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ON THE METRIC DIMENSION OF CIRCULANT GRAPHS WITH 2 GENERATORS

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Abstract. A set of vertices W resolves a connected graph G if every vertex of G is uniquely determined by its vector of distances to the vertices in W. The number of vertices in a smallest resolving set is called the metric dimension and it is denoted by $\dim(G)$. We study the circulant graphs $C_n(2,3)$ with the vertices $v_0, v_1, v_2, \dots, v_{n-1}$ and the edges $v_i v_{i+2}, v_i v_{i+3}$, where $i = 0, 1, 2, \dots, n-1$, the indices are taken modulo n. We show that for $n \geq 26$ we have $\dim(C_n(2,3)) = 3$ if $n \equiv 4 \pmod 6$, $\dim(C_n(2,3)) = 4$ if $n \equiv 0, 1, 5 \pmod 6$ and $3 \leq \dim(C_n(2,3)) \leq 4$ if $n \equiv 2, 3 \pmod 6$.

1. Introduction

Let G be a connected graph with the vertex set $V(G)$. The distance $d(u,v)$ between two vertices $u, v \in V(G)$ is the number of edges in a shortest path between them. A vertex w resolves a pair of vertices u, v if $d(u,w) \neq d(v,w)$. For an ordered set of z vertices $W = \{w_1, w_2, \dots, w_z\}$, the representation of distances of a vertex v with respect to W is the ordered z-tuple

$$r(v|W) = (d(v,w_1), d(v,w_2), \dots, d(v,w_z)).$$

A set of vertices $W \subset V(G)$ is a resolving set of G if every two vertices of G have distinct representations (if every pair of vertices of G is resolved by some vertex of W). The number of vertices in a smallest resolving set is called the metric dimension and it is denoted by $\dim(G)$. Note that the i-th coordinate in $r(v|W)$ is 0 if and only if $v = w_i$. Hence, to show that W is a resolving set of G, it suffices to verify that $r(u|W) \neq r(v|W)$ for every pair of distinct vertices $u, v \in V(G) \setminus W$.

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The concept of metric dimension was introduced by Slater [8], and Harary and Melter [3]. The metric dimension of various classes of graphs has been investigated for four decades. For example, products of graphs were considered in [7], subdivisions of Cayley graphs in [1] and from [2] it follows that the question whether the metric dimension of a graph is less than a given value, is an NP-complete problem.

We study the metric dimension of circulant graphs. Let n, m and $a_1,\ a_2, \dots, a_m$ be positive integers, such that $1 \le a_1 < a_2 < \cdots < a_m \le \frac{n}{2}$. The circulant graph $C_n(a_1, a_2, \dots, a_m)$ consists of the vertices $v_0, v_1, \dots, v_{n-1}$ and the edges $v_i v_{i+a_j}$ where $i = 0, 1, \dots, n-1$ and $j = 1, 2, \dots, m$, the indices are taken modulo n. The numbers $a_1, a_2, \dots, a_m$ are called generators. The graph $C_n(a_1, a_2, \dots, a_m)$ is a regular graph either of degree $2m$ if all generators are smaller than $\frac{n}{2}$, or of degree $2m-1$ if $\frac{n}{2}$ is one of the generators.

Javaid, Rahim and Ali [5] obtained the following results:

$$\begin{aligned} &\dim(C_n(1,2)) = 3, && \text{if } n \equiv 0, 2, 3 \pmod 4,\\ &\dim(C_n(1,2)) \le 4, && \text{if } n \equiv 1 \pmod 4. \end{aligned}$$

The metric dimension of the circulant graphs $C_n(1,3)$ was studied by Javaid, Azhar and Salman [4]. They showed that for any $n \ge 5$

$$\begin{aligned} &\dim(C_n(1,3)) = 3, && \text{if } n \equiv 1 \pmod 6,\\ &\dim(C_n(1,3)) = 4, && \text{if } n \equiv 0, 3, 4, 5 \pmod 6,\\ 4 \le\ &dim(C_n(1,3)) \le 6, && \text{if } n \equiv 2 \pmod 6. \end{aligned}$$

If n is even, then the graphs $C_n(1, \frac{n}{2})$ are 3-regular and in [6] it was proved that

$$\dim\left(C_n\left(1, \frac{n}{2}\right)\right) = \begin{cases} 3, & \text{if } n \equiv 0 \pmod 4,\\ 4, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

We extend known results on the metric dimension of circulant graphs by showing that for $n \ge 26$ we have

$$\dim(C_n(2,3)) = \begin{cases} 3, & \text{if } n \equiv 4 \pmod 6,\\ 3 \text{ or } 4, & \text{if } n \equiv 2, 3 \pmod 6,\\ 4, & \text{if } n \equiv 0, 1, 5 \pmod 6. \end{cases}$$

Let us note that the distance between two vertices v_i and v_j in $C_n(2,3)$ is

$$d(v_i, v_j) = \begin{cases} \left\lceil \dfrac{|i-j|}{3} \right\rceil, & \text{if } 2 \le |i-j| \le \dfrac{n}{2},\\[2ex] \left\lceil \dfrac{n-|i-j|}{3} \right\rceil, & \text{if } \dfrac{n}{2} < |i-j| \le n-2, \end{cases} \tag{1.1}$$

and $d(v_i, v_{i+1}) = 2$ for $i = 0, 1, 2, \dots, n-1$.

2. Lower Bounds on $C_n(2,3)$

First we present a lemma, which will be used in the proofs of main results of this section.

Lemma 2.1. *Three vertices $v_i, v_{i+1}, v_{i+2} \in V(C_n(2,3))$, where $0 \le i \le n-1$, can be resolved by one vertex w if and only if $w = v_i$ or $w = v_{i+2}$.*

Proof. Let v_j be any vertex of $C_n(2,3)$. From the definition of the graph $C_n(2,3)$ it follows that the distances $d(v_j, v_i)$, $d(v_j, v_{i+1})$, $d(v_j, v_{i+2})$ are pairwise different if and only if $j = i$ or $j = i+2$. ☐

If $4 \le n \le 6$, then the graph $C_n(2,3)$ contains multiple edges, thus we study the metric dimension of $C_n(2,3)$ for $n \ge 7$. We show that the graph $C_n(2,3)$ does not contain a resolving set, which consists of 2 vertices.

Theorem 2.1. *Let $n \ge 7$. Then* $\dim(C_n(2,3)) \ge 3$.

Proof. We prove the result by contradiction. Suppose that the graph $C_n(2,3)$ contains a resolving set, which consists of two different vertices v_i, v_j, where $i, j \in \{0, 1, 2, \ldots, n-1\}$. Without loss of generality we can assume that $i = 0$. We distinguish two cases.

Case 1: $7 \le n \le 13$.

The distance between v_0 and any vertex in $V' = \{v_2, v_3, v_{n-3}, v_{n-2}\}$ is 1. Since for $n \le 13$ the distance between any two vertices of $C_n(2,3)$ is at most 2, we have $0 \le d(v_j, v') \le 2$ for any vertex $v' \in V'$. This implies that there are two vertices in V', which are of the same distance from v_j, hence they are not resolved.

Case 2: $n \ge 14$.

Then $d(v_4, v_0) = d(v_5, v_0) = d(v_6, v_0) = 2$ and $d(v_{n-4}, v_0) = d(v_{n-5}, v_0) = d(v_{n-6}, v_0) = 2$. By Lemma 2.1, the vertices v_4, v_5, v_6 can be resolved by v_j only if $j = 4$ or 6. Similarly, the vertices $v_{n-4}, v_{n-5}, v_{n-6}$ can be resolved by v_j only if $j = n-4$ or $j = n-6$. Since $\{4, 6\} \cap \{n-4, n-6\} = \emptyset$, we have a contradiction. The proof is complete. ☐

For two vertices $v_i, v_j \in V(C_n(2,3))$ let $V_{i,j} = \{v_{i+1}, v_{i+2}, \ldots, v_j\}$ if $i < j$, and let $V_{i,j} = \{v_{i+1}, v_{i+2}, \ldots, v_{n-1}, v_0, v_1, \ldots v_j\}$ if $i > j$. Let us also define $d_{i,j}$ to be the number of vertices in $V_{i,j}$. Note that if $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$ are any vertices of $C_n(2,3)$, such that $0 \le i_1 < i_2 < \cdots < i_r \le n-1$ and $r \ge 2$, then

$$V_{i_1,i_2} \cup V_{i_2,i_3} \cup \cdots \cup V_{i_{r-1},i_r} \cup V_{i_r,i_1} = V(C_n(2,3))$$

and

$$d_{i_1,i_2} + d_{i_2,i_3} + \cdots + d_{i_{r-1},i_r} + d_{i_r,i_1} = n. \tag{2.1}$$

We improve the lower bound presented in Theorem 2.1 if $n \equiv q \pmod 6$ where $q = 0, 1, 5$.

Theorem 2.2. *Let $n \equiv q \pmod 6$ where $n \geq 29$ and $q = 0, 1, 5$. Then*

$$\dim(C_n(2,3)) \geq 4.$$

Proof. Let $n = 6k + p$ where $k \geq 5$ and $p = -1, 0, 1$. We show by contradiction that three vertices cannot resolve the graph $C_n(2,3)$. Suppose that $W = \{v_a, v_i, v_j\}$ is a resolving set of $C_n(2,3)$ where $0 \leq a < i < j \leq n-1$. Without loss of generality we can assume that $d_{a,i} \leq d_{i,j} \leq d_{j,a}$ Note that $d_{a,i} \leq \frac{n}{3}$, otherwise we have $d_{a,i} + d_{i,j} + d_{j,a} > n$, which contradicts (2.1). Due to the symmetry in the graph, we can assume that $v_a = v_0$ and $1 \leq i \leq \frac{n}{3}$ (which means that $i \leq 2k$). Then $j \leq \frac{2n}{3}$, otherwise we have $d_{i,j} > \frac{n}{3}$ and $d_{j,a} < \frac{n}{3}$, which contradicts the fact that $d_{i,j} \leq d_{j,a}$. Let $W' = \{v_0, v_i\}$. We consider three cases.

Case 1: $i \equiv 0 \pmod 3$.

By (1.1), we obtain $r(v_{3k-2}|W') = r(v_{3k-1}|W') = r(v_{3k}|W') = \left(k, k - \frac{i}{3}\right)$. From Lemma 2.1 we knopw that the vertices $v_{3k-2}, v_{3k-1}, v_{3k}$ can be resolved by v_j only if $j = 3k-2$ or $j = 3k$. This also implies that $i \leq \frac{j}{2} \leq \frac{3k}{2}$. Consequently $r(v_{3k-5}|W) = r(v_{3k-4}|W) = \left(k-1, k-1-\frac{i}{3}, 1\right)$ if $j = 3k-2$, and $r(v_{3k-5}|W) = r(v_{3k-4}|W) = \left(k-1, k-1-\frac{i}{3}, 2\right)$ if $j = 3k$, which means that the vertices v_{3k-5} and v_{3k-4} are not resolved by W.

Case 2: $i \equiv 1 \pmod 3$.

We obtain $r(v_{3k-1}|W') = r(v_{3k}|W') = r(v_{3k+1}|W') = \left(k, k + \frac{1-i}{3}\right)$. We need to resolve the vertices $v_{3k-1}, v_{3k}, v_{3k+1}$ by v_j. From Lemma 2.1, it follows that $j = 3k-1$ or $j = 3k+1$. By (1.1), if $j = 3k-1$, then $r(v_{3k-4}|W) = r(v_{3k-3}|W) = \left(k-1, k-1+\frac{1-i}{3}, 1\right)$, and if $j = 3k+1$, we have $r(v_{3k-4}|W) = r(v_{3k-3}|W) = \left(k-1, k-1+\frac{1-i}{3}, 2\right)$. A contradiction.

Case 3: $i \equiv 2 \pmod 3$.

If $p = 0$ or 1, then $r(v_{3k}|W') = r(v_{3k+1}|W') = r(v_{3k+2}|W') = \left(k, k + \frac{2-i}{3}\right)$, thus by Lemma 2.1, $v_j = v_{3k}$ or $v_j = v_{3k+2}$ $\left(\text{so } i \text{ is at most } \frac{3k}{2} + 1\right)$. Since $r(v_{3k-5}|W) = r(v_{3k-4}|W) = \left(k-1, k-1+\frac{2-i}{3}, 2\right)$ if $j = 3k$, and $r(v_{3k-4}|W) = r(v_{3k-3}|W) = \left(k-1, k-1+\frac{2-i}{3}, 2\right)$ if $j = 3k+2$, the set W cannot resolve the graph $C_n(2,3)$.

It remains to consider the case $p = -1$ (if $i \equiv 2 \pmod 3$). Then

$$\begin{aligned}
r(v_{3k-5}|W') = r(v_{3k-4}|W') &= \left(k-1, k-2+\frac{2-i}{3}\right),\\
r(v_{3k-2}|W') = r(v_{3k-1}|W') &= \left(k, k-1+\frac{2-i}{3}\right),\\
r(v_{3k}|W') = r(v_{3k+1}|W') &= \left(k, k+\frac{2-i}{3}\right),\\
r(v_{6k-4}|W') = r(v_{6k-3}|W') &= \left(1, \frac{i+1}{3}+1\right).
\end{aligned}$$

Let us distinguish 3 subcases.

Case 3a: $j \equiv 0 \pmod 3$.

Then $d(v_j, v_{3k-2}) = d(v_j, v_{3k-1}) = k - \frac{j}{3}$ if $j < 3k-3$, $d(v_j, v_{3k-5}) = d(v_j, v_{3k-4}) = \frac{j}{3} - k + 2$ if $j > 3k-3$, and $d(v_{3k-3}, v_{6k-4}) = d(v_{3k-3}, v_{6k-3}) = k$, which means that W cannot resolve the graph $C_n(2,3)$.

Case 3b: $j \equiv 1 \pmod 3$.

If $j \le 3k-2$, then $d(v_j, v_{3k}) = d(v_j, v_{3k+1}) = k + \frac{1-j}{3}$, and if $j \ge 3k+1$, then $d(v_j, v_{3k-2}) = d(v_j, v_{3k-1}) = \frac{j-1}{3} - k + 1$.

Case 3c: $j \equiv 2 \pmod 3$.

If $j < 3k-1$, then $d(v_j, v_{3k}) = d(v_j, v_{3k+1}) = k + \frac{2-j}{3}$, and if $j > 3k+2$, then $d(v_j, v_{3k}) = d(v_j, v_{3k+1}) = \frac{j+1}{3} - k$.

Let $j = 3k-1$. Then $r(v_2|W) = r(v_3|W) = \left(1, \frac{i-2}{3}, k-1\right)$ if $i \ge 5$, and $r(v_3|W) = r(v_{6k-4}|W) = (1, 2, k-1)$ if $i = 2$.

Let $j = 3k+2$. Then $r(v_2|W) = r(v_3|W) = \left(1, \frac{i-2}{3}, k\right)$ if $i \ge 5$, and $r(v_5|W) = r(v_{6k-2}|W) = (2, 1, k-1)$ if $i = 2$. Hence, $C_n(2,3)$ cannot be resolved by three vertices, which means that $\dim(C_n(2,3)) \ge 4$. $\square$

3. Resolving Sets of $C_n(2,3)$

In this section we present resolving sets, which yield upper bound on the metric dimension of $C_n(2,3)$. We show that there exists an infinite set of graphs $C_n(2,3)$ containing a resolving set, which consists of 3 vertices.

Theorem 3.1. *Let $n \equiv 4 \pmod 6$ where $n \ge 22$. Then $\dim(C_n(2,3)) \le 3$.*

Proof. Let $n = 6k+4$ where $k \ge 3$. We show that $W = \{v_0, v_4, v_8\}$ is a resolving set of $C_n(2,3)$. We give representations of distances of all vertices in $V(C_n(2,3)) \setminus W$ with respect to W:

$$
\begin{array}{ll}
r(v_{3i-2}|W) = (i, i-2, i-3), & 4 \le i \le k+1, \\
r(v_{3i-1}|W) = (i, i-1, i-3), & 4 \le i \le k+1, \\
r(v_{3i}|W) = (i, i-1, i-2), & 4 \le i \le k+1, \\
r(v_{6k-3i+4}|W) = (i, i+2, i+3), & 1 \le i \le k-2, \\
r(v_{6k-3i+5}|W) = (i, i+1, i+3), & 1 \le i \le k-2, \\
r(v_{6k-3i+6}|W) = (i, i+1, i+2), & 2 \le i \le k-1, \\
r(v_1|W) = (2, 1, 3), & r(v_2|W) = (1, 1, 2), \\
r(v_3|W) = (1, 2, 2), & r(v_5|W) = (2, 2, 1), \\
r(v_6|W) = (2, 1, 1), & r(v_7|W) = (3, 1, 2), \\
r(v_9|W) = (3, 2, 2), & r(v_{6k+3}|W) = (2, 2, 3), \\
r(v_{3k+4}|W) = (k, k, k-1), & r(v_{3k+5}|W) = (k, k+1, k-1),
\end{array}
$$

$$r(v_{3k+6}|W) = (k, k+1, k), \qquad r(v_{3k+7}|W) = (k-1, k+1, k),$$
$$r(v_{3k+8}|W) = (k-1, k, k).$$

Since no two vertices in $V(C_n(2,3))\backslash W$ have the same representations, W is a resolving set for $C_n(2,3)$. Thus $\dim(C_n(2,3)) \leq 3$. ☐

In the next three theorems we present resolving sets of $C_n(2,3)$ consisting of 4 vertices.

Theorem 3.2. *Let $n \equiv q \pmod 6$ where $n \geq 15$ and $q = 2, 3, 5$. Then*

$$\dim(C_n(2,3)) \leq 4.$$

Proof. Let $n = 6k + q$ where $n \geq 15$ and $q = 2, 3, 5$. Let us show that $W = \{v_0, v_1, v_2, v_6\}$ is a resolving set of $C_n(2,3)$. First we give representations of distances of the vertices v_i for $3 \leq i \leq 3k+1$ and $3k+q+1 \leq i \leq 6k+q-1$ with respect to $W' = \{v_0, v_1, v_2\} \subset W$ (see Table 1).

Table 1. Representations of distances of the vertices v_i for $3 \leq i \leq 3k+1$ and $3k+q+1 \leq i \leq 6k+q-1$ with respect to W'

Representation	v_0	v_1	v_2
v_3	1	1	2
v_{3i-2} $(2 \leq i \leq k+1)$	i	$i-1$	$i-1$
v_{3i-1} $(2 \leq i \leq k)$	i	i	$i-1$
v_{3i} $(2 \leq i \leq k)$	i	i	i
$v_{6k-3i+q}$ $(1 \leq i \leq k-1)$	i	$i+1$	$i+1$
$v_{6k-3i+q+1}$ $(1 \leq i \leq k)$	i	i	$i+1$
$v_{6k-3i+q+2}$ $(2 \leq i \leq k)$	i	i	i
v_{6k+q-1}	2	1	1

The only vertices, which have the same representations are the following pairs: v_3, v_{6k+q-2}; v_4, v_{6k+q-1} and $v_{3i}, v_{6k-3i+q+2}$ for $2 \leq i \leq k$. The vertex v_6 resolves all these pairs, since

$$d(v_6, v_3) = 1, \qquad d(v_6, v_{6k+q-2}) = 3,$$
$$d(v_6, v_4) = 1, \qquad d(v_6, v_{6k+q-1}) = 3,$$
$$d(v_6, v_{3i}) = i-2, \text{ for } 2 \leq i \leq k,$$
$$d(v_6, v_{6k-3i+q+2}) = i+2, \text{ for } 2 \leq i \leq k-1,$$
$$d(v_6, v_{3k+q+2}) = \begin{cases} i, & \text{if } q = 2 \text{ or } 3, \\ i+1, & \text{if } q = 5. \end{cases}$$

Let us present representations of the vertices in $V(C_n(2,3)) \setminus W'$, which are not given in Table 1.

If $n = 6k + 2$, then

$$r(v_{3k+2}|W') = (k, k+1, k),$$

and if $n = 6k + 3$, we have

$$r(v_{3k+2}|W') = (k+1, k+1, k),$$
$$r(v_{3k+3}|W') = (k, k+1, k+1).$$

Since these representations are different from the representation of any vertex in Table 1, W is a resolving set of $C_n(2,3)$ if $q = 2$ or 3.

If $n = 6k + 5$, we obtain

$$r(v_{3k+2}|W') = (k+1, k+1, k),$$
$$r(v_{3k+3}|W') = r(v_{3k+4}|W') = (k+1, k+1, k+1),$$
$$r(v_{3k+5}|W') = (k, k+1, k+1).$$

Since $d(v_6, v_{3k+3}) = k - 1$ and $d(v_6, v_{3k+4}) = k$, we have $\dim(C_n(2,3)) \leq 4$ for $q = 5$ too. □

Theorem 3.3. *Let* $n \equiv 0 \pmod 6$ *where* $n \geq 12$. *Then* $\dim(C_n(2,3)) \leq 4$.

Proof. Let $n = 6k$ where $k \geq 2$, and let $W' = \{v_0, v_1, v_2\}$. We consider representations of distances of the vertices $v_{3i-2}, v_{3i-1}, v_{3i}$ and $v_{6k-3i+q+2}$ with respect to W' given in Table 1 for $2 \leq i \leq k$ and $q = 0$. Similarly, consider representations of the vertices $v_{6k-3i+q}$ and $v_{6k-3i+q+1}$ given in Table 1 for $1 \leq i \leq k-1$ and $q = 0$. It remains to give the representation of v_{3k+1}, which is $r(v_{3k+1}|W') = (k, k, k)$.

Let us present all vertices of $C_n(2,3)$, which have the same representations of distances with respect to W':

$$r(v_3|W') = r(v_{6k-2}|W') = (1, 1, 2),$$
$$r(v_4|W') = r(v_{6k-1}|W') = (2, 1, 1),$$
$$r(v_{3i}|W') = r(v_{6k-3i+2}|W') = (i, i, i), \text{ for } 2 \leq i \leq k-1,$$
$$r(v_{3k}|W') = r(v_{3k+1}|W') = r(v_{3k+2}|W') = (k, k, k).$$

We show that the vertex v_{3k} resolves all these vertices.

$$\begin{aligned}
&d(v_{3k}, v_3) = k - 1, && d(v_{3k}, v_{6k-2}) = k,\\
&d(v_{3k}, v_4) = k - 1, && d(v_{3k}, v_{6k-1}) = k,\\
&d(v_{3k}, v_{3i}) = k - i, && d(v_{3k}, v_{6k-3i+2}) = k - i + 1, \text{ for } 2 \leq i \leq k-1,\\
&d(v_{3k}, v_{3k+1}) = 2, && d(v_{3k}, v_{6k+2}) = 1.
\end{aligned}$$

Thus $W = \{v_0, v_1, v_2, v_{3k}\}$ is a resolving set of the graph $C_n(2,3)$, which implies that $\dim(C_n(2,3)) \leq 4$. □

Theorem 3.4. *Let* $n \equiv 1 \pmod 6$ *where* $n \geq 13$. *Then* $\dim(C_n(2,3)) \leq 4$.

TABLE 2. Representations of distances of all vertices in $V(C_n(2,3))\setminus W$ with respect to W

Representation	v_0	v_1	v_{3k-1}	v_{3k}
v_2	1	2	$k-1$	k
v_{3i-2} $(2 \leq i \leq k-1)$	i	$i-1$	$k-i+1$	$k-i+1$
v_{3i-1} $(2 \leq i \leq k-1)$	i	i	$k-i$	$k-i+1$
v_{3i} $(1 \leq i \leq k-1)$	i	i	$k-i$	$k-i$
v_{3k-2}	k	$k-1$	2	1
v_{3k+1}	k	k	1	2
$v_{6k-3i+1}$ $(1 \leq i \leq k-1)$	i	$i+1$	$k-i+1$	$k-i+1$
$v_{6k-3i+2}$ $(1 \leq i \leq k)$	i	i	$k-i+1$	$k-i+1$
$v_{6k-3i+3}$ $(2 \leq i \leq k)$	i	i	$k-i+2$	$k-i+1$

Proof. Let $n = 6k+1$ where $k \geq 2$. We show that $W = \{v_0, v_1, v_{3k-1}, v_{3k}\}$ is a resolving set of $C_n(2,3)$. Representations of distances of all vertices in $V(C_n(2,3)) \setminus W$ with respect to W are given in Table 2.

Any two vertices have different representations, hence W is a resolving set of $C_n(2,3)$ and $\dim(C_n(2,3)) \leq 4$. □

4. CONCLUSION

In Section 3 we presented resolving sets of $C_n(2,3)$ except for a few small values of n. Resolving sets for those values of n, which are not included in our theorems, are given in Table 3.

TABLE 3. Resolving sets of $C_n(2,3)$ for $n = 7, 8, 9, 10, 11, 14$ and 16

$C_n(2,3)$	Resolving set
$n=7$	$\{v_0, v_1, v_2\}$
$n=8$	$\{v_0, v_1, v_4\}$
$n=9$	$\{v_0, v_1, v_2, v_4\}$
$n=10$	$\{v_0, v_1, v_2, v_3, v_4\}$
$n=11$	$\{v_0, v_1, v_3, v_4\}$
$n=14$	$\{v_0, v_1, v_2, v_6\}$
$n=16$	$\{v_0, v_1, v_2, v_6\}$

We carefully checked that these resolving sets are the smallest ones (this can be checked also by computer programs). Note that $n = 10$ is the only case, such that $\dim(C_n(2,3)) > 4$. The case $n = 14$ could be included in the proof of Theorem 3.2, but we would have to consider a short part of the proof of Theorem 3.2 separately for this case.

From our theorems presented in Sections 2 and 3 we obtain Table 4. Our results yield exact values of the metric dimension of $C_n(2,3)$ if $n \equiv q \pmod 6$ where $q = 0, 1, 4, 5$.

TABLE 4. Lower and upper bounds on $\dim(C_n(2,3))$

$\dim(C_n(2,3))$	Lower bound for $n \geq 26$	Upper bound for $n \geq 17$
$n \equiv 0 \pmod 6$	4	4
$n \equiv 1 \pmod 6$	4	4
$n \equiv 2 \pmod 6$	3	4
$n \equiv 3 \pmod 6$	3	4
$n \equiv 4 \pmod 6$	3	3
$n \equiv 5 \pmod 6$	4	4

We have

$$\dim(C_n(2,3)) = 3, \text{ for } n \equiv 4 \pmod 6, \text{ where } n \geq 22,$$
$$\dim(C_n(2,3)) = 4, \text{ for } n \equiv q \pmod 6, \text{ where } n \geq 29 \text{ and } q = 0,1,5.$$

It would be interesting to know exact values of $\dim(C_n(2,3))$ also for $n \equiv 2$ or $3 \pmod 6$. We conjecture that all resolving sets presented in this paper are the smallest ones, thus we close this section by presenting the following conjecture.

Conjecture 4.1. Let $n \equiv 2$ or $3 \pmod 6$, where $n \geq 9$. Then $\dim(C_n(2,3)) = 4$.

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REFERENCES

[1] A. Ahmad, M. Imran, O. Al-Mushayt and S. Bokhary, *On the metric dimension of barycentric subdivision of Cayley graphs* Cay$(Z_n \oplus Z_m)$, Miskolc Math. Notes **16** (2016), 637–646.
[2] M. R. Garey and D. S. Johnson, *Computers and Intractability. A Guide to the Theory of NP-completeness*, A Series of Books in the Mathematical Sciencesx, W. H. Freeman and Company X, San Francisco, 1979, 338 p.
[3] F. Harary and R. A. Melter, *On the metric dimension of a graph*, Ars Combin. **2** (1976), 191–195.
[4] I. Javaid, N. M. Azhar and M. Salman, *Metric dimension and determining number of Cayley graphs*, World Applied Sciences Journal **18** (2012), 1800–1812.
[5] I. Javaid, M. Tariq Rahim and K. Ali, *Families of regular graphs with constant metric dimension*, Util. Math. **75** (2008), 21–33.
[6] M. Salman, I. Javaid and M. A. Chaudhry, *Resolvability in circulant graphs*, Acta Math. Sin. (Engl. Ser.) **28** (2012), 1851–1864.
[7] S. Saputro, R. Simanjuntak, S. Uttunggadewa, H. Assiyatun, E. Baskoro, A. Salman and M. Bača, *The metric dimension of the lexicographic product of graphs*, Discrete Math. **313** (2013), 1045–1051.
[8] P. J. Slater, *Leaves of trees*, Proc. 6th Southeastern Conference on Combinatorics, Graph Theory, Computing, Boca Raton, 1975.

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