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LIMIT POINT ANALYSIS OF THE BROWDER SPECTRUM FOR OPERATOR MATRICES

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Abstract. In this paper, we investigate the limit point set of the Browder spectrum for upper triangular operator matrices $\mathcal{M}$ on Banach spaces. Utilizing the robust tools and comprehensive framework of local spectral theory, we offer a detailed analysis of this spectral feature. We establish that the relationship between the accumulation points of the Browder spectrum $\sigma_{Br}(\mathcal{M})$ of $\mathcal{M}$ and those of its diagonal entries is encapsulated by the equation:

$$\operatorname{Acc}\sigma_{Br}(\mathcal{M}) \cup \mathcal{W}_{\operatorname{Acc}\sigma_{Br}} = \bigcup_{i=1}^{3} \operatorname{Acc}\sigma_{Br}(A_i),$$

where $\mathcal{W}_{\operatorname{Acc}\sigma_{Br}}$ denotes a specific union of ”holes“ in $\operatorname{Acc}\sigma_{Br}(\mathcal{M})$, comprising subsets within the intersection $\bigcap_{i=1}^{3} \operatorname{Acc}\sigma_{Br}(A_i)$. Furthermore, we delineate sufficient conditions under which the limit points of the Browder spectrum for a 3×3 upper triangular block operator matrix are precisely characterized as the union of the limit points of the spectra of its diagonal entries. Our findings significantly advance the understanding of the spectral properties of operator matrices, offering crucial insights into their structure within the context of local spectral theory. Moreover, this work extends and refines the results of A. Tajmouati et al., as presented in [17], contributing to the ongoing development and enhancement of operator matrix theory.

1. Introduction

Browder operators and their associated derivative sets serve as generalizations of Fredholm and semi-Fredholm operators, as well as their derivative sets, within

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the framework of linear operators. This area has been extensively explored in the literature, leading to numerous significant advancements. For a deeper understanding, readers are encouraged to consult the references [4, 9].

In recent years, operator matrix theory has gained substantial attention in both pure and applied mathematics. As a result, numerous researchers have focused on the spectral properties of 2×2 upper triangular block operator matrices, including works by [1–7, 10, 17, 18]. This area of study is closely tied to the exploration of Browder operators and the development of local spectral theory.

In the field of mathematical physics, many linear evolution equations can be reformulated as a Cauchy problem governed by 3×3 block operator matrices. For example, such reformulations are common in equations arising from linear thermo-viscoelasticity [14] and fluid dynamics [15]. Our motivation for this paper is to investigate why the results presented by A. Tajmouati et al. [17] remain valid for 3×3 upper triangular operator matrices derived from systems of linear evolution equations.

Accordingly, after considering on the direct sum of Banach spaces, the following bounded 3×3 upper triangular operator matrix

$$\mathcal{M} := \begin{pmatrix} A_1 & B_1 & C_1 \\ 0 & A_2 & B_2 \\ 0 & 0 & A_3 \end{pmatrix}.$$

This raises an important question.

Under what conditions on the entries of the operator matrices $\mathcal{M}$ can we obtain a relation between the limit point set for the Browder spectrum of $\mathcal{M}$ and the limit point set for the Browder spectrum of the operator entries A_i, $1 \le i \le 3$, in the form:

$$\operatorname{Acc} \sigma_{Br}(\mathcal{M}) \subseteq \bigcup_{i=1}^{3} \operatorname{Acc} \sigma_{Br}(A_i)?$$

To address this question, we propose leveraging the results of A. Tajmouati et al. in [17] after reducing the matrix form $\mathcal{M}$ into the the form of block 2×2 operator matrix

$$\mathcal{M} := \begin{pmatrix} \mathcal{X} & \mathcal{Y} \\ \mathcal{Z} & A_3 \end{pmatrix},$$

where $\mathcal{X} := \begin{pmatrix} A_1 & B_1 \\ 0 & A_2 \end{pmatrix}$, $\mathcal{Y} := \begin{pmatrix} C_1 \\ B_2 \end{pmatrix}$ and $\mathcal{Z} := \begin{pmatrix} 0 & 0 \end{pmatrix}$.

While this approach might appear somewhat contrived and limited in scope, it is critical to refine our analysis to achieve more comprehensive and significant results. To achieve the best of this paper, our main objective in this work is to extend the results obtained by A. Tajmouati et al. in [17] to the case of the 3×3 block operator matrix. Specifically, we found some sufficient conditions on the entries of the operator matrices $\mathcal{M}$ to assure our interest and to prove an improvement in the theory of operators matrices and a continuation and an amelioration of the results of A. Tajmouati et al. in [17].

The paper's remaining sections are arranged as follows. In Section 2, we collect certain notes and required terminology throughout the paper and we remember some classic definitions introducing Browder's operator theory. The purpose of the work presented in Section 3 is dedicated to investigate the limit point set for Browder spectrum of a bounded 3×3 upper triangular operator matrix $\mathcal{M}$. After checking that the assertion $\operatorname{Acc} \sigma_{Br}(\mathcal{M}) \subseteq \bigcup_{i=1}^{3} \operatorname{Acc} \sigma_{Br}(A_i)$ is always true, we investigate the local spectral theory to prove that

$$\operatorname{Acc} \sigma_{Br}(\mathcal{M}) \cup \Big[\big[\mathrm{S_{vep}}(A_1^*) \cup \mathrm{S_{vep}}(A_2^*)\big] \cap \mathrm{S_{vep}}(A_3)\Big] = \bigcup_{i=1}^{3} \operatorname{Acc} \sigma_{Br}(A_i)$$

for all bounded linear operators B_1, B_2 and C_1. In addition, we impose sufficient conditions on A_i, for $i = \{1, 2, 3\}$ under which the following equality $\operatorname{Acc} \sigma_{Br}(\mathcal{M}) = \bigcup_{i=1}^{3} \operatorname{Acc} \sigma_{Br}(A_i)$ is guaranteed. Furthermore, we prove that the passage from $\operatorname{Acc} \sigma_{Br}(\mathcal{M})$ to $\bigcup_{i=1}^{3} \operatorname{Acc} \sigma_{Br}(A_i)$ can be presented as well:

$$\operatorname{Acc} \sigma_{Br}(\mathcal{M}) \cup \mathcal{W}_{\operatorname{Acc} \sigma_{Br}} = \bigcup_{i=1}^{3} \operatorname{Acc} \sigma_{Br}(A_i),$$

where $\mathcal{W}_{\operatorname{Acc} \sigma_{Br}}$ is the union of certain holes in $\operatorname{Acc} \sigma_{Br}(\mathcal{M})$, that occur to be subsets of $\bigcap_{i=1}^{3} \operatorname{Acc} \sigma_{Br}(A_i)$. Finally, we give sufficient conditions to reassure our desired equality.

2. Preliminaries

In this section, we revisit several well-known definitions and mathematical tools that are essential for the subsequent discussions. Throughout this paper, E denotes an infinite-dimensional complex Banach space, and $T \in \mathcal{L}(E)$, where $\mathcal{L}(E)$ represents the set of bounded linear operators on E. We will use the following notations and concepts:

- T^* : the adjoint of T;
- $\mathcal{N}(T)$: the null space of T;
- $\mathcal{R}(T)$: the range of T;
- $\alpha(T) = \dim \mathcal{N}(T)$: the dimension of the null space of T;
- $\beta(T = \operatorname{codim} \mathcal{R}(T))$: the codimension of the range of T;
- $\sigma(T)$: the spectrum of T.

Let us consider Υ a given subset of $\mathbb{C}$. The following notations will be clarified:

- $\operatorname{Acc} \Upsilon$: the set of all points of accumulation of Υ;
- $\operatorname{Iso}\Upsilon$: the set of all isolated points of Υ;
- Υ^c : the complement of Υ;
- $\partial\Upsilon$: the boundary of Υ;
- $\overline{\Upsilon}$: the closure of Υ;
- $\eta\Upsilon$: the polynomially convex hull of Υ.

Before passing to introduce the notion of Browder operators and their derivative sets, the local spectral theory of linear operators on a Banach space is discussed in the following basic results, which will be utilized in the rest of this work. For additional details, we may refer to [12, 13].

Definition 2.1. Let $T \in \mathcal{L}(E)$.

T is said to have the single-valued extension property (SVEP for short) at $\tau \in \mathbb{C}$ if for every open neighborhood $\mathfrak{F}_\tau$ of τ, the constant function $g \equiv 0$ is the only analytic solution of the equation:

$$(\lambda I - T)g(\lambda) = 0, \quad \text{for all } \lambda \in \mathfrak{F}_\tau.$$

We denote by $\mathrm{S_{vep}}(T)$, the open set of $\tau \in \mathbb{C}$ where T fails to have SVEP at τ.

Remark 2.1. We would like to note also that T has SVEP if $\mathrm{S_{vep}}(T) = \emptyset$.

Now, we introduce an important class of operators which involves the concept of Fredholm theory.

Definition 2.2. Let E be a Banach space, we define the following.

(i) The set of upper semi-Fredholm operators on E as:

$$\Phi_+(E) = \{T \in \mathcal{L}(E) : \alpha(T) < +\infty \text{ and } \mathcal{R}(T) \text{ is closed in } E\}.$$

(ii) The set lower semi-Fredholm operators on E as:

$$\Phi_-(E) = \{T \in \mathcal{L}(E) : \beta(T) < +\infty \text{ and } \mathcal{R}(T) \text{ is closed in } E\}.$$

Accordingly to Definition 2.2, the set of Fredholm operators on E is defined by:

$$\begin{aligned}\Phi(E) :=& \Phi_+(E) \cap \Phi_-(E)\\ :=& \{T \in \mathcal{L}(E) : \beta(T) < +\infty,\ \mathcal{R}(T) \text{ is closed in } E \text{ and } \beta(T) < +\infty\}.\end{aligned}$$

Before moving to introduce some sets of Browder operators, the following quantities will be omitted for $T \in \mathcal{L}(E)$:

- the ascent of T, denoted by $a(T)$ and defined as:

$$a(T) := \inf\{q \in \mathbb{N} : \mathcal{N}(T^q) = \mathcal{N}(T^{q+1})\};$$

- the descent of T, denoted by $d(T)$ and defined as:

$$d(T) := \inf\{q \in \mathbb{N} : \mathcal{R}(T^q) = \mathcal{R}(T^{q+1})\}.$$

Before proceeding further, we introduce the following definition.

Definition 2.3. **(i)** Set of upper semi-Browder operators on E is defined as:

$$Br_+(E) := \{T \in \mathcal{L}(E) : T \in \Phi_+(E) : \ a(T) < +\infty\}.$$

(ii) Set of lower semi-Browder operators on E is defined as:

$$Br_-(E) := \{T \in \mathcal{L}(E) : T \in \Phi_-(E) : \ d(T) < +\infty\}.$$

(iii) Set of Browder operators on E is defined by:

$$\begin{aligned}Br(E) :=& Br_-(E) \cap Br_+(E)\\ :=& \{T \in \mathcal{L}(E) : T \in \Phi(E), a(T) < +\infty \text{ and } d(T) < +\infty\}.\end{aligned}$$

Continuing in this context, we introduce the definitions of the upper semi-Browder spectrum, the lower semi-Browder spectrum, and the Browder spectrum.

Definition 2.4. Let $T \in \mathcal{L}(E)$. Then, we define the following spectra:

$$\sigma_{Br^+}(T) := \{\tau \in \mathbb{C} : \tau I - T \notin Br_+(E)\} \text{ – upper semi-Browder spectrum of } T,$$
$$\sigma_{Br^-}(T) := \{\tau \in \mathbb{C} : \tau I - T \notin Br_-(E)\} \text{ – lower semi-Browder spectrum of } T,$$
$$\sigma_{Br}(T) := \{\tau \in \mathbb{C} : \tau I - T \notin Br(E)\} \text{ – Browder spectrum of } T.$$

Remark 2.2. **(i)** It is important to point out that $\sigma_{Br^+}(T) = \sigma_{Br^-}(T^*)$ and $\sigma_{Br^-}(T) = \sigma_{Br^+}(T^*)$, where $T^* \in \mathcal{L}(E^*)$ and $\sigma_{Br}(T) = \sigma_{Br^+}(T) \cup \sigma_{Br^-}(T)$. This proves that $\sigma_{Br}(T) = \sigma_{Br}(T^*)$.

(ii) On the basis of Theorem 3.52 in [4], we have:

$$\sigma_{Br}(T) = \sigma_{Br^+}(T) \cup \mathrm{S}_{\mathrm{vep}}(T^*) = \sigma_{Br^-}(T) \cup \mathrm{S}_{\mathrm{vep}}(T).$$

At the moment, we introduce the following upper triangular block 3×3 operator matrix defined on the direct sum of Banach spaces $E \oplus F \oplus G$ as well:

$$\mathcal{M} := \begin{pmatrix} A_1 & B_1 & C_1 \\ 0 & A_2 & B_2 \\ 0 & 0 & A_3 \end{pmatrix}.$$

Each operator entry of such kind of operator matrix is bounded and act on their corresponding spaces as:

$$\begin{array}{lll} A_1 : E \to E, & A_2 : F \to F, & A_3 : G \to G, \\ B_1 : F \to E, & B_2 : G \to F\ , & C_1 : G \to E. \end{array}$$

Considering the case of infinite dimensions, it is widely acknowledged that the inclusion, $\sigma(\mathcal{M}) \subset \bigcup_{i=1}^{3} \sigma(A_i)$, can be strict. Many mathematicians have drawn the attention to study the defect set $\left(\bigcup_{i=1}^{3} \tilde{\sigma}(A_i)\right) \setminus \tilde{\sigma}(\mathcal{M})$, where $\tilde{\sigma}$ goes through various types of spectra.

The subsequent analysis requires a crucial conclusion that stems from the following theorem, whose formulation is inspired by the framework developed in [9].

Theorem 2.1. *Let $(A_1, A_2, A_3) \in \Big(\mathcal{L}(E), \mathcal{L}(F), \mathcal{L}(G)\Big)$. Then,*

$$\sigma_{Br}(\mathcal{M}) \cup \Big[\Big[\mathrm{S}_{\mathrm{vep}}(A_1^*) \cup \mathrm{S}_{\mathrm{vep}}(A_2^*)\Big] \cap \mathrm{S}_{\mathrm{vep}}(A_3)\Big] = \bigcup_{i=1}^{3} \sigma_{Br}(A_i),$$

for every $B_1 \in \mathcal{L}(F, E)$, $B_2 \in \mathcal{L}(G, F)$ and $C_1 \in \mathcal{L}(G, E)$.

3. Main Results and Proofs

In order to obtain our first result. We start by introducing the following diagonal operator matrix denoted by $\mathcal{M}_{\mathbf{d}}$ expressed as well: $\mathcal{M}_{\mathbf{d}} := \operatorname{diag}\,(A_1, A_2, A_3)$.

Our first lemma reads as follows.

Lemma 3.1. *Let* $(A_1, A_2, A_3) \in \Big(\mathcal{L}(E), \mathcal{L}(F), \mathcal{L}(G)\Big)$. *Then, for every* $B_1 \in \mathcal{L}(F, E)$, $B_2 \in \mathcal{L}(G, F)$ *and* $C_1 \in \mathcal{L}(G, E)$ *we have:*

$$\sigma_{Br}(\mathcal{M}) \subseteq \sigma_{Br}(\mathcal{M}_{\mathbf{d}}) = \bigcup_{i=1}^{3} \sigma_{Br}(A_i).$$

Proof. We start with the following factorization of the operator matrices $\mathcal{M}_{\frac{1}{k}}$ written as follows:

$$\mathcal{M}_{\frac{1}{k}} := \mathfrak{D}_1 \ \mathcal{M} \ \mathfrak{D}_2, \ \text{for every } k \in \mathbb{N}^* \tag{3.1}$$

where: $\mathfrak{D}_1 := \begin{pmatrix} I & 0 & 0 \\ 0 & kI & 0 \\ 0 & 0 & kI \end{pmatrix}$ and $\mathfrak{D}_2 := \begin{pmatrix} I & 0 & 0 \\ 0 & \frac{1}{k}I & 0 \\ 0 & 0 & \frac{1}{k}I \end{pmatrix}$.

Since $\|\mathcal{M}_{1/k} - \mathcal{M}_{\mathbf{d}}\| \to 0$ as $k \to +\infty$, and given that the Browder spectrum σ_{Br} is upper semi-continuous (see Theorem 2 in [16]), it follows that $\sigma_{Br}(\mathcal{M}) \subseteq \sigma_{Br}(\mathcal{M}_{\mathbf{d}})$.

The second equality is evident. □

Currently, we express the following proposition which will be widely used in the sequel.

Proposition 3.1. *Presume that:*

(**i**) $(A_1, A_2, A_3) \in \Big(\mathcal{L}(E), \mathcal{L}(F), \mathcal{L}(G)\Big)$;

(**ii**) $(B_1, B_2, C_1) \in \Big(\mathcal{L}(F, E), \mathcal{L}(G, F), \mathcal{L}(G, E)\Big)$.

Then, we have:

$$\operatorname{Acc} \sigma_{\mathrm{Br}}(\mathcal{M}) \subseteq \operatorname{Acc} \sigma_{\mathrm{Br}}(\mathcal{M}_{\mathbf{d}}) = \bigcup_{i=1}^{3} \operatorname{Acc} \sigma_{\mathrm{Br}}(A_i).$$

Proof. It is obvious that $\operatorname{Acc} \sigma_{Br}(\mathcal{M}_{\mathbf{d}}) = \bigcup_{i=1}^{3} \operatorname{Acc} \sigma_{Br}(A_i)$, since we have $\sigma_{Br}(\mathcal{M}_{\mathbf{d}}) = \bigcup_{i=1}^{3} \sigma_{Br}(A_i)$. Without loss of generality, let consider $0 \notin \bigcup_{i=1}^{3} \operatorname{Acc} \sigma_{Br}(A_i)$, then, there exists $\alpha > 0$ such that $\tau I - A_1$, $\tau I - A_2$ and $\tau I - A_3$ are Browder operators for every τ, $0 < |\tau| < \alpha$. Furthermore, according to Lemma 3.1, we get $\tau I - \mathcal{M}$ is Browder for every τ, $0 < |\tau| < \alpha$. This concludes that $0 \notin \operatorname{Acc} \sigma_{Br}(\mathcal{M})$. □

The presented example illustrates that the inclusion $\operatorname{Acc} \sigma_{Br}(\mathcal{M}) \subseteq \bigcup_{i=1}^{3} \operatorname{Acc} \sigma_{Br}(A_i)$ can indeed be strict, meaning that the accumulation points of the Browder spectrum of $\mathcal{M}$ may not coincide with the union of those of its diagonal entries.

Example 3.1. Let $A_1, A_2, A_3, C_1 \in \mathcal{L}(\ell^2)$ be defined by:

$$A_1 e_k = A_2 e_k = e_{k+1}, \quad B_1 = B_2 = 0, \quad C_1 = e_0 \otimes e_0, \quad A_3 = A_1^*,$$

where $\{e_k\}_{k \in \mathbb{N}}$ is the canonical orthonormal basis of the Hilbert space ℓ^2.

It is well known that $\sigma_{Br}(A_1) = \sigma_{Br}(A_2) = \{\tau \in \mathbb{C} : |\tau| \leq 1\}$, hence

$$\operatorname{Acc} \sigma_{Br}(A_1) = \operatorname{Acc} \sigma_{Br}(A_2) = \{\tau \in \mathbb{C} : |\tau| \leq 1\}.$$

Since the operator matrix $\mathcal{M}$ constructed from these entries is unitary, it follows that $\operatorname{Acc}\sigma_{Br}(\mathcal{M}) \subseteq \{\tau \in \mathbb{C} : |\tau| = 1\}$. This implies that $0 \notin \operatorname{Acc}\sigma_{Br}(\mathcal{M})$, although $0 \in \bigcup_{i=1}^{3} \operatorname{Acc}\sigma_{Br}(A_i)$.

We also remark that the operator $A_3 = A_1^*$ does not possess the Single-Valued Extension Property (SVEP).

In what follows, we introduce the following definition and we state our next lemma which plays a crucial role in proving our next result.

Definition 3.1. Let $T \in \mathcal{L}(E)$. We say that the operator T has the property (Br^a) at $\tau \in \mathbb{C}$ if $\tau \notin \operatorname{Acc}\sigma_{Br}(T)$.

Lemma 3.2. *If any three of operators A_1, A_2, A_3, and $\mathcal{M}$ have the property (Br^a) at 0, then the fourth does as well.*

Proof. (**i**) If the operators A_1, A_2 and A_3 have the property (Br^a) at 0, then, according to Proposition 3.1, $\mathcal{M}$ has the property (Br^a) at 0.

(**ii**) If the operators A_1, A_2 and $\mathcal{M}$ have the property (Br^a) at 0, this means $0 \notin \operatorname{Acc}\sigma_{Br}(A_1)$, $0 \notin \operatorname{Acc}\sigma_{Br}(A_2)$ and $0 \notin \operatorname{Acc}\sigma_{Br}(\mathcal{M})$. Hence, there exists $\alpha > 0$ such that $\tau I - A_1$, $\tau I - A_2$ and $\tau I - \mathcal{M}$ are Browder operators for every τ, $0 < |\tau| < \alpha$. Thus, by combining Proposition 3.5 in [8] and Corollary 5 in [11], we obtain that $\tau I - A_3$ is Browder operator for every τ, $0 < |\tau| < \alpha$, that is $0 \notin \operatorname{Acc}\sigma_{Br}(A_3)$.

(**iii**) If the operators A_2, A_3 and $\mathcal{M}$ have the property (Br^a) at 0, then A_1 has also the property (Br^a) at 0. We adopt the same reasoning treated in the previous case.

(**iv**) If A_1, A_3 and $\mathcal{M}$ have the property (Br^a) at 0, then A_2 has the property (Br^a) at 0. The proof is similar to (**ii**). ☐

Our next aim in this section is to prove our first main result by means of localized SVEP. This theorem will guide us to a sufficient condition that guaranties the desired equality shown in Corollary 3.1.

Theorem 3.1. *Assume that:*

(**i**) $(A_1, A_2, A_3) \in \Big(\mathcal{L}(E), \mathcal{L}(F), \mathcal{L}(G)\Big)$;

(**ii**) $(B_1, B_2, C_1) \in \Big(\mathcal{L}(F,E), \mathcal{L}(G,F), \mathcal{L}(G,E)\Big)$.

Then, we have:

$$\operatorname{Acc}\sigma_{\mathrm{Br}}(\mathcal{M}) \cup \Big[\Big[\mathrm{Svep}(A_1^*) \cup \mathrm{Svep}(A_2^*)\Big] \cap \mathrm{Svep}(A_3)\Big] = \bigcup_{i=1}^{3} \operatorname{Acc}\sigma_{\mathrm{Br}}(A_i).$$

Proof. Theorem 2.1 shows that

$$\sigma_{Br}(\mathcal{M}) \cup \Big[\Big[\mathrm{Svep}(A_1^*) \cup \mathrm{Svep}(A_2^*)\Big] \cap \mathrm{Svep}(A_3)\Big] = \bigcup_{i=1}^{3} \sigma_{Br}(A_i),$$

for every $B_1 \in \mathcal{L}(F,E)$, $B_2 \in \mathcal{L}(G,F)$ and $C_1 \in \mathcal{L}(G,E)$. Hence,

$$\bigcup_{i=1}^{3} \operatorname{Acc}\sigma_{Br}(A_i) = \Big\{\sigma_{Br}(\mathcal{M}) \cup \Big[\,(\mathrm{S}_{\mathrm{vep}}(A_1^*) \cup \mathrm{S}_{\mathrm{vep}}(A_2^*)) \cap \mathrm{S}_{\mathrm{vep}}(A_3)\Big]\Big\}$$

$$\cap \left\{ \bigcup_{i=1}^{3} \operatorname{Iso} \sigma_{Br}(A_i) \right\}^c . \tag{3.2}$$

In addition, it is obvious that

$$\Big[(\mathrm{S}_{\mathrm{vep}}(A_1^*) \cup \mathrm{S}_{\mathrm{vep}}(A_2^*)) \cap \mathrm{S}_{\mathrm{vep}}(A_3) \Big] \subseteq \bigcup_{i=1}^{3} \operatorname{Acc} \sigma(A_i) \subseteq \bigcup_{i=1}^{3} \operatorname{Acc} \sigma_{Br}(A_i) \tag{3.3}$$

and

$$\begin{aligned} \Big[(\mathrm{S}_{\mathrm{vep}}(A_1^*) \cup \mathrm{S}_{\mathrm{vep}}(A_2^*)) \cap \mathrm{S}_{\mathrm{vep}}(A_3) \Big] = {} & \Big[(\mathrm{S}_{\mathrm{vep}}(A_1^*) \cup \mathrm{S}_{\mathrm{vep}}(A_2^*)) \cap \mathrm{S}_{\mathrm{vep}}(A_3) \Big] \\ & \cap \left\{ \bigcup_{i=1}^{3} \operatorname{Iso} \sigma_{Br}(A_i) \right\}^c . \end{aligned} \tag{3.4}$$

First, we need to demonstrate that

$$\begin{aligned} & \Big\{ \sigma_{Br}(\mathcal{M}) \cup \Big[\Big[\mathrm{Svep}(A_1^*) \cup \mathrm{Svep}(A_2^*) \Big] \cap \mathrm{Svep}(A_3) \Big] \Big\} \cap \left\{ \bigcup_{i=1}^{3} \mathrm{Iso} \sigma_{Br}(A_i) \right\}^c \\ & \subseteq \operatorname{Acc} \sigma_{Br}(\mathcal{M}) \cup \Big[\Big[\mathrm{Svep}(A_1^*) \cup \mathrm{Svep}(A_2^*) \Big] \cap \mathrm{Svep}(A_3) \Big]. \end{aligned}$$

As a matter of fact, let $\tau \notin \operatorname{Acc} \sigma_{Br}(\mathcal{M}) \cup \Big[\Big[\mathrm{Svep}(A_1^*) \cup \mathrm{Svep}(A_2^*) \Big] \cap \mathrm{Svep}(A_3) \Big]$. Without loss of generality, we can suppose that $\tau = 0$. Hence, $0 \notin \operatorname{Acc} \sigma_{Br}(\mathcal{M})$ and $0 \notin \Big[\Big[\mathrm{Svep}(A_1^*) \cup \mathrm{Svep}(A_2^*) \Big] \cap \mathrm{Svep}(A_3) \Big]$. Thus, there exists $\alpha > 0$ such that $\tau I - \mathcal{M}$ is Browder for every $0 < |\tau| < \alpha$. Therefore, for every $0 < |\tau| < \alpha$, $\tau I - A_3$ is lower semi-Browder operator and $\tau I - A_1$ is upper semi-Browder operator. Consequently, $0 \notin \Big[\operatorname{Acc} \sigma_{Br^+}(A_1) \cap \operatorname{Acc} \sigma_{Br^+}(A_2) \Big] \cup \operatorname{Acc} \sigma_{Br^-}(A_3)$. On the other side

$$0 \notin \Big[\Big[\mathrm{Svep}(A_1^*) \cup \mathrm{Svep}(A_2^*) \Big] \cap \mathrm{Svep}(A_3) \Big].$$

There are two possible cases.

- <u>CASE 1</u>: $0 \notin \mathrm{Svep}(A_1^*) \cap \mathrm{Svep}(A_3)$.

. $0 \in \sigma_{Br}(A_1^*) \cap \sigma_{Br}(A_3)$.

We keep into account that $\sigma_{Br}(A_1^*) = \mathrm{Svep}(A_1^*) \cup \sigma_{Br^-}(A_1^*)$. So, if $0 \in \sigma_{Br}(A_1^*)$ then $0 \in \sigma_{Br^-}(A_1^*)$. Since $0 \notin \operatorname{Acc} \sigma_{Br^+}(A_1) = \operatorname{Acc} \sigma_{Br^-}(A_1^*)$, we obtain that 0 is an isolated point of $\sigma_{Br^-}(A_1^*)$. Furthermore, $\overline{\mathrm{Svep}(A_1^*)} \subseteq \sigma_{Br}(A_1^*) = \sigma_{Br^-}(A_1^*) \cup \mathrm{Svep}(A_1^*)$, thus $\partial \mathrm{Svep}(A_1^*) \subseteq \sigma_{Br^-}(A_1^*)$. Since $\sigma_{Br}(A_1^*) = \mathrm{Svep}(A_1^*) \cup \sigma_{Br^-}(A_1^*)$ and $0 \in \mathrm{Iso}\sigma_{Br^-}(A_1^*)$. Therefore, 0 is an isolated point of $\sigma_{Br}(A_1) = \sigma_{Br}(A_1^*)$. On the other side, if $0 \in \sigma_{Br}(A_3)$ then $0 \in \sigma_{Br^-}(A_3)$. As $0 \notin \operatorname{Acc} \sigma_{Br^-}(A_3)$, it follows that $0 \in \mathrm{Iso}\sigma_{Br^-}(A_3)$. This shows that $0 \in \mathrm{Iso}\sigma_{Br}(A_3)$.

. $0 \notin \sigma_{Br}(A_1^*) \cap \sigma_{Br}(A_3)$.

If $0 \notin \sigma_{Br}(A_1^*) \cap \sigma_{Br}(A_3)$. Then, $0 \notin \sigma_{Br}(A_1^*) = \sigma_{Br}(A_1)$ and $0 \notin \sigma_{Br}(A_3)$. Hence, $0 \notin \operatorname{Acc} \sigma_{Br}(A_1)$ and $0 \notin \operatorname{Acc} \sigma_{Br}(A_3)$. This proves that $0 \notin \operatorname{Acc} \sigma_{Br}(A_2)$. So either $0 \notin \sigma_{Br}(A_2)$ or $0 \in \mathrm{Iso}\sigma_{Br}(A_2)$, i.e., $0 \notin \sigma_{Br}(\mathcal{M})$ or $0 \in \mathrm{Iso}\sigma_{Br}(A_2)$.

- <u>CASE 2</u>: $0 \notin \mathrm{Svep}(A_2^*) \cap \mathrm{Svep}(A_3)$.

We adopt the same reasoning treated in the first case. It suffices to replace A_1 by A_2.

On the basis of Proposition 3.1, (3.2), (3.3) and (3.4), we get:

$$\begin{aligned}\bigcup_{i=1}^{3}\operatorname{Acc}\sigma_{Br}(A_i) =&\Big\{\sigma_{Br}(\mathcal{M})\cup\Big[\,(\mathrm{S}_{\mathrm{vep}}(A_1^*)\cup\mathrm{S}_{\mathrm{vep}}(A_2^*))\cap\mathrm{S}_{\mathrm{vep}}(A_3)\Big]\Big\}\cap\Big\{\bigcup_{i=1}^{3}\operatorname{Iso}\sigma_{Br}(A_i)\Big\}^c\\ &\subseteq \operatorname{Acc}\sigma_{Br}(\mathcal{M})\cup\Big[\,(\mathrm{S}_{\mathrm{vep}}(A_1^*)\cup\mathrm{S}_{\mathrm{vep}}(A_2^*))\cap\mathrm{S}_{\mathrm{vep}}(A_3)\Big]\\ &\subseteq \bigcup_{i=1}^{3}\operatorname{Acc}\sigma_{Br}(A_i).\end{aligned}$$

Thus, we complete the proof. □

Taking into account the previous theorem, the following corollary provides a sufficient condition which ensures that $\operatorname{Acc}\sigma_{Br}(\mathcal{M})=\bigcup_{i=1}^{3}\operatorname{Acc}\sigma_{Br}(A_i)$ for every bounded operators $(B_1,B_2,C_1)\in\Big(\mathcal{L}(F,E),\mathcal{L}(G,F),\mathcal{L}(G,E)\Big)$.

Corollary 3.1. *Suppose that:*

(**i**) $(A_1,A_2,A_3)\in\Big(\mathcal{L}(E),\mathcal{L}(F),\mathcal{L}(G)\Big)$*;*

(**ii**) $\Big[S_{\mathrm{vep}}(A_1^*)\cup S_{\mathrm{vep}}(A_2^*)\Big]\cap S_{\mathrm{vep}}(A_3)=\emptyset.$

Then, for every $B_1\in\mathcal{L}(F,E), B_2\in\mathcal{L}(G,F)$ *and* $C_1\in\mathcal{L}(G,E)$ *we have:*

$$\operatorname{Acc}\sigma_{\mathrm{Br}}(\mathcal{M})=\bigcup_{i=1}^{3}\operatorname{Acc}\sigma_{\mathrm{Br}}(A_i).$$

Remark 3.1. The above result makes it easy to confirm that the limit point set of the Browder spectrum of the operator matrix $\mathcal{M}$ is the union of the limit point set of the Browder spectrum of its diagonal entries if A_3 has SVEP or A_1^* and A_2^* have SVEP.

The efficacy of the previously presented result is illustrated by the following example, which serves as a direct application of Corollary 3.1.

Example 3.2. Let consider $\mathcal{S}_u$ the simple unilateral shift operator defined on $\ell^2(\mathbb{N})$. We define also the following operators as follows:

$$A_1:=(\mathcal{S}_u\oplus{\mathcal{S}_u}^*)+2I,\quad A_2:=\mathcal{S}_u\oplus{\mathcal{S}_u}^*\quad\text{and}\quad A_3:=(\mathcal{S}_u\oplus{\mathcal{S}_u}^*)-2I.$$

Then,

$$\begin{aligned}\sigma_{Br}(A_1)&:=\{\tau\in\mathbb{C}:0\le|\tau-2|\le 1\},\\ \sigma_{Br}(A_2)&:=\{\tau\in\mathbb{C}:0\le|\tau|\le 1\},\\ \sigma_{Br}(A_3)&:=\{\tau\in\mathbb{C}:0\le|\tau+2|\le 1\}.\end{aligned}$$

It follows that:

$$\begin{aligned}\mathrm{S}_{\mathrm{vep}}(A_1)&:=\{\tau\in\mathbb{C}:0\le|\tau-2|<1\},\\ \mathrm{S}_{\mathrm{vep}}(A_2)&:=\{\tau\in\mathbb{C}:0\le|\tau|<1\},\\ \mathrm{S}_{\mathrm{vep}}(A_3)&:=\{\tau\in\mathbb{C}:0\le|\tau+2|<1\}.\end{aligned}$$

Consequently, $\left[\mathrm{Svep}(A_1^*) \cup \mathrm{Svep}(A_2^*)\right] \cap \mathrm{Svep}(A_3) = \emptyset$. This proves that

$$\mathrm{Acc}\,\sigma_{Br}(\mathcal{M}) = \bigcup_{i=1}^{3} \mathrm{Acc}\,\sigma_{Br}(A_i).$$

The next two lemmas are the key to our second main outcome and are founded in [17]. We consider two compact subsets $\mathcal{U}, \mathcal{V} \subseteq \mathbb{C}$ such that $\mathcal{U} \subseteq \mathcal{V}$.

Lemma 3.3. *Assume further that* $\partial\mathcal{V} \subseteq \mathcal{U}$. *Then,* $\partial \mathrm{Acc}\,\mathcal{V} \subseteq \mathrm{Acc}\,\mathcal{U}$.

Lemma 3.4. *Assume further that* $\eta(\mathcal{U}) = \eta(\mathcal{V})$. *Then,* $\eta(\mathrm{Acc}\,\mathcal{U}) = \eta(\mathrm{Acc}\,\mathcal{V})$.

Finally, we are in a position to prove the following theorem which says that the passage from $\bigcup_{i=1}^{3} \mathrm{Acc}\,\sigma_{Br}(A_i)$ to $\mathrm{Acc}\,\sigma_{Br}(\mathcal{M})$ is the punching of some set in $\bigcap_{i=1}^{3} \mathrm{Acc}\,\sigma_{Br}(A_i)$.

Theorem 3.2. *Imply that:*
(**i**) $(A_1, A_2, A_3) \in \Big(\mathcal{L}(E), \mathcal{L}(F), \mathcal{L}(G)\Big)$;
(**ii**) $(B_1, B_2, C_1) \in \Big(\mathcal{L}(F,E), \mathcal{L}(G,F), \mathcal{L}(G,E)\Big)$.
Then, we obtain:

$$\mathrm{Acc}\,\sigma_{\mathrm{Br}}(\mathcal{M}) \cup \mathcal{W}_{\mathrm{Acc}\,\sigma_{\mathrm{Br}}} = \bigcup_{i=1}^{3} \mathrm{Acc}\,\sigma_{\mathrm{Br}}(A_i),$$

where $\mathcal{W}_{\mathrm{Acc}\,\sigma_{\mathrm{Br}}}$ *is the union of certain holes in* $\mathrm{Acc}\,\sigma_{\mathrm{Br}}(\mathcal{M})$, *that occur to be subsets of* $\bigcap_{i=1}^{3} \mathrm{Acc}\,\sigma_{\mathrm{Br}}(A_i)$.

Proof. First notice that the inclusion

$$\mathrm{Acc}\,\sigma_{Br}(\mathcal{M}) \cup \left\{\bigcap_{i=1}^{3} \mathrm{Acc}\,\sigma_{Br}(A_i)\right\} \subseteq \bigcup_{i=1}^{3} \mathrm{Acc}\,\sigma_{Br}(A_i)$$

holds for every $B_1 \in \mathcal{L}(F,E)$, $B_2 \in \mathcal{L}(G,F)$ and $C_1 \in \mathcal{L}(G,E)$.

The reverse inclusion follows from the following equivalence:

$$\begin{aligned}
&\tau \notin \mathrm{Acc}\,\sigma_{Br}(\mathcal{M}) \cup \left\{\bigcap_{i=1}^{3} \mathrm{Acc}\,\sigma_{Br}(A_i)\right\}\\
\Leftrightarrow\ &\tau \in \{\mathrm{Acc}\,\sigma_{Br}(\mathcal{M})^c \cap \mathrm{Acc}\,\sigma_{Br}(A_1)^c\} \quad \text{or} \quad \tau \in \{\mathrm{Acc}\,\sigma_{Br}(\mathcal{M})^c \cap \mathrm{Acc}\,\sigma_{Br}(A_2)^c\}\\
&\quad \text{or} \quad \tau \in \{\mathrm{Acc}\,\sigma_{Br}(\mathcal{M})^c \cap \mathrm{Acc}\,\sigma_{Br}(A_3)^c\}\\
\Leftrightarrow\ &\tau \in \mathrm{Acc}\,\sigma_{Br}(A_1)^c \text{ and } \tau \in \mathrm{Acc}\,\sigma_{Br}(A_2)^c \text{ and } \tau \in \mathrm{Acc}\,\sigma_{Br}(A_3)^c \quad \text{(Lemma 3.2)}\\
\Leftrightarrow\ &\tau \notin \bigcup_{i=1}^{3} \mathrm{Acc}\,\sigma_{Br}(A_i).
\end{aligned}$$

We deduce that

$$\mathrm{Acc}\,\sigma_{Br}(\mathcal{M}) \cup \left\{\bigcap_{i=1}^{3} \mathrm{Acc}\,\sigma_{Br}(A_i)\right\} = \bigcup_{i=1}^{3} \mathrm{Acc}\,\sigma_{Br}(A_i). \tag{3.5}$$

Moreover, getting inspired by the proof of Theorem 2.6 in [18] and by applying Lemmas 3.3 and 3.4, we get

$$\partial\left(\bigcap_{i=1}^{3}\sigma_{Br}(A_i)\right)\subseteq\sigma_{Br}(\mathcal{M}),$$

$$\eta\left(\operatorname{Acc}\sigma_{Br}(\mathcal{M})\right)=\eta\left(\bigcup_{i=1}^{3}\operatorname{Acc}\sigma_{Br}(A_i)\right). \tag{3.6}$$

Consequently, (3.6) proves that the passage from $\operatorname{Acc}\sigma_{Br}(\mathcal{M})$ to $\bigcup_{i=1}^{3}\operatorname{Acc}\sigma_{Br}(A_i)$ is the filling in certain of the holes in $\operatorname{Acc}\sigma_{Br}(\mathcal{M})$. But, $\left(\bigcup_{i=1}^{3}\operatorname{Acc}\sigma_{Br}(A_i)\right)\backslash\operatorname{Acc}\sigma_{Br}(\mathcal{M})$ is contained in $\bigcap_{i=1}^{3}\operatorname{Acc}\sigma_{Br}(A_i)$, according to (3.5). This shows that the filling in certain of the holes in $\operatorname{Acc}\sigma_{Br}(\mathcal{M})$ should take place in $\bigcap_{i=1}^{3}\operatorname{Acc}\sigma_{Br}(A_i)$. □

Nevertheless, we have the following corollary.

Corollary 3.2. *Suppose that:*
(**i**) $(A_1, A_2, A_3) \in \left(\mathcal{L}(E), \mathcal{L}(F), \mathcal{L}(G)\right)$;
(**ii**) $\bigcap_{i=1}^{3}\operatorname{Acc}\sigma_{\mathrm{Br}}(A_i)$ *has no interior points.*
Then, for every $B_1 \in \mathcal{L}(F,E), B_2 \in \mathcal{L}(G,F)$ *and* $C_1 \in \mathcal{L}(G,E)$ *we have:*

$$\operatorname{Acc}\sigma_{\mathrm{Br}}(\mathcal{M}) = \bigcup_{i=1}^{3}\operatorname{Acc}\sigma_{\mathrm{Br}}(A_i).$$

We would like to finish this work with the following question.

Question 3.3. Let consider the following bounded full 3×3 block operator matrix

$$\widehat{\mathcal{M}} := \begin{pmatrix} A_1 & B_1 & B_2 \\ C_1 & A_2 & B_3 \\ C_2 & C_3 & A_3 \end{pmatrix}.$$

The following question is being asked: what are the conditions that will be placed on the input parts of the operator matrices $\widehat{\mathcal{M}}$ to achieve that

$$\operatorname{Acc}\sigma_{Br}(\widehat{\mathcal{M}}) = \bigcup_{i=1}^{3}\operatorname{Acc}\sigma_{Br}(A_i)?$$

4. Conclusion

In this paper, we investigated the accumulation points of the Browder spectrum for 3×3 upper triangular operator matrices acting on Banach spaces. By applying the tools of local spectral theory, we established a precise relationship between the accumulation set of the matrix and those of its diagonal entries. In particular, we described the structure of a supplementary set of "holes" that appear within the intersection of the accumulation points of the diagonal Browder spectra, allowing us to characterize the accumulation behavior of the full matrix spectrum. Our results provide a refined description that extends the work of A. Tajmouati et al. [17] by considering a more general setting and offering a deeper analysis of the spectral accumulation structure. This contribution enhances the understanding of spectral properties of operator matrices and may serve as a foundation for further developments in the theory.

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