

EXTENSIONS OF MEIR-KEELER CONTRACTION VIA w -DISTANCES WITH AN APPLICATION

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ABSTRACT. In this article, we conceive the notion of a generalized (α, ψ, q) -Meir-Keeler contractive mapping and then we investigate a fixed point theorem involving such kind of contractions in the setting of a complete metric space via a w -distance. Our obtained result extends and generalizes some of the previously derived fixed point theorems in the literature via w -distances. In addition, to validate the novelty of our findings, we illustrate a couple of constructive numerical examples. Moreover, as an application, we employ the achieved result to earn the existence criteria of the solution of a kind of non-linear Fredholm integral equation.

1. INTRODUCTION AND PRELIMINARIES

In this paper, we introduce the notion of a generalized (α, ψ, q) -Meir-Keeler contractive mapping and investigate fixed points for such operators in the context of complete metric spaces via a w -distance. For this purpose we first recall the outstanding result of Meir-Keeler [14] (see also [10]).

Theorem 1.1 ([14]). *Let f be a self-map defined on a complete metric space (M, d) . Also assume that for any $\varepsilon > 0$ we can find a $\delta > 0$ such that*

$$\varepsilon \leq d(\rho, \varrho) < \varepsilon + \delta \quad \text{implies} \quad d(f\rho, f\varrho) < \varepsilon,$$

for all $\rho, \varrho \in M$. Then f has a unique fixed point.

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This result is also known as a uniform contraction and it has been studied and extended by a number of researchers in many directions (see [16, 20]). Now we recall the notion of w -distance introduced by Kada et al. [12].

Definition 1.1 ([12]). Let (M, d) be a metric space. A mapping $q : M \times M \rightarrow [0, \infty)$ is said to be a w -distance on M if

- (i) $q(\rho, \sigma) \leq q(\rho, \varrho) + q(\varrho, \sigma)$ for any $\rho, \varrho, \sigma \in M$;
- (ii) q is a lower semi-continuous map in the second variable, that is, when $\rho \in M$ and $\sigma_n \rightarrow \sigma$ in M , then we have $q(\rho, \sigma) \leq \liminf_n q(\rho, \sigma_n)$;
- (iii) for every $\epsilon > 0$, there is a $\delta > 0$ which $q(\sigma, \rho) \leq \delta$ and $q(\sigma, \varrho) \leq \delta$ imply that $d(\rho, \varrho) \leq \epsilon$.

Let $T : M \rightarrow M$ and $\alpha : M \times M \rightarrow [0, \infty)$. We say that T is α -orbital admissible (see [17]) if

$$\alpha(p, Tp) \geq 1 \quad \text{implies} \quad \alpha(Tp, T^2p) \geq 1,$$

for all $p \in M$. By using this auxiliary function, it is possible to combine several existing results in the literature, see, e.g. [9, 15, 18, 19] and the related references therein. In particular, Lakzian et al. [13] introduced the concept of (α, ψ, q) -contractive mappings in metric spaces via w -distances and proved fixed point results via this notion.

On the other hand, inspired by the notion of Meir-Keeler contractions, Chen [11] introduced the concept of a weaker Meir-Keeler function as follows.

Definition 1.2 ([11]). A mapping $\psi : [0, \infty) \rightarrow [0, \infty)$ is said to be a weaker Meir-Keeler function if, for every $\epsilon > 0$, there is a $\delta > 0$ such that for every $\tau \in [0, \infty)$ with $\epsilon \leq \tau < \epsilon + \delta$, we have an $n_0 \in \mathbb{N}$ satisfying $\psi^{n_0}(\tau) < \epsilon$.

Regarding [11], we also consider the family Ψ of weaker Meir-Keeler functions $\psi : [0, \infty) \rightarrow [0, \infty)$ fulfilling the subsequent properties:

- (ψ_1) $\psi(\tau) > 0$ whenever $\tau > 0$ and $\psi(0) = 0$;
- (ψ_2) $\sum_{n=1}^{\infty} \psi^n(\tau) < \infty$, $\tau \in (0, \infty)$;
- (ψ_3) for each $y_n \in [0, \infty)$, the following hold:
 - (i) when $\lim_{n \rightarrow \infty} y_n = \ell > 0$, then $\lim_{n \rightarrow \infty} \psi(y_n) < \ell$;
 - (ii) whenever $\lim_{n \rightarrow \infty} y_n = 0$, we have $\lim_{n \rightarrow \infty} \psi(y_n) = 0$.

Along with the aforementioned terminologies, the following lemma is also playing a crucial role in our subsequent studies.

Lemma 1.1 ([12]). *Suppose that (M, d) is a metric space with a w -distance q .*

- (i) *For any sequence $\{\rho_n\}$ in M with $\lim_n q(\rho_n, \rho) = \lim_n q(\rho_n, \varrho) = 0$, we have $\rho = \varrho$. Additionally, $q(\sigma, \rho) = q(\sigma, \varrho) = 0$ implies $\rho = \varrho$.*
- (ii) *For two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0, \infty)$ converging to 0, whenever $q(\rho_n, \varrho_n) \leq \alpha_n$, $q(\rho_n, \varrho) \leq \beta_n$ hold for each $n \in \mathbb{N}$, then the sequence $\{\varrho_n\}$ converges to ϱ .*

- (iii) Suppose that $\{\rho_n\}$ is a sequence in M such that for every $\varepsilon > 0$ there is an $N_\varepsilon \in \mathbb{N}$ with $m > n > N_\varepsilon$ implies that $q(\rho_n, \rho_m) < \varepsilon$ (or $\lim_{m,n} q(\rho_n, \rho_m) = 0$). Then $\{\rho_n\}$ is a Cauchy sequence.

In this paper, we define the concept of generalized (α, ψ, q) -Meir-Keeler contractive mappings and by using this new concept, we give some fixed point results. Furthermore, some significant non-trivial numerical examples are investigated to authenticate our findings. Moreover, as an application, the existence of the solution for a non-linear Fredholm integral equation is investigated.

2. (α, ψ, q) -MEIR-KEELER CONTRACTIONS

This section brings the idea of generalized (α, ψ, q) -Meir-Keeler contractive mappings with the help of a weaker Meir-Keeler function. Also, we conceive a fixed point result concerning such kinds of mappings. Now we consider the following expressions:

$$M_q(\rho, \varrho) = \max \left\{ q(\rho, \varrho), q(\rho, f\rho), q(\varrho, f\varrho), \frac{q(\rho, f\varrho) + q(f\rho, \varrho)}{2} \right\}$$

and

$$m(\rho, \varrho) = \max \left\{ d(\rho, \varrho), d(\rho, f\rho), d(\varrho, f\varrho), \frac{d(\rho, f\varrho) + d(f\rho, \varrho)}{2} \right\}.$$

Here, we propose the idea of generalized (α, ψ, q) -Meir-Keeler contractive mappings.

Definition 2.1. Suppose that (M, d) is a metric space with a w -distance q and consider the functions $\psi \in \Psi$, $\alpha : M \times M \rightarrow [0, \infty)$ and an α -orbital admissible map f . Then f is called a generalized (α, ψ, q) -Meir-Keeler contractive mapping if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$, when $\eta \leq \psi(M_q(\rho, \varrho)) < \eta + \delta$, we have $\alpha(\rho, \varrho)q(f\rho, f\varrho) < \eta$.

In addition, for $q = d$ and $M_q(\rho, \varrho) = m(\rho, \varrho)$, the mapping f is said to be a generalized (α, ψ) -Meir-Keeler-contractive. Furthermore, f is a (α, ψ, q) -Meir-Keeler contractive map, when $M_q(\rho, \varrho) = q(\rho, \varrho)$ for each $\rho, \varrho \in M$.

The succeeding theorem deals with an interesting fixed point result involving the previously discussed type of maps.

Theorem 2.1. Suppose that (M, d) is a complete metric space with a w -distance q . Also assume that f is a generalized (α, ψ, q) -Meir-Keeler contractive map such that there is $\rho_0 \in M$ with $q(f^n \rho_0, f^n \rho_0) = 0$ for all non-negative integers n and $\alpha(\rho_0, f\rho_0) \geq 1$. Suppose that one of the following conditions holds.

- (i) For each $w \in M$ satisfying $w \neq fw$, we have $\inf\{q(\rho, w) + q(\rho, f\rho) : \rho \in M\} > 0$.
- (ii) f is continuous.
- (iii) If for some sequence $\{\rho_n\}$, $\lim_{n \rightarrow \infty} q(\rho_n, \rho) = \lim_{n \rightarrow \infty} q(f\rho_n, \rho)$, then $f\rho = \rho$.

Then f owns a fixed point $u \in M$, with $q(u, u) = 0$.

Proof. We construct a sequence $\{\rho_n\}$ in M such that $\rho_{n+1} = f\rho_n = f^{n+1}\rho_0$ for each $n \in \mathbb{N}$. When $\rho_{n_0} = \rho_{n_0+1}$ for some positive integer n_0 , then $u = \rho_{n_0}$ is a fixed point of f . Hence, without loss of generality consider that,

$$\rho_n \neq \rho_{n+1}, \quad \text{for all } n \in \mathbb{N}.$$

As f is α -orbital admissible, we have

$$\alpha(\rho_0, \rho_1) = \alpha(\rho_0, f\rho_0) \geq 1 \quad \text{implies} \quad \alpha(f\rho_0, f\rho_1) = \alpha(\rho_1, \rho_2) \geq 1.$$

Using mathematical induction, it follows that $\alpha(\rho_n, \rho_{n+1}) \geq 1$ for each $n \in \mathbb{N}$. Now, we divide the entire proof into four steps and discuss one by one.

Step 1. We first prove that for each $n \in \mathbb{N}$

$$q(\rho_n, \rho_{n+1}) < M_q(\rho_{n-1}, \rho_n).$$

Note that for every natural number n , we have $q(\rho_n, \rho_{n+1}) > 0$. Since, otherwise by the combination of $q(\rho_n, \rho_{n+1}) = 0$ and the assumption $q(\rho_n, \rho_n) = 0$ and applying Lemma 1.1 we get $\rho_n = \rho_{n+1}$, which is a contradiction. Therefore, we find that

$$M_q(\rho_{n-1}, \rho_n) = \max \left\{ q(\rho_{n-1}, \rho_n), q(\rho_{n-1}, \rho_n), q(\rho_n, \rho_{n+1}), \frac{q(\rho_{n-1}, \rho_{n+1}) + q(\rho_n, \rho_n)}{2} \right\} > 0.$$

Hence, we obtain $\psi(M_q(\rho_{n-1}, \rho_n)) > 0$. Now, from the hypothesis and Definition 2.1 for $\eta = \psi(M_q(\rho_{n-1}, \rho_n))$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$, when $\eta \leq \psi(M_q(\rho, \varrho)) < \eta + \delta$, we have $\alpha(\rho, \varrho)q(f\rho, f\varrho) < \eta$.

In particular, since for each $\tau > 0$, $\psi(\tau) < \tau$, we have

$$(2.1) \quad q(\rho_n, \rho_{n+1}) \leq \alpha(\rho_{n-1}, \rho_n)q(\rho_n, \rho_{n+1}) < \eta = \psi(M_q(\rho_{n-1}, \rho_n)) < M_q(\rho_{n-1}, \rho_n).$$

Since

$$\frac{q(\rho_{n-1}, \rho_{n+1})}{2} \leq \frac{q(\rho_{n-1}, \rho_n) + q(\rho_n, \rho_{n+1})}{2} \leq \max\{q(\rho_{n-1}, \rho_n), q(\rho_n, \rho_{n+1})\},$$

we have

$$\begin{aligned} M_q(\rho_{n-1}, \rho_n) &= \max \left\{ q(\rho_{n-1}, \rho_n), q(\rho_{n-1}, \rho_n), q(\rho_n, \rho_{n+1}), \frac{q(\rho_{n-1}, \rho_{n+1}) + q(\rho_n, \rho_n)}{2} \right\} \\ &= \max \left\{ q(\rho_{n-1}, \rho_n), q(\rho_n, \rho_{n+1}), \frac{q(\rho_{n-1}, \rho_{n+1})}{2} \right\} \\ &= \max\{q(\rho_{n-1}, \rho_n), q(\rho_n, \rho_{n+1})\}. \end{aligned}$$

So, $q(\rho_n, \rho_{n+1}) < M_q(\rho_{n-1}, \rho_n) = \max\{q(\rho_{n-1}, \rho_n), q(\rho_n, \rho_{n+1})\}$ and this implies that

$$M_q(\rho_{n-1}, \rho_n) = q(\rho_{n-1}, \rho_n) \quad \text{and} \quad q(\rho_n, \rho_{n+1}) < q(\rho_{n-1}, \rho_n).$$

Now, since $\{q(\rho_{n-1}, \rho_n)\}$ is decreasing and bounded below, it is convergent to $t \geq 0$ such that $q(\rho_n, \rho_{n+1}) \geq t$ for each n . Assume that $t \neq 0$ and $\xi = \lim_n \psi(q(\rho_n, \rho_{n+1}))$. Then by (ψ_3) , $0 < \xi < t$ and by Definition 2.1, we can find $\delta > 0$ satisfying

$$(2.2) \quad \text{when } \xi \leq \psi(M_q(\rho, \varrho)) < \xi + \delta, \quad \text{we have } \alpha(\rho, \varrho)q(f\rho, f\varrho) < \xi,$$

for $\rho, \varrho \in M$. Consider $k_0 \in \mathbb{N}$ such that $\frac{1}{k_0} < \delta$ and $\frac{1}{k_0} < \xi$. Then for each $k \geq k_0$ there is $\delta_k \leq \frac{1}{k}$ such that

$$(2.3) \quad \xi - \frac{1}{k} \leq \psi(M_q(\rho, \varrho)) < \xi - \frac{1}{k} + \delta_k \quad \text{implies} \quad \alpha(\rho, \varrho)p(f\rho, f\varrho) < \xi - \frac{1}{k} < \xi.$$

Also there is $k_2 \in \mathbb{N}$ such that for each $n \geq k_2$ one obtains

$$\xi - \frac{1}{k_0} < \psi(q(\rho_{n-1}, \rho_n)) = \psi(M_q(\rho_{n-1}, \rho_n)) < \xi + \frac{1}{k_0} < \xi + \delta.$$

Now, when $\xi \leq \psi(M_q(\rho_{n-1}, \rho_n)) \leq \xi + \frac{1}{k_0}$, by (2.2), we have

$$q(\rho_n, \rho_{n+1}) \leq \alpha(\rho_{n-1}, \rho_n)q(\rho_n, \rho_{n+1}) < \xi < t,$$

and when $\xi - \frac{1}{k_0} \leq \psi(M_q(\rho_{n-1}, \rho_n)) < \xi$ by (2.3) and since

$$[\xi - \frac{1}{k_0}, \xi] \subseteq \cup_{k \geq k_0} [\xi - \frac{1}{k}, \xi - \frac{1}{k} + \delta_k),$$

we have $q(\rho_n, \rho_{n+1}) \leq \alpha(\rho_{n-1}, \rho_n)q(\rho_n, \rho_{n+1}) < \xi < t$, which is a contradiction. Therefore, $t = 0$ and so

$$(2.4) \quad \lim_n q(\rho_n, \rho_{n+1}) = 0.$$

Step 2. We prove that $\{\rho_n\}$ is a Cauchy sequence. Alternatively, from the inequality (2.1), we arrive at

$$(2.5) \quad q(\rho_n, \rho_{n+1}) \leq \psi(q(\rho_{n-1}, \rho_n)), \quad \text{for all } n \in \mathbb{N}.$$

Indeed, if there exists some n^* such that

$$q(\rho_{n^*}, \rho_{n^*+1}) \leq \psi(q(\rho_{n^*}, \rho_{n^*+1})) < q(\rho_{n^*}, \rho_{n^*+1}),$$

we get a contradiction. Hence, (2.5) holds. Inductively, we derive, from (2.5), that

$$q(\rho_n, \rho_{n+1}) \leq \psi^n(q(\rho_0, \rho_1)), \quad \text{for all } n \in \mathbb{N}.$$

Fix ε and let $n_\varepsilon \in \mathbb{N}$ such that $\sum_{k \geq n_\varepsilon} \psi^k(q(\rho_1, \rho_0)) < \varepsilon$. Furthermore, for $m > n > n_\varepsilon$ we can find that

$$\begin{aligned} q(\rho_n, \rho_m) &\leq q(\rho_n, \rho_{n+1}) + \cdots + q(\rho_{m-1}, \rho_m) \\ &\leq \sum_{k=n}^{m-1} \psi^k(q(\rho_1, \rho_0)) \\ &\leq \sum_{k \geq n_\varepsilon} \psi^k(q(\rho_1, \rho_0)). \end{aligned}$$

Hence, we conclude that the sequence $\{\rho_n\}$ is Cauchy. Now, since (M, d) is complete, we can get $u \in M$ with $\rho_n \rightarrow u$ in M .

Step 3. u is a fixed point of f .

Case (i). For each $\varrho \in M$ satisfying $\varrho \neq f\varrho$, we have $\inf\{q(\rho, \varrho) + q(\rho, f\rho) : \rho \in M\} > 0$. It implies that for every $\varepsilon > 0$, there is a natural number N such that for

$n > N_\varepsilon$, we have $q(\rho_{N_\varepsilon}, \rho_n) < \varepsilon$. Since, $\rho_n \rightarrow u$ and $q(\rho, \cdot)$ is a lower semi-continuous map, we have

$$q(\rho_{N_\varepsilon}, u) \leq \liminf_{n \rightarrow \infty} q(\rho_{N_\varepsilon}, \rho_n) \leq \varepsilon.$$

Putting $\varepsilon = \frac{1}{k}$ and $N_\varepsilon = n_k$, we have

$$(2.6) \quad \lim_{k \rightarrow \infty} q(\rho_{n_k}, u) = 0.$$

Assume that $u \neq fu$. Then

$$0 < \inf\{q(\sigma, u) + q(\sigma, f\sigma) : \sigma \in M\} \leq \inf\{q(\rho_{n_k}, u) + q(\rho_{n_k}, \rho_{n_k+1}) : k \in \mathbb{N}\}.$$

From (2.4) and (2.6), we derive $\inf\{q(\sigma, u) + q(\sigma, f\sigma) : \sigma \in M\} = 0$, which contradicts the given hypothesis. Therefore, $fu = u$.

Case (ii). Let f be continuous.

Using the triangular inequality, we have

$$q(\rho_n, f^2\rho_n) \leq q(\rho_n, f\rho_n) + q(f\rho_n, f^2\rho_n).$$

Accordingly, letting $n \rightarrow \infty$, we obtain $q(\rho_n, f^2\rho_n) \rightarrow 0$. Further, Lemma 1.1 confirms that $\{f^2\rho_n\} \rightarrow u$ as $n \rightarrow \infty$. As f is continuous, we have

$$fu = f(\lim_{n \rightarrow \infty} f\rho_n) = \lim_{n \rightarrow \infty} f^2\rho_n = u.$$

Hence, u is a fixed point of f .

Case (iii). Here, $\lim_{n \rightarrow \infty} q(f\rho_n, u) = \lim_{n \rightarrow \infty} q(\rho_{n+1}, u) = \lim_{n \rightarrow \infty} q(\rho_n, u)$. Hence, $fu = u$.

Step 4. u is a fixed point with $q(u, u) = 0$.

Conversely, suppose that $q(u, u) > 0$. Then from (2.1), we get

$$0 < q(u, u) = q(fu, fu) \leq \psi(M_q(u, u)) < M_q(u, u) = q(u, u),$$

and this is impossible. Hence, our claim is verified. □

The fixed point obtained in the previous theorem may be not unique. The following examples validate our claim.

Example 2.1. Suppose that G is a locally compact group, $M = L^1(G)$ and

$$q(f, g) = \|g\|_1, \quad f, g \in L^1(G).$$

Then q is a w -distance. Define $\psi(t) = \begin{cases} \frac{t}{2}, & t \in [0, 1], \\ \frac{1}{2}, & t \in (1, \infty), \end{cases}$ and

$$\alpha(f, g) = \begin{cases} 2, & g = 0 \quad (a.e.), \\ \frac{\psi(M_q(f, g))}{2\|g\|_1}, & \text{otherwise,} \end{cases}$$

and for an arbitrary $x \in G$

$$\begin{aligned} T_x : L^1(G) &\rightarrow L^1(G), \\ f &\mapsto \frac{1}{8}L_x f, \end{aligned}$$

where $L_x f(y) = f(x^{-1}y)$. Then for each $f \in L^1(G)$ and $x \in G$, since $\|L_x f\|_1 = \|f\|_1$, we conclude that $M_q(f, g) = \max\{\frac{1}{8}\|f\|_1, \|g\|_1\}$ and so

$$\alpha(f, g) = \frac{\psi(M_q(f, g))}{2\|g\|_1} = \frac{\psi(\max\{\frac{1}{8}\|f\|_1, \|g\|_1\})}{2\|g\|_1} \geq 1.$$

In each of the cases $0 \leq \max\{\frac{1}{8}\|f\|_1, \|g\|_1\} \leq 1$, $1 \leq \max\{\frac{1}{8}\|f\|_1, \|g\|_1\} \leq 8$ and $8 \leq \max\{\frac{1}{8}\|f\|_1, \|g\|_1\}$ we conclude that

$$\alpha(T_x f, T_x g) = \frac{\psi(\max\{\frac{1}{64}\|f\|_1, \frac{1}{8}\|g\|_1\})}{\frac{2}{8}\|g\|_1} > 1.$$

So, T_x is α -orbital admissible. Now for each $\eta > 0$ and $\delta = \eta$, if $\eta \leq \psi(M_q(f, g)) < 2\eta$, then for $g \neq 0$ we have

$$\alpha(f, g)q\left(\frac{1}{8}L_x f, \frac{1}{8}L_x g\right) \leq \frac{\psi(M_q(f, g))}{2\|g\|_1} \left(\frac{1}{8}\right)\|g\|_1 \leq \frac{1}{8}\eta < \eta,$$

and for $g = 0$, since $q(T_x f, T_x g) = 0$, we are done. So, T_x is a generalized (α, ψ, q) -Meir-keeler contractive map. Moreover, $\alpha(0, T_x 0) = \alpha(0, 0) = 2 > 1$,

$$q(T_x^n 0, T_x^n 0) = q(0, 0) = \|0\|_1 = 0,$$

and T_x is continuous. Therefore, all the hypotheses of Theorem 2.1 hold and so, T_x has a fixed point (which is $f = 0$, satisfying $q(0, 0) = 0$). Note that for each $f \in L^1(G)$, we have

$$\lim \|T_x^n f\|_1 = \lim \frac{1}{8^n} \|f\|_1 = 0.$$

Therefore, $T_x^n f$ converges to 0 and so 0 is the only fixed point of T_x .

Example 2.2. Suppose that $M = \{\frac{1}{2^n} : n \in \mathbb{N}\} \cup \{0\}$ is equipped with the usual metric on \mathbb{R} . Consider

$$q(\rho, \varrho) = \begin{cases} \frac{1}{n} + \frac{1}{m}, & \varrho = \frac{1}{2^m}, \rho = \frac{1}{2^n}, \\ 0, & \rho = 0 \text{ or } \varrho = 0, \end{cases}$$

$$\alpha(\rho, \varrho) = \begin{cases} \frac{m}{n}, & \varrho = \frac{1}{2^m}, \rho = \frac{1}{2^n} \text{ and } 2n \geq m \geq n, \\ 1, & \rho = 0 \text{ or } \varrho = 0, \\ \frac{1}{n}, & \text{otherwise,} \end{cases}$$

and $f\rho = \rho^8$. Then $\alpha(0, f0) = 1$, $q(f^n 0, f^n 0) = 0$ for each $n \in \mathbb{N}$ and f is continuous and also α -orbital admissible. Since if $\alpha(\rho, f\rho) \geq 1$, then $\rho = 0$, since if $\rho = \frac{1}{2^n}$ for some n , then $n \leq 8n \leq 2n$ is impossible. Therefore, $\alpha(f\rho, f^2\rho) \geq 1$. Also if

$$\psi(t) = \begin{cases} \frac{t}{2}, & t \in [0, 1], \\ \frac{1}{2}, & t \in (1, \infty), \end{cases}$$

then, since $0 \leq M_q(\rho, \varrho) \leq 1$, we have $\psi(M_q(\rho, \varrho)) = \frac{1}{2}M_q(\rho, \varrho)$. On the other hand, for each $\eta > 0$ and for $\delta = \eta$, if $\eta \leq \psi(M_q(\rho, \varrho)) < 2\eta$, then ρ or ϱ is non-zero. So if $\rho = \frac{1}{2^n}$ and $\varrho = \frac{1}{2^m}$, then since $\alpha(\rho, \varrho) \leq 2$, we conclude that

$$\alpha(\rho, \varrho)q(f\rho, f\varrho) \leq 2 \left(\frac{1}{8n} + \frac{1}{8m} \right) = \frac{1}{4}q(\rho, \varrho) \leq \frac{1}{4}M_p(\rho, \varrho) = \frac{1}{2}\psi(M_p(\rho, \varrho)) < \eta.$$

Also, if one of ρ or ϱ is zero, then $\alpha(\rho, \varrho)q(f\rho, f\varrho) = 0 \leq \eta$. So, f is a generalized (α, ψ, q) -Meir-Keeler contractive map. Therefore, all the conditions of Theorem 2.1 hold. Hence, $\rho = 0$ is the unique fixed point of f .

Example 2.3. Let $M = [0, 1]$ be equipped with the usual metric. Also let us consider the w -distance as $q(\rho, \varrho) = |\rho - \varrho|$ for each $\rho, \varrho \in M$. Further, we define

$$f\rho = \begin{cases} \frac{\rho}{20}, & \rho \in [0, 1), \\ 1, & \rho = 1, \end{cases} \quad \alpha(\rho, \varrho) = \begin{cases} 1, & \rho, \varrho \in [0, 1), \\ 0, & \rho = 1, \end{cases} \quad \psi(\rho) = \begin{cases} 0, & \rho = 0, \\ \frac{1}{3}, & \rho \in \left(0, \frac{1}{2}\right), \\ \frac{\rho}{2}, & \rho \in \left[\frac{1}{2}, 1\right], \\ \frac{1}{2}, & \rho \in (1, \infty). \end{cases}$$

Hence, for every $w \in M$ with $fw \neq w$, we obtain $w \neq 0, 1$ and so

$$\lim_{\rho \rightarrow w} (|w - \rho| + |\rho - f\rho|) \geq \frac{19}{20}w > 0.$$

Again,

$$\lim_{\rho \rightarrow \varrho} (|w - \rho| + |\rho - f\rho|) \geq |w - \varrho| > 0, \quad \varrho \neq w.$$

Therefore, we have $\inf\{q(\rho, w) + q(\rho, f\rho) : \rho \in M\} > 0$ for each $w \in M$ satisfying $w \neq fw$. Besides, for every $\rho \in M$, we obtain $|f^n\rho - f\rho| = 0$. Now for each $\eta > 0$, put $\delta = \eta$. Then, $\rho = \varrho$ implies $M_q(\rho, \varrho) = 0$ and when $\rho \neq \varrho$, $M_q(\rho, \varrho) \neq 0$ and further, $\psi(M_q(\rho, \varrho)) \geq \frac{1}{4}$. Therefore, for $\eta > \frac{1}{8}$, there is no $\rho, \varrho \in M$ satisfying

$$\frac{1}{8} \leq \psi(M_q(\rho, \varrho)) < \frac{1}{4}.$$

On the other hand, for $\eta \leq \frac{1}{8}$, if $\eta \leq \psi(M_q(\rho, \varrho)) < \eta + \eta = 2\eta$, we have

$$\alpha(\rho, \varrho)|f\rho - f\varrho| \leq |f\rho - f\varrho| = \left| \frac{\rho}{20} - \frac{\varrho}{20} \right| \leq \frac{2}{20} < \frac{1}{8} < \eta.$$

That is for each ρ, ϱ , if $\eta \leq \psi(M_q(\rho, \varrho)) < \eta + \eta = 2\eta$, then $\alpha(\rho, \varrho)|f\rho - f\varrho| \leq \eta$. Note that 0, 1 are the fixed points of f .

Remark 2.1. In the case where $q(\rho, \varrho) = \varrho$ for each $\rho, \varrho \in M$, the assumption $q(f^n\rho, f^n\rho) = 0$, for some $\rho \in M$ and for each $n \in \mathbb{N}$, imply that $f^n\rho = 0$ for each n . Therefore, in this case without any another condition, since $\rho_n = 0 = \rho_{n+1}$,

the first part of the Theorem 2.1 implies that f possesses a fixed point. For example, let $M = \{\frac{1}{2^n} : n \in \mathbb{N}\} \cup \{0\}$,

$$\alpha(\rho, \varrho) = \begin{cases} 0, & \varrho \in \left\{ \frac{1}{2^{2k}} : k \in \mathbb{N} \right\}, \\ 1, & \text{otherwise,} \end{cases} \quad \text{and} \quad f\rho = \begin{cases} \frac{\rho}{2}, & \rho \in \left\{ \frac{1}{2^{2k}} : k \in \mathbb{N} \right\}, \\ 1, & \text{otherwise.} \end{cases}$$

Then f is continuous, $q(f^n 0, f^n 0) = 0$ for each $n \in \mathbb{N}$ and $\rho, \varrho \in M$ and $\eta, \delta > 0$, if $\eta \leq \psi(M_q(\rho, \varrho)) < \eta + \delta$, then we have

$$0 = \alpha(\rho, \varrho)q(f\rho, f\varrho) \leq \eta.$$

Note that 0 is a fixed point of f , since here we require only $q(f^n 0, f^n 0) = 0$.

Now we put down the following additional hypothesis. To attest the uniqueness of the fixed point of f , this condition along with those of Theorem 2.1 is required.

Property U. Let $\alpha(u, v) < 1$, implies that at least one of u or v is not a fixed point of f .

For example if $\alpha(u, v) \geq 1$ for each $u, v \in M$, then the property U is valid.

Theorem 2.2. *Suppose that (M, d) is a metric space with a w -distance q . Also assume that f is a generalized (α, ψ, p) -Meir-Keeler contractive mapping and satisfies all the hypotheses of Theorem 2.1 along with the additional property U . Then we can claim the uniqueness of the fixed point of f obtained in Theorem 2.1.*

Proof. We suppose that $u, v \in M$ are two distinct fixed points of f . Then $\alpha(u, v) \geq 1$, $fu = u$, $fv = v$, $q(u, u) = 0$ and $q(v, v) = 0$. Using the aforementioned criteria and (2.1), we obtain

$$q(u, v) = q(fu, fv) \leq \alpha(u, v)q(fu, fv) \leq \psi(M_q(u, v)) = \psi(q(u, v)) < q(u, v),$$

and this is impossible. Hence, f possesses a unique fixed point. \square

3. CONSEQUENCES

This section deals with a few immediate corollaries of our obtained Theorem 2.1. First, we give the following important result for an (α, ψ, q) -Meir-Keeler contractive mapping.

Corollary 3.1. *Suppose that (M, d) is a complete metric space with a w -distance q . Also let f be an (α, ψ, q) -Meir-Keeler contractive mapping with the fact that there is some $\rho_0 \in M$, with $q(f^n \rho_0, f^n \rho_0) = 0$ for all non-negative integers n and $\alpha(\rho_0, f\rho_0) \geq 1$. Suppose that one of the following holds.*

(i) *For each $w \in M$ satisfying $w \neq fw$, we have $\inf\{q(\rho, w) + q(\rho, f\rho) : \rho \in M\} > 0$.*

(ii) *f is continuous.*

(iii) *If for some sequence $\{\rho_n\}$, $\lim_{n \rightarrow \infty} q(\rho_n, \rho) = \lim_{n \rightarrow \infty} q(f\rho_n, \rho)$, then $f\rho = \rho$.*

Then f possesses a fixed point $u \in M$, with $q(u, u) = 0$.

Putting $\alpha \equiv 1$ in Theorem 2.1, we obtain the trailing important corollary.

Corollary 3.2. *Suppose that (M, d) is a complete metric space with a w -distance q . Also let f be a (ψ, q) -Meir-Keeler contractive mapping with the fact that there is some $\rho_0 \in M$, with $q(f^n \rho_0, f^n \rho_0) = 0$ for all non-negative integers n . Suppose that one of the following conditions holds.*

- (i) *For each $w \in M$ satisfying $w \neq fw$, we have $\inf\{q(\rho, w) + q(\rho, f\rho) : \rho \in M\} > 0$.*
- (ii) *f is continuous.*
- (iii) *If for some sequence $\{\rho_n\}$, $\lim_{n \rightarrow \infty} q(\rho_n, \rho) = \lim_{n \rightarrow \infty} q(f\rho_n, \rho)$, then $f\rho = \rho$.*

Then f possesses a fixed point $u \in M$.

Considering $q = d$ in Theorem 2.1, we deduce the subsequent corollary.

Corollary 3.3. *Suppose that (M, d) is a complete metric space and f be an (α, ψ) -Meir-Keeler contractive mapping with the fact that there is some $\rho_0 \in M$ with $\alpha(\rho_0, f\rho_0) \geq 1$ or $\alpha(f\rho_0, \rho_0) \geq 1$. Suppose that one of the following conditions holds.*

- (i) *For each $w \in M$ satisfying $w \neq fw$, we have $\inf\{d(\rho, w) + d(\rho, f\rho) : \rho \in M\} > 0$.*
- (ii) *f is continuous.*
- (iii) *For some sequence $\{\rho_n\}$ in M with $\alpha(\rho_n, \rho_{n+1}) \geq 1$ for all natural numbers n and $\rho_n \rightarrow \rho \in M$ as $n \rightarrow \infty$, then $\alpha(\rho_n, \rho) \geq 1$ for every $n \in \mathbb{N}$.*

Then f possesses a fixed point $u \in M$.

Taking $\alpha \equiv 1$ in Corollary 3.3, we get the succeeding consequence.

Corollary 3.4. *Suppose that (M, d) is a complete metric space and f be a ψ -Meir-Keeler contractive mapping. Suppose that either f is continuous or $\inf\{d(\rho, w) + d(\rho, f\rho) : \rho \in M\} > 0$ for each $w \in M$ with $w \neq fw$. Then f possesses a fixed point $u \in M$.*

Definition 3.1. Suppose that (M, d) is a metric space with a w -distance q and consider the functions $\psi \in \Psi$, $\alpha : M \times M \rightarrow [0, \infty)$ and a self-map f . Then f is said to be a generalized (α, ψ, q) -Meir-Keeler contractive mapping of

- (a) Banach type if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$
when $\eta \leq \psi(q(\rho, \varrho)) < \eta + \delta$, we have $\alpha(\rho, \varrho)p(f\rho, f\varrho) < \eta$;
- (b) Kannan type I if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$
when $\eta \leq \psi\left(\frac{q(\rho, f\rho) + q(\varrho, f\varrho)}{2}\right) < \eta + \delta$, we have $\alpha(\rho, \varrho)p(f\rho, f\varrho) < \eta$;
- (c) Kannan type II if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$
when $\eta \leq \psi(\max\{q(\rho, f\rho), q(\varrho, f\varrho)\}) < \eta + \delta$, we have $\alpha(\rho, \varrho)p(f\rho, f\varrho) < \eta$;

- (d) Chatterjea type I if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$ when $\eta \leq \psi\left(\frac{q(\rho, f\varrho) + q(\varrho, f\rho)}{2}\right) < \eta + \delta$, we have $\alpha(\rho, \varrho)p(f\rho, f\varrho) < \eta$;
- (e) Chatterjea type II if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$ when $\eta \leq \psi(\max\{q(\rho, f\varrho), q(\varrho, f\rho)\}) < \eta + \delta$, we have $\alpha(\rho, \varrho)p(f\rho, f\varrho) < \eta$;
- (f) Reich type I if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$ when $\eta \leq \psi\left(\frac{q(\rho, \varrho) + q(\rho, f\rho) + q(\varrho, f\varrho)}{3}\right) < \eta + \delta$, we have $\alpha(\rho, \varrho)p(f\rho, f\varrho) < \eta$;
- (g) Reich type II if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$ when $\eta \leq \psi(\max\{q(\rho, \varrho), q(\rho, f\rho), q(\varrho, f\varrho)\}) < \eta + \delta$, we have $\alpha(\rho, \varrho)p(f\rho, f\varrho) < \eta$;
- (h) Reich type III if for every $\eta > 0$, there exists a $\delta > 0$ such that for $\rho, \varrho \in M$ when $\eta \leq \psi(\max\{q(\rho, \varrho), q(\rho, f\varrho), q(\varrho, f\rho)\}) < \eta + \delta$, we have $\alpha(\rho, \varrho)p(f\rho, f\varrho) < \eta$.

In addition, for taking $q = d$ in the inequalities above, we can get several other kind of contractions in the context of metric spaces.

If in Theorem 2.1, we change the contraction condition ‘generalized (α, ψ, q) -Meir-Keeler contractive mapping’ with one of the new contractions defined in Definition 3.1, then we may obtain a similar result as Theorem 2.1. Furthermore, as in Corollary 3.3 and Corollary 3.4, we may get some more results by letting $q = d$. Also, notice that by choosing the auxiliary function α in a proper way in Theorem 2.1, we can deduce more consequences related to cyclic contractions and results in metric spaces endowed with a partially ordered set, see for example [1–8].

4. AN APPLICATION

In this section, we discuss an application of our obtained fixed point result to a certain kind of non-linear Fredholm integral equations. First of all, we prove a proposition which is going to play a crucial role here.

Proposition 4.1. *Suppose that (M, d) is a metric space with a w -distance q . Also, assume that f is a self-mapping on M satisfying*

$$(4.1) \quad \alpha(\rho, \varrho)q(f\rho, f\varrho) \leq k\psi(M_q(\rho, \varrho)),$$

for all $\rho, \varrho \in M$ and for some $k \in (0, 1)$. Then f is a generalized (α, ψ, q) -Meir-Keeler contractive mapping.

Proof. Consider $\delta = (\frac{1}{k} - 1)\eta$ in Definition 2.1. Accordingly, we derive

$$\eta \leq \psi(M_q(\rho, \varrho)) < \eta + \delta < \eta + \left(\frac{1}{k} - 1\right)\eta = \frac{\eta}{k},$$

and so, for every $\rho, \varrho \in M$, we obtain $k\eta \leq k\psi(M_q(\rho, \varrho)) < \eta$. Using (4.1), we get

$$\alpha(\rho, \varrho)q(f\rho, f\varrho) \leq k\psi(M_q(\rho, \varrho)) < \eta.$$

Hence, $\alpha(\rho, \varrho)q(f\rho, f\varrho) < \eta$ and therefore, f is an (α, ψ, q) -Meir-Keeler contractive mapping. \square

Now, we try to obtain a criterion to ensure the existence of a solution for a type of non-linear Fredholm integral equation.

Theorem 4.1. *Let us consider the non-linear Fredholm integral equation*

$$(4.2) \quad (fx)(t) = g(t) + \int_a^b H(t, s, x(s))ds,$$

for some $a, b \in \mathbb{R}$, with $a < b$, $g : [a, b] \rightarrow \mathbb{R}$ and $H : [a, b]^2 \times \mathbb{R} \rightarrow \mathbb{R}$ be two continuous maps. Also, assume that the subsequent properties hold:

- (i) $f : C[a, b] \rightarrow C[a, b]$ is a continuous mapping;
- (ii) there exists a weaker Meir-Keeler function ψ and $k \in [0, 1)$ satisfying

$$\begin{aligned} & |H(t, s, x(s))| + |H(t, s, y(s))| \\ & \leq \frac{k \left[\psi \left(\max \left\{ |x(t)| + |y(t)|, |x(t)| + |(fx)(t)|, |y(t)| + |(fy)(t)|, \right. \right. \right. \\ & \quad \left. \left. \left. \frac{(|x(t)| + |(fy)(t)|) + (|(fx)(t)| + |y(t)|)}{2} \right\} \right) \right] - 2|g(t)|}{b - a}, \end{aligned}$$

for all $t, s \in [a, b]$. Then the non-linear Fredholm integral equation (4.2) owns a unique solution in $C[a, b]$.

Proof. Suppose $M = C[a, b]$. Obviously, M is complete with respect to the metric $d : M \times M \rightarrow \mathbb{R}^+$ defined as

$$d(x, y) = \sup_{t \in [a, b]} |x(t) - y(t)|,$$

where $x, y \in M$. Now, we consider the map $q : M \times M \rightarrow \mathbb{R}^+$ given by

$$q(x, y) = \sup_{t \in [a, b]} |x(t)| + \sup_{t \in [a, b]} |y(t)|,$$

where $x, y \in M$. One can easily check that, q is a w -distance on M . Here we have

$$\begin{aligned} & |(fx)(t)| + |(fy)(t)| \\ & = \left| g(t) + \int_a^b H(t, s, x(s))ds \right| + \left| g(t) + \int_a^b H(t, s, y(s))ds \right| \\ & \leq |g(t)| + \left| \int_a^b H(t, s, x(s))ds \right| + |g(t)| + \left| \int_a^b H(t, s, y(s))ds \right| \\ & \leq 2|g(t)| + \left| \int_a^b H(t, s, x(s))ds \right| + \left| \int_a^b H(t, s, y(s))ds \right| \end{aligned}$$

$$\begin{aligned}
&\leq 2|g(t)| + \int_a^b |H(t, s, x(s))| ds + \int_a^b |H(t, s, y(s))| ds \\
&\leq 2|g(t)| + \int_a^b (|H(t, s, x(s))| + |H(t, s, y(s))|) ds \\
&\leq 2|g(t)| + \int_a^b \left(\frac{k \left[\psi \left(\max \left\{ |x(t)| + |y(t)|, |x(t)| + |(fx)(t)|, |y(t)| + |(fy)(t)|, \right. \right. \right. \right. \\
&\quad \left. \left. \left. \frac{(|x(t)| + |(fy)(t)) + (|(fx)(t)| + |y(t)|)}{2} \right\} \right) \right] - 2|g(t)|}{b-a} \right) ds \\
&= 2|g(t)| + \frac{k \left[\psi \left(\max \left\{ |x(t)| + |y(t)|, |x(t)| + |(fx)(t)|, |y(t)| + |(fy)(t)|, \right. \right. \right. \right. \\
&\quad \left. \left. \left. \frac{(|x(t)| + |(fy)(t)) + (|(fx)(t)| + |y(t)|)}{2} \right\} \right) \right] - 2|g(t)|}{b-a} \int_a^b ds \\
&= k \left[\psi \left(\max \left\{ |x(t)| + |y(t)|, |x(t)| + |(fx)(t)|, |y(t)| + |(fy)(t)|, \right. \right. \right. \\
&\quad \left. \left. \left. \frac{(|x(t)| + |(fy)(t)) + (|(fx)(t)| + |y(t)|)}{2} \right\} \right) \right] \\
&\leq k \left[\psi \left(\max \left\{ q(x, y), q(x, fx), q(y, fy), \frac{q(x, fy) + q(y, fx)}{2} \right\} \right) \right] \\
&= k [\psi (M_q(x, y))],
\end{aligned}$$

for all $x, y \in M$ and $t \in [0, \infty]$. Thus,

$$\sup_{t \in [a, b]} |(fx)(t)| + \sup_{t \in [a, b]} |(Ty)(t)| \leq k [\psi (M_q(x, y))],$$

and therefore for each $x, y \in M$

$$q(fx, fy) \leq k [\psi (M_q(x, y))].$$

This implies that f satisfies Proposition 4.1 and hence it is an (α, ψ, q) -Meir-Keeler contractive mapping. Therefore, by Theorem 2.1, the non-linear Fredholm integral equation (4.2) has a solution. \square

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