij} = (\mathcal{L}_Y g)(\partial_i, \partial_j)$ for $i = 1, 2, 3, 4$.

3. Riemann solitons on Kantowski-Sachs spacetimes

In this section, we investigate the existence of Riemann solitons on Kantowski-Sachs spacetimes. From (1.1), we have

$$\begin{aligned}2R(U_1,U_2,U_3,U_4) = &-2\mu\Big[g(U_1,U_4)g(U_2,U_3) - g(U_1,U_3)g(U_2,U_4)\Big]\\ &- \Big[g(U_1,U_4)\mathcal{L}_V g(U_2,U_3) + g(U_2,U_3)\mathcal{L}_V g(U_1,U_4)\Big]\\ &+ \Big[g(U_1,U_3)\mathcal{L}_V g(U_2,U_4) + g(U_2,U_4)\mathcal{L}_V g(U_1,U_3)\Big],\end{aligned} \tag{3.1}$$

for any vector fields U_1,U_2,U_3,U_4. By using (3.1), Kantowski-Sachs spacetime is a Riemann soliton with potential vector field Y if and only if

$$\begin{aligned}2R_{1212} &= 2\mu g_{11}g_{22} + g_{11}(\mathcal{L}_Y g)_{22} + g_{22}(\mathcal{L}_Y g)_{11},\\ 2R_{1313} &= 2\mu g_{11}g_{33} + g_{11}(\mathcal{L}_Y g)_{33} + g_{33}(\mathcal{L}_Y g)_{11},\\ 2R_{1414} &= 2\mu g_{11}g_{44} + g_{11}(\mathcal{L}_Y g)_{44} + g_{44}(\mathcal{L}_Y g)_{11},\\ 2R_{2323} &= 2\mu g_{22}g_{33} + g_{22}(\mathcal{L}_Y g)_{33} + g_{33}(\mathcal{L}_Y g)_{22},\\ 2R_{2424} &= 2\mu g_{22}g_{44} + g_{22}(\mathcal{L}_Y g)_{44} + g_{44}(\mathcal{L}_Y g)_{22},\\ 2R_{3434} &= 2\mu g_{33}g_{44} + g_{44}(\mathcal{L}_Y g)_{33} + g_{33}(\mathcal{L}_Y g)_{44},\\ 2R_{1213} &= g_{11}(\mathcal{L}_Y g)_{23},\\ 2R_{1214} &= g_{11}(\mathcal{L}_Y g)_{24},\\ 2R_{1314} &= g_{11}(\mathcal{L}_Y g)_{34},\\ 2R_{1232} &= g_{22}(\mathcal{L}_Y g)_{13},\\ 2R_{1242} &= g_{22}(\mathcal{L}_Y g)_{14},\\ 2R_{1323} &= g_{33}(\mathcal{L}_Y g)_{12}.\end{aligned}$$

Applying (2.1) and (2.2) in the above equations, we get

$$\begin{aligned}2e^{2A}(A'^2 + A'') &= -2\mu e^{2A} + (\mathcal{L}_Y g)_{22} - e^{2A}(\mathcal{L}_Y g)_{11},\\ 2e^{2B}(B'^2 + B'') &= -2\mu e^{2B} + (\mathcal{L}_Y g)_{33} - e^{2B}(\mathcal{L}_Y g)_{11},\\ 2e^{2B}(B'^2 + B'')\sin^2 x &= -2\mu e^{2B}\sin^2 x + (\mathcal{L}_Y g)_{44} - e^{2B}\sin^2 x(\mathcal{L}_Y g)_{11},\\ -2A'B'e^{2A+2B} &= 2\mu e^{2A+2B} - e^{2A}(\mathcal{L}_Y g)_{33} - e^{2B}(\mathcal{L}_Y g)_{22},\\ -2A'B'e^{2A+2B}\sin^2 x &= 2\mu e^{2A+2B}\sin^2 x - e^{2A}(\mathcal{L}_Y g)_{44} - e^{2B}\sin^2 x(\mathcal{L}_Y g)_{22},\\ -2(e^{2B} + e^{4B}B'^2)\sin^2 x &= 2\mu e^{4B}\sin^2 x - e^{2B}\sin^2 x(\mathcal{L}_Y g)_{33} - e^{2B}(\mathcal{L}_Y g)_{44},\\ 0 = (\mathcal{L}_Y g)_{23} &= (\mathcal{L}_Y g)_{24} = (\mathcal{L}_Y g)_{34} = (\mathcal{L}_Y g)_{13}(\mathcal{L}_Y g)_{14} = (\mathcal{L}_Y g)_{12}.\end{aligned}$$

Equivalently, we have

$$(\mathcal{L}_Y g)_{11} = C, \qquad (\mathcal{L}_Y g)_{22} = D, \qquad (\mathcal{L}_Y g)_{33} = E, \qquad (\mathcal{L}_Y g)_{44} = E\sin^2 x,$$
$$(\mathcal{L}_Y g)_{23} = (\mathcal{L}_Y g)_{24} = (\mathcal{L}_Y g)_{34} = (\mathcal{L}_Y g)_{13}(\mathcal{L}_Y g)_{14} = (\mathcal{L}_Y g)_{12} = 0,$$
$$-A'B' = e^{-2B} + B' - A'^2 - A'' + B'^2 + B'',$$

where

$$C = -e^{-2B} - (\mu + B' + 2B'^2 + 2B''),$$
$$D = e^{2A}\Big(2A'^2 + 2A'' + 2\mu + C\Big),$$
$$E = \mu e^{2B} - 1 - e^{2B}B'.$$

Inserting (2.3) in the above equation, we conclude

$$2\partial_1 Y^1 = C, \tag{3.2}$$
$$-e^{2A}\partial_1 Y^2 + \partial_2 Y^1 = 0, \tag{3.3}$$
$$-e^{2B}\partial_1 Y^3 + \partial_3 Y^1 = 0, \tag{3.4}$$
$$-e^{2B}(\sin^2 x)\partial_1 Y^4 + \partial_4 Y^1 = 0, \tag{3.5}$$
$$-2e^{2A}(\partial_2 Y^2 + A'Y^1) = D, \tag{3.6}$$
$$-e^{2B}\partial_2 Y^3 - e^{2A}\partial_3 Y^2 = 0, \tag{3.7}$$
$$-e^{2B}(\sin^2 x)\partial_2 Y^4 - e^{2A}\partial_4 Y^2 = 0, \tag{3.8}$$
$$-2e^{2B}(\partial_3 Y^3 + B'Y^1) = E, \tag{3.9}$$
$$(\sin^2 x)\partial_3 Y^4 + \partial_4 Y^3 = 0, \tag{3.10}$$
$$-2(\sin x)e^{2B}\Big((\sin x)\partial_4 Y^4 + (\sin x)B'Y^1 + (\cos x)Y^3\Big) = E\sin^2 x, \tag{3.11}$$
$$A'B' + e^{-2B} + B' - A'^2 - A'' + B'^2 + B'' = 0. \tag{3.12}$$

Now we solve the above system of partial differential equations. Integrating equation (3.2) with respect to t, we have

$$Y^1 = \frac{1}{2}\int C dt + F(r, x, y), \tag{3.13}$$

for some smooth function F.

Applying (3.13) in (3.3) and integrating the resulting equation with respect to t, we obtain

$$Y^2 = (\partial_2 F)\int e^{-2A} dt + G(r, x, y), \tag{3.14}$$

for some smooth map G. Also, inserting (3.13) in (3.4) and integrating the resulting equation with respect to t, we arrive at

$$Y^3 = (\partial_3 F)\int e^{-2B} dt + H(r, x, y), \tag{3.15}$$

for some smooth map H. Putting (3.13) in (3.5) and integrating the resulting equation regarding to t, we get

$$Y^4 = \frac{1}{\sin^2 x}(\partial_4 F)\int e^{-2B}dt + K(r,x,y), \tag{3.16}$$

for some smooth function K.

Plugging (3.15) and (3.16) into (3.10), we can write

$$\Big(-2\cot x\partial_4 F + 2\partial^2_{43}F\Big)\int e^{-2B}dt + (\sin^2 x)\partial_3 K(r,x,y) + \partial_4 H(r,x,y) = 0.$$

Differentiating the last equation with respect to t, we provide

$$-(\cot x)\partial_4 F + \partial^2_{43}F = 0, \quad (\sin^2 x)\partial_3 K + \partial_4 H = 0. \tag{3.17}$$

Substituting (3.13) and (3.15) in (3.9), we find

$$(\partial^2_{33}F)\int e^{-2B}dt + \partial_3 H(r,x,y) + \frac{1}{2}B'\int Cdt + B'F(r,x,y) = -\frac{1}{2}e^{-2B}E. \tag{3.18}$$

Differentiating equation (3.18) with respect to t, we conclude

$$(\partial^2_{33}F) + e^{2B}B''F(r,x,y) = \mathcal{E}, \tag{3.19}$$

where $\mathcal{E} = -\frac{1}{2}e^{2B}(e^{-2B}E + B'\int Cdt)'$. Again, differentiating equation (3.19) with respect to t, we deduce

$$(e^{2B}B'')'F(r,x,y) = \mathcal{E}'. \tag{3.20}$$

Equation (3.20) suggest the following cases.

Case I. If $(e^{2B}B'')' \neq 0$, $F(r,x,y) = \frac{\mathcal{E}'}{(e^{2B}B'')'} = a_1$ for some constant a_1. In this case, $Y^1,\dots,Y^4$ become

$$Y^1 = \frac{1}{2}\int Cdt + a_1, \quad Y^2 = G(r,x,y), \quad Y^3 = H(r,x,y), \quad Y^4 = K(r,x,y).$$

Equation (3.18) gives

$$\partial_3 H(r,x,y) = -\frac{1}{2}\left(e^{-2B}E + B'\int Cdt\right) - B'a_1 = a_2, \tag{3.21}$$

for some constant a_2. Thus, $H = a_2 x + H_1(r,y)$ for some smooth function H_1. From (3.19), we get $\mathcal{E} = a_1 e^{2B}B''$. Inserting (3.13) and (3.14) in (3.6), we obtain

$$\partial_2 G(r,x,y) = -\frac{1}{2}\left(e^{-2A}D + A'\int Cdt\right) - A'a_1 = a_3,$$

for some constant a_3. Hence, $G = a_3 r + G_1(x,y)$ for some smooth function G_1. Equation (3.17), yields

$$(\sin^2 x)\partial_3 K = -\partial_4 H_1 = a_4,$$

for some constant a_4. Therefore,

$$H_1(r,y) = -a_4 y + H_2(r), \quad K(r,x,y) = -a_4(\cot x) + K_1(r,y),$$

for some functions H_2 and K_1. Using (3.11) and (3.21), it follows that

$$\partial_4 K_1(r,y) + (\cot x)(a_2 x - a_4 y + H_2(r)) = a_2.$$

The last equation leads to $a_2 = a_4 = H_2(r) = 0$ and $K_1(r, y) = K_2(r)$ for some smooth function K_2. Also, from (3.7), we infer $\partial_3 G_1(x, y) = 0$. Then, $G_1(x, y) = G_2(y)$ for some smooth map G_2. Applying (3.8), we conclude that

$$(\sin^2 x)K_2'(r) = -e^{2A-2B}G_2'(y).$$

Therefore, $K_2'(r) = G_2'(y) = 0$ and $K_2(r) = a_5$ and $G_2(y) = a_6$ for some constants a_5 and a_6. Thus, we have

$$\begin{aligned}
&Y^1 = \frac{1}{2}\int C\,dt + a_1,\\
&Y^2 = a_3 r + a_6,\\
&Y^3 = 0,\\
&Y^4 = a_5,\\
&A'B' + e^{-2B} + B' - A'^2 - A'' + B'^2 + B'' = 0,\\
&-\frac{1}{2}\Big(e^{-2B}E + B'\int C\,dt\Big) = B'a_1,\\
&-\frac{1}{2}\Big(e^{-2A}D + A'\int C\,dt\Big) - A'a_1 = a_3,\\
&(e^{2B}B'')' \neq 0.
\end{aligned} \tag{3.22}$$

Case II. Now, we assume that $(e^{2B}B'')' = 0$ and $e^{2B}B'' = b_1^2 > 0$ for some constant b_1. Equation (3.20) leads to $-\frac{1}{2}e^{2B}(e^{-2B}E + B'\int C dt)' = b_2$ for some constant b_2. Using (3.19), we get $(\partial_{33}^2 F) + b_1^2 F(r, x, y) = b_2$ and

$$F(r, x, y) = F_1(r, y)\sin b_1 x + F_2(r, y)\cos b_1 x + \frac{b_2}{b_1^2},$$

for some smooth functions F_1 and F_2. From (3.18), we can write $\partial_3 H = 0$. Thus, $H(r, x, y) = H_4(r, y)$ for some smooth function H_4. Using equation (3.17), it deduces that

$$-(\cos x \sin b_1 x + b_1 \sin x \cos b_1 x)\partial_4 F_1 - (\cos x \cos b_1 x + b_1 \sin x \sin b_1 x)\partial_4 F_2 = 0.$$

and consequently $\partial_4 F_1 = \partial_4 F_2 = 0$. Then, $F_1(r, y) = F_3(r)$ and $F_2(r, y) = F_4(r)$ for some smooth function F_3 and F_4. Also, using the second equation of (3.17), we obtain

$$K(r, x, y) = (\cot x)\partial_4 H_4(r, y) + K_3(r, y),$$

for some smooth function K_3. Taking derivative of (3.11) with respect to t, one gets

$$b_1(\cos x \cos b_1 x + b_1 \sin x \sin b_1 x)F_3 + b_1(-\cos x \sin b_1 x + b_1 \sin x \cos b_1 x)F_4 = b_2 \sin x.$$

Then, $F_3 = F_4 = b_2 = 0$ and $F = 0$.

Case III. Now, we consider $e^{2B}B'' = -c_1^2 < 0$ for some constant c_1. Equation (3.20) leads to $-\frac{1}{2}e^{2B}(e^{-2B}E + B'\int C dt)' = c_2$ for some constant c_2. Using (3.19), we get $(\partial_{33}^2 F) - c_1^2 F(r, x, y) = c_2$ and

$$F(r, x, y) = F_5(r, y)e^{c_1 x} + F_6(r, y)e^{-c_1 x} + \frac{c_2}{c_1^2},$$

for some smooth functions F_5 and F_6. From (3.18), we can write $\partial_3 H = 0$ for some constant c_3. Thus, $H(r, x, y) = H_7(r, y)$ for some smooth function H_7. Using equation (3.17), it follows that

$$(c_1 - \cot x)e^{c_1 x}\partial_4 F_5 - (\cot x + c_1)e^{c-1x}\partial_4 F_6 = 0,$$

and consequently $\partial_4 F_5 = \partial_4 F_6 = 0$. Then, $F_5(r, y) = F_7(r)$ and $F_6(r, y) = F_8(r)$ for some smooth function F_7 and F_8. Also, using the second equation of (3.17), we obtain

$$K(r, x, y) = (\cot x)\partial_4 H_7(r, y) + K_5(r, y),$$

for some smooth function K_5. Taking derivative of (3.11) with respect to t, one gets

$$c_1(\cos x e^{c_1 x} - c_1 \sin x e^{c_1 x})F_7 - c_1(\cos x e^{-c_1 x} + c_1 \sin x e^{-c_1 x})F_8 = 2c_2 \sin x.$$

Then, $F_7 = F_8 = c_2 = 0$ and $F = 0$.

Case IV. If $e^{2B}B'' = 0$, then $B = d_1 t + d_2$, $e^{-2B}E = \mu - e^{-2d_1 t - 2d_2} - d_1$, and $C = -e^{-2d_1 t + 2d_2} - \mu - d_1 - 2d_1^2$ for some constants d_1 and d_2. Therefore,

$$\mathcal{E} = -\frac{1}{2}e^{2d_1 t + 2d_2}\left(d_1 e^{-2d_1 t - 2d_2} - d_1(\mu + d_1 + 2d_1^2)\right).$$

Equation (3.19) implies that $\partial^2_{33}F = \mathcal{E}$ and consequently $\mathcal{E}' = 0$. Thus, $d_1 = 0$ or $\mu = -d_1 - 2d_1^2$. If $d_1 = 0$, then $B = d_2$. Hence, $B' = 0$ and

$$F(r, x, y) = F_9(r, y)x + F_{10}(r, y),$$

for some smooth functions F_9 and F_{10}. From (3.18) we get $\partial_3 H = -\frac{1}{2}(\mu - e^{-2d_2})$ and it follows that $H = -\frac{1}{2}(\mu - e^{-2d_2})x + H_8(r, y)$ for smooth function H_8. Using equation (3.17), it follows that

$$(1 - x\cot x)\partial_4 F_9 - (\cot x)\partial_4 F_{10} = 0,$$

and consequently $\partial_4 F_9 = \partial_4 F_{10} = 0$. Then, $F_9(r, y) = F_{11}(r)$ and $F_{10}(r, y) = F_{12}(r)$ for some smooth functions F_{11} and F_{12}. Also, using the second equation of (3.17), we obtain

$$K(r, x, y) = (\cot x)\partial_4 H_8(r, y) + K_7(r, y),$$

for some smooth function K_7. Taking derivative of (3.11) with respect to t, one gets $F_{11}(r) = 0$. Also, (3.11) becomes $(\cot x)(\partial^2_{44}H_8 + H_8) + \partial_4 K_7 = 0$. Thus, $\partial^2_{44}H_8 + H_8 = 0$ and $\partial_4 K_7 = 0$. Therefore,

$$H_8(r, y) = H_9(r)\sin y + H_{10}(r)\cos y, \quad K_7(r, y) = K_8(r),$$

for some smooth functions H_9, H_{10} and K_8. Equation (3.7) leads to

$$-F'_{12}(r)t + H'_9(r)\sin y + H'_{10}(r)\cos y - e^{-2A}\partial_3 G = 0.$$

When the last equation is differentiated with respect to the variable t, it leads to the conclusion

$$-F'_{12}(r) + 2A'e^{-2A}\partial_3 G = 0.$$

When the last equation is differentiated with respect to the variable t, it follows that

$$(A'' - 2A'^2)\,\partial_3 G = 0.$$

From (3.12), we get $A'^2 + A'' = e^{-2d_2}$. Since $A'' - 2A'^2 \neq 0$, we have

$$\partial_3 G = F'_{12}(r) = H'_9(r) = H'_{10}(r) = 0.$$

Therefore, $G(r, x, y) = G_3(r, y)$, for some smooth function G_3, and

$$F_{12}(r) = d_3, \quad H_9(r) = d_4, \quad H_{10}(r) = d_5,$$

for some constants d_3, d_4, d_5. Also, we have $D = e^{2A}(\mu + e^{-2d_2})$, $C = -\mu - e^{-2d_2}$. Equation (3.6) yields

$$\partial_2 G_3 = -\frac{1}{2}(\mu + e^{-2d_2}) + \frac{1}{2}A'(\mu + e^{-2d_2})t.$$

Then, $(A't)' = d_6$ for some constant d_6. This is a contradiction with

$$A'^2 + A'' = e^{-2d_2}.$$

Now, we assume that $d_1 \neq 0$ and $\mu = -d_1 - 2d_1^2$. In this case, equation (3.19) becomes $\partial^2_{33} F = -\frac{1}{2}d_1$ and consequently

$$F(r, x, y) = -\frac{1}{4}d_1 x^2 + F_{13}(r, y)x + F_{14}(r, y),$$

for some smooth functions F_{13} and F_{14}. From (3.18) we get

$$\partial_3 H = -d_1\left(-\frac{1}{4}d_1 x^2 + F_{13}(r, y)x + F_{14}(r, y)\right) + d_1 + d_1^2,$$

and it follows that

$$H = -d_1\left(-\frac{1}{12}d_1 x^3 + F_{13}(r, y)\frac{x^2}{2} + F_{14}(r, y)x\right) + (d_1 + d_1^2)x + H_{11}(r, y),$$

for smooth function H_{11}. Using equation (3.17), it follows that

$$(1 - x\cot x)\partial_4 F_{13} - (\cot x)\partial_4 F_{14} = 0,$$

and consequently $\partial_4 F_{13} = \partial_4 F_{14} = 0$. Then, $F_{13}(r, y) = F_{15}(r)$ and $F_{14}(r, y) = F_{16}(r)$ for some smooth function F_{15} and F_{14}. Also, using the second equation of (3.17), we obtain

$$K(r, x, y) = (\cot x)\partial_4 H_{11}(r, y) + K_9(r, y),$$

for some smooth function K_9. Taking derivative of (3.11) with respect to t, one gets $d_1 x\cot x - 2(\cot x)F_{13}(r) = d_1$. This yields $d_1 = 0$, which is a contradiction.

Theorem 3.1. *A Kantowski-Sachs spacetime with metric* (2.1) *admits a Riemann soliton* (M^4, g, μ, Y) *if and only if* μ, A, B *and* Y *satisfy* (3.22).

From the Theorem 3.1, the Riemann soliton (M^4, g, μ, Y) is a gradient Riemann soliton with

$$Y = \nabla f = \partial_1 f\partial_1 - e^{-2A}\partial_2 f\partial_2 - e^{-2B}\partial_3 f\partial_3 - e^{-2B}\frac{1}{\sin^2 x}\partial_4 f\partial_4$$

if and only if

$$\partial_1 f = \frac{1}{2}\int C\,dt + a_1, \quad \partial_2 f = a_3 r + a_6, \quad \partial_3 f = 0, \quad \partial_4 f = a_5.$$

Thus, with direct integration, we get

$$f = \frac{1}{2}\int\int C dt dt + a_1 t + \frac{1}{2}a_3 r^2 + a_6 r + a_5 y + a_7, \tag{3.23}$$

for some constant a_7.

Corollary 3.1. *The Riemann soliton* (M^4, g, μ, Y) *is a gradient Riemann soliton with potential function* f *satisfying* (3.23).

Remark 3.1. From Theorem 3.1, we conclude that no Riemann soliton vector field on Kantowski-Sachs spacetimes is a Killing vector field.

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