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JORDAN HIGHER DERIVATIONS ON PRIME HILBERT C*-MODULES

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Abstract. Let $\mathcal{M}$ be a Hilbert C*-module. A sequence of linear mappings $\{\varphi_n : \mathcal{M} \to \mathcal{M}\}_{n=0}^{+\infty}$ with $\varphi_0 = I$, is said to be a Hilbert C*-module Jordan higher derivation on $\mathcal{M}$, if it satisfies the equation

$$\varphi_n(\langle a, b\rangle a) = \sum_{i+j+k=n} \langle \varphi_i(a), \varphi_j(b)\rangle \varphi_k(a),$$

for all $a, b \in \mathcal{M}$ and each non-negative integer n. In this paper, we show that, if $\mathcal{M}$ is prime, then every Hilbert C*-module Jordan higher derivation $\{\varphi_n\}_{n=0}^{+\infty}$ on $\mathcal{M}$, is a Hilbert C*-module higher derivation on $\mathcal{M}$. As a consequence, we show that every Hilbert C*-module Jordan derivation on $\mathcal{M}$, is a Hilbert C*-module derivation on $\mathcal{M}$.

1. Introduction

The notion of a Hilbert C*-module initiated as a generalization of a Hilbert space in which the inner product takes its values in a C*-algebra (see [13]). Let $\mathcal{A}$ be a C*-algebra. An inner product $\mathcal{A}$-module is a complex linear space $\mathcal{M}$ which is a left $\mathcal{A}$-module with compatible scalar multiplication $\lambda(ax) = (\lambda a)x = a(\lambda x)$ $(\lambda \in \mathbb{C}, x \in \mathcal{M}, a \in \mathcal{A})$, together with an $\mathcal{A}$-valued inner product $(x, y) \mapsto \langle x, y\rangle : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ such that for each $x, y, z \in \mathcal{M}$, $\alpha, \beta \in \mathbb{C}$ and $a \in \mathcal{A}$,

(i) $\langle x, x\rangle \geq 0$ and the equality holds if and only if $x = 0$;
(ii) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$;
(iii) $\langle ax, y\rangle = a\langle x, y\rangle$;

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(iv) $\langle x, y\rangle^* = \langle y, x\rangle$.

It follows from the above conditions that $\langle x, x\rangle$ is a positive element in C*-algebra $\mathcal{A}$, the inner product is conjugate-linear in its second variable and $\langle x, ay\rangle = \langle x, y\rangle a^*$ for all $x, y \in \mathcal{M}$ and $a \in \mathcal{A}$. An inner product $\mathcal{A}$-module $\mathcal{M}$ which is complete with respect to the norm $\|x\|_{\mathcal{M}} = \|\langle x, x\rangle\|_{\mathcal{A}}^{\frac{1}{2}}$ is called a Hilbert $\mathcal{A}$-module or a Hilbert C*-module over the C*-algebra $\mathcal{A}$. For example, every C*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module under the $\mathcal{A}$-valued inner product $\langle a, b\rangle = ab^*$ $(a, b \in \mathcal{A})$. Every complex Hilbert space is a left Hilbert $\mathbb{C}$-module. The notion of a right Hilbert $\mathcal{A}$-module can be defined similarly.

A Hilbert C*-module $\mathcal{M}$ is said to be *prime*, if for elements a, b of $\mathcal{M}$, $\langle a, \mathcal{M}\rangle b = 0$ implies that $a = 0$ or $b = 0$. Equivalently, $\mathcal{M}$ is called prime, if for elements a, b of $\mathcal{M}$, validity the equation $\langle a, x\rangle b = 0$ for all $x \in \mathcal{M}$, implies that $a = 0$ or $b = 0$. $\mathcal{M}$ is said to be *semiprime*, if $\langle a, \mathcal{M}\rangle a = 0$ implies that $a = 0$. Trivially any prime Hilbert C*-module $\mathcal{M}$ is semiprime.

Let $\mathcal{M}$ and $\mathcal{N}$ be Hilbert C*-modules over a C*-algebra $\mathcal{A}$. A mapping $T : \mathcal{M} \to \mathcal{N}$ is said to be adjointable, if there exists a mapping $S : \mathcal{N} \to \mathcal{M}$ such that $\langle T(x), y\rangle = \langle x, S(y)\rangle$ for all $x \in D_T \subseteq \mathcal{M}$, $y \in D_S \subseteq \mathcal{N}$. The unique mapping S is denoted by T^* and is called the adjoint of T. It is well known that any adjointable mapping $T : \mathcal{M} \to \mathcal{N}$ is $\mathcal{A}$-linear (that is $T(ax + \lambda y) = aT(x) + \lambda T(y)$ for all $x, y \in \mathcal{M}, a \in \mathcal{A}, \lambda \in \mathbb{C}$) and bounded.

A linear mapping $\psi : \mathcal{M} \to \mathcal{M}$ is called a *Hilbert C*-module derivation* on $\mathcal{M}$, if it satisfies the equation

$$\psi(\langle a, b\rangle c) = \langle \psi(a), b\rangle c + \langle a, \psi(b)\rangle c + \langle a, b\rangle \psi(c),$$

for all $a, b, c \in \mathcal{M}$. ψ is called a *Hilbert C*-module Jordan derivation* on $\mathcal{M}$, if it satisfies the equation

$$\psi(\langle a, b\rangle a) = \langle \psi(a), b\rangle a + \langle a, \psi(b)\rangle a + \langle a, b\rangle \psi(a),$$

for all $a, b \in \mathcal{M}$. Note that every Hilbert C*-module derivation is a Hilbert C*-module Jordan derivation. But the converse is not true in general.

Remark 1.1. Every adjointable mapping $\psi : \mathcal{M} \to \mathcal{M}$ satisfying $\psi^* = -\psi$ is a Hilbert C*-module derivation. Infact if $\psi^* = -\psi$, then $\langle \psi(a), b\rangle c + \langle a, \psi(b)\rangle c = 0$ for all $a, b, c \in \mathcal{M}$. Moreover

$$\begin{aligned}\langle \psi(\langle a, b\rangle c), x\rangle &= \Big\langle \langle a, b\rangle c, \psi^*(x)\Big\rangle = \langle a, b\rangle \langle c, \psi^*(x)\rangle = \langle a, b\rangle \langle \psi(c), x\rangle \\ &= \Big\langle \langle a, b\rangle \psi(c), x\Big\rangle,\end{aligned}$$

for all $a, b, c, x \in \mathcal{M}$ which implies that $\psi(\langle a, b\rangle c) = \langle a, b\rangle \psi(c)$ for all $a, b, c \in \mathcal{M}$.

Example 1.1. Let $M_2(\mathbb{C})$ be the C*-algebra of 2×2 complex matrices. The mapping $\psi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ defined by

$$\psi(A) = \begin{bmatrix} a_{21} & a_{22} \\ -a_{11} & -a_{12} \end{bmatrix},$$

for all $A = [a_{ij}] \in M_2(\mathbb{C})$, is a Hilbert C*-module derivation on $M_2(\mathbb{C})$.

A sequence of linear mappings $\{\varphi_n : \mathcal{M} \to \mathcal{M}\}_{n=0}^{+\infty}$, with $\varphi_0 = I$ (the identity mapping on $\mathcal{M}$) is called a Hilbert C*-module higher derivation on $\mathcal{M}$, if it satisfies the equation

$$\varphi_n(\langle a, b\rangle c) = \sum_{i+j+k=n} \langle \varphi_i(a), \varphi_j(b)\rangle \varphi_k(c),$$

for all $a, b, c \in \mathcal{M}$ and each non-negative integer n.

Example 1.2. Let ψ be a Hilbert C*-module derivation on $\mathcal{M}$. Then the sequence $\{\varphi_n\}_{n=0}^{+\infty}$ of linear mappings on $\mathcal{M}$ defined by $\varphi_0 = I$ and

$$\varphi_n(\langle a, b\rangle c) = \sum_{\substack{i+j+k=n\\ 0\le i,j,k\le n}} \frac{1}{i!j!k!} \langle \psi^i(a), \psi^j(b)\rangle \psi^k(c),$$

for all $a, b, c \in \mathcal{M}$ and each non-negative integer n (in which $\psi^0 = I$), is a Hilbert C*-module higher derivation on $\mathcal{M}$. The four terms of this Hilbert C*-module higher derivation are

$$\begin{aligned}
\varphi_0(\langle a, b\rangle c) =& \langle a, b\rangle c,\\
\varphi_1(\langle a, b\rangle c) =& \langle \psi(a), b\rangle c + \langle a, \psi(b)\rangle c + \langle a, b\rangle \psi(c),\\
\varphi_2(\langle a, b\rangle c) =& \frac{1}{2}\langle \psi^2(a), b\rangle c + \frac{1}{2}\langle a, \psi^2(b)\rangle c + \frac{1}{2}\langle a, b\rangle \psi^2(c)\\
&+ \langle \psi(a), \psi(b)\rangle c + \langle \psi(a), b\rangle \psi(c) + \langle a, \psi(b)\rangle \psi(c),\\
\varphi_3(\langle a, b\rangle c) =& \frac{1}{6}\langle \psi^3(a), b\rangle c + \frac{1}{6}\langle a, \psi^3(b)\rangle c + \frac{1}{6}\langle a, b\rangle \psi^3(c)\\
&+ \frac{1}{2}\langle \psi^2(a), \psi(b)\rangle c + \frac{1}{2}\langle \psi^2(a), b\rangle \psi(c) + \frac{1}{2}\langle \psi(a), \psi^2(b)\rangle c\\
&+ \frac{1}{2}\langle a, \psi^2(b)\rangle \psi(c) + \frac{1}{2}\langle \psi(a), b\rangle \psi^2(c) + \frac{1}{2}\langle a, \psi(b)\rangle \psi^2(c)\\
&+ \langle \psi(a), \psi(b)\rangle \psi(c).
\end{aligned}$$

A sequence of linear mappings $\{\varphi_n : \mathcal{M} \to \mathcal{M}\}_{n=0}^{+\infty}$, with $\varphi_0 = I$, is called a Hilbert C*-module Jordan higher derivation on $\mathcal{M}$, if

$$\varphi_n(\langle a, b\rangle a) = \sum_{i+j+k=n} \langle \varphi_i(a), \varphi_j(b)\rangle \varphi_k(a),$$

for all $a, b \in \mathcal{M}$ and each non-negative integer n.

When $\{\varphi_n\}_{n=0}^{+\infty}$ is a Hilbert C*-module higher derivation (Jordan higher derivation), φ_1 is a Hilbert C*-module derivation (Jordan derivation). Trivially every Hilbert C*-module higher derivation is a Hilbert C*-module Jordan higher derivation. But the converse is not true in general.

The classical result due to Herstein [11] was extended for higher derivations by Haetinger [9], who proved that every Jordan higher derivation on a prime ring of characteristic different from two is a higher derivation. Further, Ferrero and Haetinger

[8] established that on a 2-torsion free semiprime ring every Jordan triple higher derivation, is a higher derivation. In this paper we prove that if $\mathcal{M}$ is a prime Hilbert C*-module, then every Hilbert C*-module Jordan higher derivation on $\mathcal{M}$, is a Hilbert C*-module higher derivation on $\mathcal{M}$. As a consequence, we show that every Hilbert C*-module Jordan derivation on $\mathcal{M}$, is a Hilbert C*-module derivation on $\mathcal{M}$.

For more information about Hilbert C*-module derivations and Hilbert C*-module higher derivations the reader can see [6, 16]. Also for information about derivations and higher derivations on algebras, the reader refer to [1–5, 7, 10, 12, 14, 15, 17, 18].

2. The Result

Let $\mathcal{M}$ be a Hilbert C*-module and I be the identity mapping on $\mathcal{M}$. A sequence of linear mappings $\{\varphi_n : \mathcal{M} \to \mathcal{M}\}_{n=0}^{+\infty}$, with $\varphi_0 = I$, is said to be a

(i) *Hilbert C*-module higher derivation* on $\mathcal{M}$, if it satisfies the equation

$$\varphi_n(\langle a, b\rangle c) = \sum_{i+j+k=n} \langle \varphi_i(a), \varphi_j(b)\rangle \varphi_k(c), \tag{2.1}$$

for all $a, b, c \in \mathcal{M}$ and each non-negative integer n;

(ii) *Hilbert C*-module Jordan higher derivation* on $\mathcal{M}$, if it satisfies the equation

$$\varphi_n(\langle a, b\rangle a) = \sum_{i+j+k=n} \langle \varphi_i(a), \varphi_j(b)\rangle \varphi_k(a), \tag{2.2}$$

for all $a, b \in \mathcal{M}$ and each non-negative integer n.

Trivially every Hilbert C*-module higher derivation is a Hilbert C*-module Jordan higher derivation. But the converse is not true in general. In this section, we prove that on a prime Hilbert C*-module $\mathcal{M}$, every Hilbert C*-module Jordan higher derivation is a Hilbert C*-module higher derivation. Before proving the result, we need some lemmas.

Lemma 2.1. *Let $\mathcal{M}$ be a Hilbert C*-module and $\{\varphi_n : \mathcal{M} \to \mathcal{M}\}_{n=0}^{+\infty}$ be a Hilbert C*-module Jordan higher derivation on $\mathcal{M}$. Then,*

$$\varphi_n(\langle a, b\rangle c + \langle c, b\rangle a) = \sum_{i+j+k=n} \left(\langle \varphi_i(a), \varphi_j(b)\rangle \varphi_k(c) + \langle \varphi_i(c), \varphi_j(b)\rangle \varphi_k(a)\right), \tag{2.3}$$

for all $a, b, c \in \mathcal{M}$ and each non-negative integer n.

Proof. Replacing a by $a + c$ in (2.2), we get

$$\varphi_n(\langle a + c, b\rangle (a + c)) = \sum_{i+j+k=n} \langle \varphi_i(a + c), \varphi_j(b)\rangle \varphi_k(a + c),$$

which implies that

$$\begin{aligned} &\varphi_n(\langle a, b\rangle a + \langle c, b\rangle a + \langle a, b\rangle c + \langle c, b\rangle c) \\ &= \sum_{i+j+k=n} \Big(\langle \varphi_i(a), \varphi_j(b)\rangle \varphi_k(a) + \langle \varphi_i(a), \varphi_j(b)\rangle \varphi_k(c) \\ &\quad + \langle \varphi_i(c), \varphi_j(b)\rangle \varphi_k(a) + \langle \varphi_i(c), \varphi_j(b)\rangle \varphi_k(c)\Big), \end{aligned}$$

for all $a, b, c \in \mathcal{M}$. Since $\varphi_n(\langle a, b\rangle a) = \sum_{i+j+k=n}\langle \varphi_i(a), \varphi_j(b)\rangle \varphi_k(a)$ and $\varphi_n(\langle c, b\rangle c) = \sum_{i+j+k=n}\langle \varphi_i(c), \varphi_j(b)\rangle \varphi_k(c)$, canceling these terms from both sides of the above equation, we get the equation (2.3). $\square$

Lemma 2.2. *Let $\mathcal{M}$ be a 2-torsion-free semiprime Hilbert C^*-module and $a, b \in \mathcal{M}$. If $\langle a, x\rangle b + \langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$, then $\langle a, x\rangle b = \langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$. If $\langle a, x\rangle b = 0$ for all $x \in \mathcal{M}$, then $\langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$.*

Proof. Let $a, b \in \mathcal{M}$. Suppose that $\langle a, x\rangle b + \langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$. Then, we have

$$\begin{aligned}\Big\langle \langle a, x\rangle b, y\Big\rangle \langle a, x\rangle b &= -\Big\langle \langle b, x\rangle a, y\Big\rangle \langle a, x\rangle b = -\langle b, x\rangle \langle a, y\rangle \langle a, x\rangle b = -\Big\langle b, \langle y, a\rangle x\Big\rangle \langle a, x\rangle b \\ &= -\Big\langle \langle b, \langle y, a\rangle x\rangle a, x\Big\rangle b = \Big\langle \langle a, \langle y, a\rangle x\rangle b, x\Big\rangle b = \Big\langle \langle a, x\rangle \langle a, y\rangle b, x\Big\rangle b \\ &= \langle a, x\rangle \langle a, y\rangle \langle b, x\rangle b = \langle a, x\rangle \Big\langle \langle a, y\rangle b, x\Big\rangle b = -\langle a, x\rangle \Big\langle \langle b, y\rangle a, x\Big\rangle b \\ &= -\langle a, x\rangle \langle b, y\rangle \langle a, x\rangle b = -\Big\langle \langle a, x\rangle b, y\Big\rangle \langle a, x\rangle b,\end{aligned}$$

for all $y \in \mathcal{M}$, which implies that $\Big\langle \langle a, x\rangle b, y\Big\rangle \langle a, x\rangle b = 0$ for all $y \in \mathcal{M}$. Since $\mathcal{M}$ is semiprime, we get $\langle a, x\rangle b = 0$ and so $\langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$.

Now suppose that $\langle a, x\rangle b = 0$ for all $x \in \mathcal{M}$. Then, we have

$$\Big\langle \langle b, x\rangle a, y\Big\rangle \langle b, x\rangle a = \langle b, x\rangle \langle a, y\rangle \langle b, x\rangle a = \langle b, x\rangle \Big\langle \langle a, y\rangle b, x\Big\rangle a = 0,$$

for all $y \in \mathcal{M}$. Then semiprimeness of $\mathcal{M}$ implies that $\langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$. $\square$

Lemma 2.3. *Let $\mathcal{M}$ be a 2-torsion-free Hilbert C^*-module. Then the following conditions are equivalent.*

(i) *$\mathcal{M}$ is a prime Hilbert C^*-module.*
(ii) *For $a, b \in \mathcal{M}$, validity of $\langle a, x\rangle b + \langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$, implies that $a = 0$ or $b = 0$.*
(iii) *For $a, b \in \mathcal{M}$, validity of $\langle a, x\rangle a = \langle b, x\rangle b$ for all $x \in \mathcal{M}$, implies that $a = b$ or $a = -b$.*

Proof. (i)⇒(ii) If $\mathcal{M}$ is a prime Hilbert C*-module and $\langle a, x\rangle b + \langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$, then by Lemma 2.2, $\langle a, x\rangle b = 0$ for all $x \in \mathcal{M}$ and then by primeness of $\mathcal{M}$, $a = 0$ or $b = 0$.

(ii)⇒(i) Suppose that $\langle a, x\rangle b = 0$ for all $x \in \mathcal{M}$. Then $\Big\langle \langle b, x\rangle a, y\Big\rangle \langle b, x\rangle a = \langle b, x\rangle \langle a, y\rangle \langle b, x\rangle a = \langle b, x\rangle \Big\langle \langle a, y\rangle b, x\Big\rangle a = 0$ which implies that $\langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$. Hence $\langle a, x\rangle b + \langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$ and therefore $a = 0$ or $b = 0$. Thus, $\mathcal{M}$ is a prime.

(ii)⇒(iii) Let $\langle a, x\rangle a = \langle b, x\rangle b$ for all $x \in \mathcal{M}$. Then $\langle a-b, x\rangle (a+b) + \langle a+b, x\rangle (a-b) = 0$ for all $x \in \mathcal{M}$. Thus, $a - b = 0$ or $a + b = 0$.

(iii)⇒(ii) Let $\langle a, x\rangle b + \langle b, x\rangle a = 0$ for all $x \in \mathcal{M}$. Then, $\langle a - b, x\rangle (a - b) = \langle a + b, x\rangle (a + b)$ for all $x \in \mathcal{M}$. Hence, $a - b = a + b$ or $a - b = -(a + b)$. That is $a = 0$ or $b = 0$. $\square$

Lemma 2.4. *Let $\mathcal{M}$ be a 2-torsion-free semiprime Hilbert C^*-module and $\Delta, \Omega : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ be mappings which are additive in each variable and $\Delta(a,b,a) = \Omega(a,b,a) = 0$ for all $a, b \in \mathcal{M}$. If*

$$\langle \Delta(a,b,c), x\rangle \Omega(a,b,c) = 0, \tag{2.4}$$

for all $a, b, c, x \in \mathcal{M}$, then $\langle \Delta(a,b,c), x\rangle \Omega(c,b,a) = 0$ for all $a, b, c, x \in \mathcal{M}$.

Proof. Suppose that $\langle \Delta(a,b,c), x\rangle \Omega(a,b,c) = 0$ for all $a, b, c, x \in \mathcal{M}$. Then, by Lemma 2.2, we get $\langle \Omega(a,b,c), x\rangle \Delta(a,b,c) = 0$ for all $a, b, c, x \in \mathcal{M}$.

Replacing a and c by $a + c$ in (2.4), we have

$$\langle \Delta(a+c, b, a+c), x\rangle \Omega(a+c, b, a+c) = 0,$$

which implies that

$$\langle \Delta(a,b,c), x\rangle \Omega(c,b,a) + \langle \Delta(c,b,a), x\rangle \Omega(a,b,c) = 0,$$

for all $a, b, c, x \in \mathcal{M}$. It follows from

$$\begin{aligned}
&\Big\langle \langle \Delta(a,b,c), x\rangle \Omega(c,b,a), y\Big\rangle \langle \Delta(a,b,c), x\rangle \Omega(c,b,a) \\
&= - \Big\langle \langle \Delta(a,b,c), x\rangle \Omega(c,b,a), y\Big\rangle \langle \Delta(c,b,a), x\rangle \Omega(a,b,c) \\
&= - \langle \Delta(a,b,c), x\rangle \langle \Omega(c,b,a), y\rangle \langle \Delta(c,b,a), x\rangle \Omega(a,b,c) \\
&= - \langle \Delta(a,b,c), x\rangle \Big\langle \langle \Omega(c,b,a), y\rangle \Delta(c,b,a), x\Big\rangle \Omega(a,b,c) = 0,
\end{aligned}$$

and semiprimeness of $\mathcal{M}$ that $\langle \Delta(a,b,c), x\rangle \Omega(c,b,a) = 0$ for all $a, b, c, x \in \mathcal{M}$. □

Lemma 2.5. *Let $\mathcal{M}$ be a Hilbert C^*-module. Then for all $a, b, c, x \in \mathcal{M}$ we have*

$$\Big\langle a, \Big\langle b, \langle c, x\rangle c\Big\rangle b\Big\rangle a = \Big\langle \langle a, b\rangle c, x\Big\rangle \langle c, b\rangle a.$$

Proof. Let $a, b, c, x \in \mathcal{M}$, then

$$\begin{aligned}
\Big\langle a, \Big\langle b, \langle c, x\rangle c\Big\rangle b\Big\rangle a &= \Big\langle a, \langle b, c\rangle \langle x, c\rangle b\Big\rangle a = \Big\langle a, \langle x, c\rangle b\Big\rangle \langle c, b\rangle a \\
&= \langle a, b\rangle \langle c, x\rangle \langle c, b\rangle a = \Big\langle \langle a, b\rangle c, x\Big\rangle \langle c, b\rangle a.
\end{aligned}$$

□

Theorem 2.1. *Let $\mathcal{M}$ be a 2-torsion-free prime Hilbert C^*-module. Then, every Hilbert C^*-module Jordan higher derivation on $\mathcal{M}$ is a Hilbert C^*-module higher derivation on $\mathcal{M}$.*

Proof. Let $\{\varphi_n\}_{n=0}^{+\infty}$ be a Hilbert C*-module Jordan higher derivation on $\mathcal{M}$ and $a, b, c \in \mathcal{M}$. Define

$$\Delta_n(a,b,c) := \varphi_n(\langle a, b\rangle c) - \sum_{i+j+k=n} \langle \varphi_i(a), \varphi_j(b)\rangle \varphi_k(c), \tag{2.5}$$

for each non-negative integer n and $\Omega(a,b,c) := \langle a, b\rangle c - \langle c, b\rangle a$. Trivially $\Delta_n(a,b,a) = \Omega(a,b,a) = 0$ for all $n \in \mathbb{N}$, $\Delta_n(a,b,c) + \Delta_n(c,b,a) = 0$ and $\Omega(a,b,c) + \Omega(c,b,a) = 0$.

We have

$$\begin{aligned}
S =&\varphi_n\Big(\Big\langle a, \Big\langle b, \langle c, x\rangle c\Big\rangle b\Big\rangle a + \Big\langle c, \Big\langle b, \langle a, x\rangle a\Big\rangle b\Big\rangle c\Big)\\
=&\sum_{i+j+k=n}\Big(\Big\langle \varphi_i(a), \varphi_j(\Big\langle b, \langle c, x\rangle c\Big\rangle b)\Big\rangle \varphi_k(a) + \Big\langle \varphi_i(c), \varphi_j(\Big\langle b, \langle a, x\rangle a\Big\rangle b)\Big\rangle \varphi_k(c)\Big)\\
=&\sum_{i+p+q+r+k=n}\Big(\Big\langle \varphi_i(a), \Big\langle \varphi_p(b), \varphi_q(\langle c, x\rangle c)\Big\rangle \varphi_r(b)\Big\rangle \varphi_k(a)\\
&+ \Big\langle \varphi_i(c), \Big\langle \varphi_p(b), \varphi_q(\langle a, x\rangle a)\Big\rangle \varphi_r(b)\Big\rangle \varphi_k(c)\Big)\\
=&\sum_{i+p+s+t+u+r+k=n}\Big(\Big\langle \varphi_i(a), \Big\langle \varphi_p(b), \langle \varphi_s(c), \varphi_t(x)\rangle \varphi_u(c)\Big\rangle \varphi_r(b)\Big\rangle \varphi_k(a)\\
&+ \Big\langle \varphi_i(c), \Big\langle \varphi_p(b), \langle \varphi_s(a), \varphi_t(x)\rangle \varphi_u(a)\Big\rangle \varphi_r(b)\Big\rangle \varphi_k(c)\Big)\\
=&\sum_{i+p+s+t+u+r+k=n}\Big(\Big\langle \langle \varphi_i(a), \varphi_p(b)\rangle \varphi_s(c), \varphi_t(x)\Big\rangle \langle \varphi_u(c), \varphi_r(b)\rangle \varphi_k(a)\\
&+ \Big\langle \langle \varphi_i(c), \varphi_p(b)\rangle \varphi_s(a), \varphi_t(x)\Big\rangle \langle \varphi_u(a), \varphi_r(b)\rangle \varphi_k(c)\Big),
\end{aligned}$$

for all $x \in \mathcal{M}$. On the other hand, using Lemmas 2.5 and 2.1, we get

$$\begin{aligned}
S =& \varphi_n\Big(\Big\langle \langle a, b\rangle c, x\Big\rangle \langle c, b\rangle a + \Big\langle \langle c, b\rangle a, x\Big\rangle \langle a, b\rangle c\Big)\\
=&\sum_{i+j+k=n}\Big(\Big\langle \varphi_i(\langle a, b\rangle c), \varphi_j(x)\Big\rangle \varphi_k(\langle c, b\rangle a) + \Big\langle \varphi_i(\langle c, b\rangle a), \varphi_j(x)\Big\rangle \varphi_k(\langle a, b\rangle c)\Big),
\end{aligned}$$

for all $x \in \mathcal{M}$. It follows from above equations that

$$\begin{aligned}
&\sum_{i+j+k=n}\Big(\Big\langle \varphi_i(\langle a, b\rangle c), \varphi_j(x)\Big\rangle \varphi_k(\langle c, b\rangle a) + \Big\langle \varphi_i(\langle c, b\rangle a), \varphi_j(x)\Big\rangle \varphi_k(\langle a, b\rangle c)\Big)\\
&= \sum_{i+p+s+t+u+r+k=n}\Big(\Big\langle \langle \varphi_i(a), \varphi_p(b)\rangle \varphi_s(c), \varphi_t(x)\Big\rangle \langle \varphi_u(c), \varphi_r(b)\rangle \varphi_k(a)\\
&\quad + \Big\langle \langle \varphi_i(c), \varphi_p(b)\rangle \varphi_s(a), \varphi_t(x)\Big\rangle \langle \varphi_u(a), \varphi_r(b)\rangle \varphi_k(c)\Big),
\end{aligned}\tag{2.6}$$

for all $x \in \mathcal{M}$.

Now we use induction on n. Putting $n = 1$ in the above equation and canceling the like terms from both sides of this equation and then arranging them, we get

$$\begin{aligned}
&\langle \Delta_1(a, b, c), x\rangle \langle c, b\rangle a + \langle \langle c, b\rangle a, x\rangle \Delta_1(a, b, c)\\
&+ \langle \Delta_1(c, b, a), x\rangle \langle a, b\rangle c + \langle \langle a, b\rangle c, x\rangle \Delta_1(c, b, a) = 0,
\end{aligned}$$

for all $x \in \mathcal{M}$. Since $\Delta_1(c, b, a) = -\Delta_1(a, b, c)$, we get

$$\begin{aligned}
&\langle \Delta_1(a, b, c), x\rangle \langle c, b\rangle a + \langle \langle c, b\rangle a, x\rangle \Delta_1(a, b, c)\\
&- \langle \Delta_1(a, b, c), x\rangle \langle a, b\rangle c - \langle \langle a, b\rangle c, x\rangle \Delta_1(a, b, c) = 0,
\end{aligned}$$

which implies that

$$\Big\langle \Delta_1(a,b,c), x\Big\rangle \Omega(c,b,a) + \Big\langle \Omega(c,b,a), x\Big\rangle \Delta_1(a,b,c) = 0,$$

for all $x \in \mathcal{M}$ and since $\Omega(c,b,a) = -\Omega(a,b,c)$, then

$$\Big\langle \Delta_1(a,b,c), x\Big\rangle \Omega(a,b,c) + \Big\langle \Omega(a,b,c), x\Big\rangle \Delta_1(a,b,c) = 0,$$

for all $x \in \mathcal{M}$. Since $\mathcal{M}$ is semiprime, it follows from Lemma 2.2, that

$$\Big\langle \Delta_1(a,b,c), x\Big\rangle \Omega(a,b,c) = \Big\langle \Omega(a,b,c), x\Big\rangle \Delta_1(a,b,c) = 0,$$

for all $x \in \mathcal{M}$. Since $\mathcal{M}$ is prime, it follows from Lemma 2.3 that $\Delta_1(a,b,c) = 0$ or $\Omega(a,b,c) = 0$. If $\Delta_1(a,b,c) = 0$, then $\varphi_1(\langle a,b\rangle c) = \langle \varphi_1(a), b\rangle c + \langle a, \varphi_1(b)\rangle c + \langle a,b\rangle \varphi_1(c)$, and so φ_1 is a Hilbert C*-module derivation. If $\Omega(a,b,c) = 0$, then $\langle a,b\rangle c = \langle c,b\rangle a$. Thus it follows from Lemma 2.1 that φ_1 is a Hilbert C*-module derivation.

Now suppose that for all $1 \le \ell \le n-1$, φ_ℓ satisfies the equation

$$\varphi_\ell(\langle a,b\rangle c) = \sum_{i+j+k=\ell} \langle \varphi_i(a), \varphi_j(b)\rangle \varphi_k(c). \tag{2.7}$$

We will show that the equation (2.7) is true for $\ell = n$.

Note that equation (2.6) can be written as

$$\begin{aligned} &\sum_{j=0}^{n} \sum_{i+k=n-j} \Big(\Big\langle \varphi_i(\langle a,b\rangle c), \varphi_j(x)\Big\rangle \varphi_k(\langle c,b\rangle a) + \Big\langle \varphi_i(\langle c,b\rangle a), \varphi_j(x)\Big\rangle \varphi_k(\langle a,b\rangle c)\Big) \\ &= \sum_{t=0}^{n} \sum_{i+p+s+u+r+k=n-t} \Big(\Big\langle \langle \varphi_i(a), \varphi_p(b)\rangle \varphi_s(c), \varphi_t(x)\Big\rangle \langle \varphi_u(c), \varphi_r(b)\rangle \varphi_k(a) \\ &\quad + \Big\langle \langle \varphi_i(c), \varphi_p(b)\rangle \varphi_s(a), \varphi_t(x)\Big\rangle \langle \varphi_u(a), \varphi_r(b)\rangle \varphi_k(c)\Big), \end{aligned} \tag{2.8}$$

for all $x \in \mathcal{M}$. In (2.8), for $1 \le j \le n$ we have $i+k = n-j < n$ and then $i,k < n$. This implies that φ_i, φ_k satisfy (2.7). Thus we can cancel the like terms of both sides of equation (2.8). In fact the equation (2.8) reduces to the following equation for the case that $j=0$:

$$\begin{aligned} &\sum_{i+k=n} \Big(\Big\langle \varphi_i(\langle a,b\rangle c), x\Big\rangle \varphi_k(\langle c,b\rangle a) + \Big\langle \varphi_i(\langle c,b\rangle a), x\Big\rangle \varphi_k(\langle a,b\rangle c)\Big) \\ &= \sum_{i+p+s+u+r+k=n} \Big(\Big\langle \langle \varphi_i(a), \varphi_p(b)\rangle \varphi_s(c), x\Big\rangle \langle \varphi_u(c), \varphi_r(b)\rangle \varphi_k(a) \\ &\quad + \Big\langle \langle \varphi_i(c), \varphi_p(b)\rangle \varphi_s(a), x\Big\rangle \langle \varphi_u(a), \varphi_r(b)\rangle \varphi_k(c)\Big), \end{aligned}$$

which implies that

$$
\begin{aligned}
&\Big\langle \varphi_n(\langle a,b\rangle c),x\Big\rangle\langle c,b\rangle a+\Big\langle \varphi_n(\langle c,b\rangle a),x\Big\rangle\langle a,b\rangle c\\
&+\Big\langle \langle a,b\rangle c,x\Big\rangle\varphi_n(\langle c,b\rangle a)+\Big\langle \langle c,b\rangle a,x\Big\rangle\varphi_n(\langle a,b\rangle c)\\
&+\sum_{\substack{i+k=n\\1\leq i,k\leq n-1}}\Big(\Big\langle \varphi_i(\langle a,b\rangle c),x\Big\rangle\varphi_k(\langle c,b\rangle a)+\Big\langle \varphi_i(\langle c,b\rangle a),x\Big\rangle\varphi_k(\langle a,b\rangle c)\Big)\\
=&\sum_{i+p+s=n}\Big\langle \langle\varphi_i(a),\varphi_p(b)\rangle\varphi_s(c),x\Big\rangle\langle c,b\rangle a+\Big\langle \langle\varphi_i(c),\varphi_p(b)\rangle\varphi_s(a),x\Big\rangle\langle a,b\rangle c\\
&+\sum_{u+r+k=n}\Big\langle \langle a,b\rangle c,x\Big\rangle\langle\varphi_u(c),\varphi_r(b)\rangle\varphi_k(a)+\Big\langle \langle c,b\rangle a,x\Big\rangle\langle\varphi_u(a),\varphi_r(b)\rangle\varphi_k(c)\\
&+\sum_{\substack{i+p+s+u+r+k=n\\1\leq i+p+s,u+r+k\leq n-1}}\Big(\Big\langle \langle\varphi_i(a),\varphi_p(b)\rangle\varphi_s(c),x\Big\rangle\langle\varphi_u(c),\varphi_r(b)\rangle\varphi_k(a)\\
&+\Big\langle \langle\varphi_i(c),\varphi_p(b)\rangle\varphi_s(a),x\Big\rangle\langle\varphi_u(a),\varphi_r(b)\rangle\varphi_k(c)\Big).
\end{aligned}
$$

Canceling the like terms from both sides of the above equation and then arranging them, we get

$$
\begin{aligned}
&\langle\Delta_n(a,b,c),x\rangle\langle c,b\rangle a+\langle\langle c,b\rangle a,x\rangle\Delta_n(a,b,c)\\
&+\langle\Delta_n(c,b,a),x\rangle\langle a,b\rangle c+\langle\langle a,b\rangle c,x\rangle\Delta_n(c,b,a)=0,
\end{aligned}
$$

for all $x\in\mathcal{M}$. A similar argument as used for $n=1$, shows that

$$\Big\langle\Delta_n(a,b,c),x\Big\rangle\Omega(a,b,c)=\Big\langle\Omega(a,b,c),x\Big\rangle\Delta_n(a,b,c)=0,$$

for all $x\in\mathcal{M}$. It follows from primeness of $\mathcal{M}$ that $\Delta_n(a,b,c)=0$ or $\Omega(a,b,c)=0$. In each case, it follows that the equation (2.7) holds for $\ell=n$. This proves that$\{\varphi_n\}_{n=0}^{+\infty}$ is a Hilbert C*-module higher derivation on $\mathcal{M}$. □

The next corollary follows from Theorem 2.1.

Corollary 2.1. *Let $\mathcal{M}$ be a 2-torsion-free prime Hilbert C*-module. Then every Hilbert C*-module Jordan derivation on $\mathcal{M}$ is a Hilbert C*-module derivation on $\mathcal{M}$.*

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