ON THE METRIC DIMENSION OF CIRCULANT GRAPHS WITH 2 GENERATORS

L. DU TOIT\textsuperscript{1} AND T. VETRÍK\textsuperscript{2}

Abstract. A set of vertices $W$ resolves a connected graph $G$ if every vertex of $G$ is uniquely determined by its vector of distances to the vertices in $W$. The number of vertices in a smallest resolving set is called the metric dimension and it is denoted by $\dim(G)$. We study the circulant graphs $C_n(2,3)$ with the vertices $v_0, v_1, v_2, \ldots, v_{n-1}$ and the edges $v_iv_{i+2}, v_iv_{i+3}$, where $i = 0, 1, 2, \ldots, n-1$, the indices are taken modulo $n$. We show that for $n \geq 26$ we have $\dim(C_n(2,3)) = 3$ if $n \equiv 4 \pmod{6}$, $\dim(C_n(2,3)) = 4$ if $n \equiv 0, 1, 5 \pmod{6}$ and $3 \leq \dim(C_n(2,3)) \leq 4$ if $n \equiv 2, 3 \pmod{6}$.

1. Introduction

Let $G$ be a connected graph with the vertex set $V(G)$. The distance $d(u, v)$ between two vertices $u, v \in V(G)$ is the number of edges in a shortest path between them. A vertex $w$ resolves a pair of vertices $u, v$ if $d(u, w) \neq d(v, w)$. For an ordered set of $z$ vertices $W = \{w_1, w_2, \ldots, w_z\}$, the representation of distances of a vertex $v$ with respect to $W$ is the ordered $z$-tuple

$$r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_z)).$$

A set of vertices $W \subseteq V(G)$ is a resolving set of $G$ if every two vertices of $G$ have distinct representations (if every pair of vertices of $G$ is resolved by some vertex of $W$). The number of vertices in a smallest resolving set is called the metric dimension and it is denoted by $\dim(G)$. Note that the $i$-th coordinate in $r(v|W)$ is 0 if and only if $v = w_i$. Hence, to show that $W$ is a resolving set of $G$, it suffices to verify that $r(u|W) \neq r(v|W)$ for every pair of distinct vertices $u, v \in V(G) \setminus W$.

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The concept of metric dimension was introduced by Slater [8], and Harary and Melter [3]. The metric dimension of various classes of graphs has been investigated for four decades. For example, products of graphs were considered in [7], subdivisions of Cayley graphs in [1] and from [2] it follows that the question whether the metric dimension of a graph is less than a given value, is an NP-complete problem.

We study the metric dimension of circulant graphs. Let \( a_1, a_2, \ldots, a_m \) be positive integers, such that \( 1 \leq a_1 < a_2 < \cdots < a_m \leq \frac{n}{2} \). The circulant graph \( C_n(a_1, a_2, \ldots, a_m) \) consists of the vertices \( v_0, v_1, \ldots, v_{n-1} \) and the edges \( v_i v_{i+a_j} \) where \( i = 0, 1, \ldots, n-1 \) and \( j = 1, 2, \ldots, m \), the indices are taken modulo \( n \). The numbers \( a_1, a_2, \ldots, a_m \) are called generators. The graph \( C_n(a_1, a_2, \ldots, a_m) \) is a regular graph of degree \( 2m \) if all generators are smaller than \( \frac{n}{2} \), or of degree \( 2m - 1 \) if \( \frac{n}{2} \) is one of the generators.

Javaid, Rahim and Ali [5] obtained the following results:

\[
\begin{align*}
\dim(C_n(1, 2)) &= 3, & \text{if } n \equiv 0, 2, 3 \pmod{4}, \\
\dim(C_n(1, 2)) &\leq 4, & \text{if } n \equiv 1 \pmod{4}.
\end{align*}
\]

The metric dimension of the circulant graphs \( C_n(1, 3) \) was studied by Javaid, Azhar and Salman [4]. They showed that for any \( n \geq 5 \)

\[
\begin{align*}
\dim(C_n(1, 3)) &= 3, & \text{if } n \equiv 1 \pmod{6}, \\
\dim(C_n(1, 3)) &= 4, & \text{if } n \equiv 0, 3, 4, 5 \pmod{6}, \\
4 &\leq \dim(C_n(1, 3)) \leq 6, & \text{if } n \equiv 2 \pmod{6}.
\end{align*}
\]

If \( n \) is even, then the graphs \( C_n(1, \frac{n}{2}) \) are 3-regular and in [6] it was proved that

\[
\dim\left(C_n\left(1, \frac{n}{2}\right)\right) = \begin{cases} 
3, & \text{if } n \equiv 0 \pmod{4}, \\
4, & \text{if } n \equiv 2 \pmod{4}.
\end{cases}
\]

We extend known results on the metric dimension of circulant graphs by showing that for \( n \geq 26 \) we have

\[
\dim(C_n(2, 3)) = \begin{cases} 
3, & \text{if } n \equiv 4 \pmod{6}, \\
3 \text{ or } 4, & \text{if } n \equiv 2, 3 \pmod{6}, \\
4, & \text{if } n \equiv 0, 1, 5 \pmod{6}.
\end{cases}
\]

Let us note that the distance between two vertices \( v_i \) and \( v_j \) in \( C_n(2, 3) \) is

\[
d(v_i, v_j) = \begin{cases} 
\left\lfloor \frac{|i - j|}{3} \right\rfloor, & \text{if } 2 \leq |i - j| \leq \frac{n}{2}, \\
\left\lfloor \frac{n - |i - j|}{3} \right\rfloor, & \text{if } \frac{n}{2} < |i - j| \leq n - 2,
\end{cases}
\]

and \( d(v_i, v_{i+1}) = 2 \) for \( i = 0, 1, 2, \ldots, n - 1 \).
2. Lower Bounds on \( C_n(2, 3) \)

First we present a lemma, which will be used in the proofs of main results of this section.

**Lemma 2.1.** Three vertices \( v_i, v_{i+1}, v_{i+2} \in V(C_n(2, 3)) \), where \( 0 \leq i \leq n - 1 \), can be resolved by one vertex \( w \) if and only if \( w = v_i \) or \( w = v_{i+2} \).

**Proof.** Let \( v_j \) be any vertex of \( C_n(2, 3) \). From the definition of the graph \( C_n(2, 3) \) it follows that the distances \( d(v_j, v_i), d(v_j, v_{i+1}), d(v_j, v_{i+2}) \) are pairwise different if and only if \( j = i \) or \( j = i + 2 \).

If \( 4 \leq n \leq 6 \), then the graph \( C_n(2, 3) \) contains multiple edges, thus we study the metric dimension of \( C_n(2, 3) \) for \( n \geq 7 \). We show that the graph \( C_n(2, 3) \) does not contain a resolving set, which consists of 2 vertices.

**Theorem 2.1.** Let \( n \geq 7 \). Then \( \dim(C_n(2, 3)) \geq 3 \).

**Proof.** We prove the result by contradiction. Suppose that the graph \( C_n(2, 3) \) contains a resolving set, which consists of two different vertices \( v_i, v_j \), where \( i, j \in \{0, 1, 2, \ldots, n - 1\} \). Without loss of generality we can assume that \( i = 0 \). We distinguish two cases.

Case 1: \( 7 \leq n \leq 13 \).

The distance between \( v_0 \) and any vertex in \( V' = \{v_2, v_3, v_{n-3}, v_{n-2}\} \) is 1. Since for \( n \leq 13 \) the distance between any two vertices of \( C_n(2, 3) \) is at most 2, we have \( 0 \leq d(v_j, v') \leq 2 \) for any vertex \( v' \in V' \). This implies that there are two vertices in \( V' \), which are of the same distance from \( v_j \), hence they are not resolved.

Case 2: \( n \geq 14 \).

Then \( d(v_4, v_0) = d(v_5, v_0) = d(v_6, v_0) = 2 \) and \( d(v_{n-4}, v_0) = d(v_{n-5}, v_0) = d(v_{n-6}, v_0) = 2 \). By Lemma 2.1, the vertices \( v_4, v_5, v_6 \) can be resolved by \( v_j \) only if \( j = 4 \) or 6. Similarly, the vertices \( v_{n-4}, v_{n-5}, v_{n-6} \) can be resolved by \( v_j \) only if \( j = n - 4 \) or \( j = n - 6 \). Since \( \{4, 6\} \cap \{n - 4, n - 6\} = \emptyset \), we have a contradiction. The proof is complete.

For two vertices \( v_i, v_j \in V(C_n(2, 3)) \) let \( V_{i,j} = \{v_{i+1}, v_{i+2}, \ldots, v_j\} \) if \( i < j \), and let \( V_{i,j} = \{v_{j-1}, v_{j-2}, \ldots, v_0, v_1, v_2, \ldots, v_j\} \) if \( i > j \). Let us also define \( d_{i,j} \) to be the number of vertices in \( V_{i,j} \). Note that if \( v_{i_1}, v_{i_2}, \ldots, v_{i_r} \) are any vertices of \( C_n(2, 3) \), such that \( 0 \leq i_1 < i_2 < \cdots < i_r \leq n - 1 \) and \( r \geq 2 \), then

\[
V_{i_1,i_2} \cup V_{i_2,i_3} \cup \cdots \cup V_{i_{r-1},i_r} \cup V_{i_r,i_1} = V(C_n(2, 3))
\]

and

\[
d_{i_1,i_2} + d_{i_2,i_3} + \cdots + d_{i_{r-1},i_r} + d_{i_r,i_1} = n.
\]

We improve the lower bound presented in Theorem 2.1 if \( n \equiv q \pmod{6} \) where \( q = 0, 1, 5 \).
Theorem 2.2. Let $n \equiv q \ (\text{mod} \ 6)$ where $n \geq 29$ and $q = 0, 1, 5$. Then
\[ \dim(C_n(2,3)) \geq 4. \]

Proof. Let $n = 6k + p$ where $k \geq 5$ and $p = -1, 0, 1$. We show by contradiction that three vertices cannot resolve the graph $C_n(2,3)$. Suppose that $W = \{v_a, v_i, v_j\}$ is a resolving set of $C_n(2,3)$ where $0 \leq a < i < j \leq n - 1$. Without loss of generality we can assume that $d_{a,i} \leq d_{i,j} \leq d_{j,a}$. Note that $d_{a,i} \leq \frac{n}{3}$, otherwise we have $d_{a,i} + d_{i,j} + d_{j,a} > n$, which contradicts (2.1). Due to the symmetry in the graph, we can assume that $v_a = v_0$ and $1 \leq i \leq \frac{n}{3}$ (which means that $i \leq 2k$). Then $j \leq \frac{2n}{3}$, otherwise we have $d_{i,j} > \frac{n}{3}$ and $d_{j,a} < \frac{n}{3}$, which contradicts the fact that $d_{i,j} \leq d_{j,a}$.

Let $W' = \{v_0, v_i\}$. We consider three cases.

Case 1: $i \equiv 0 \ (\text{mod} \ 3)$.

By (1.1), we obtain $r(v_{3k-2}|W') = r(v_{3k-1}|W') = (k, k - \frac{i}{3})$. From Lemma 2.1, we know that the vertices $v_{3k-2}, v_{3k-1}, v_{3k}$ can be resolved by $v_j$ only if $j = 3k - 2$ or $j = 3k$. This also implies that $i \leq \frac{3k}{2} \leq 3k$. Consequently $r(v_{3k-5}|W') = r(v_{3k-4}|W') = (k - 1, k - 1 - \frac{3}{3}, 1)$ if $j = 3k - 2$, and $r(v_{3k-5}|W') = r(v_{3k-4}|W') = (k - 1, k - 1 - \frac{3}{3}, 2)$ if $j = 3k$, which means that the vertices $v_{3k-5}$ and $v_{3k-4}$ are not resolved by $W$.

Case 2: $i \equiv 1 \ (\text{mod} \ 3)$.

We obtain $r(v_{3k-1}|W') = r(v_{3k}|W') = r(v_{3k+1}|W') = (k, k + \frac{1-i}{3})$. We need to resolve the vertices $v_{3k-1}, v_{3k}, v_{3k+1}$ by $v_j$. From Lemma 2.1, it follows that $j = 3k - 1$ or $j = 3k + 1$. By (1.1), if $j = 3k - 1$, then $r(v_{3k-4}|W') = r(v_{3k-3}|W') = (k - 1, k - 1 + \frac{1-i}{3}, 1)$, and if $j = 3k + 1$, we have $r(v_{3k-4}|W') = r(v_{3k-3}|W') = (k - 1, k - 1 + \frac{1-i}{3}, 2)$. A contradiction.

Case 3: $i \equiv 2 \ (\text{mod} \ 3)$.

If $p = 0$ or 1, then $r(v_{3k}|W') = r(v_{3k+1}|W') = r(v_{3k+2}|W') = (k, k + \frac{2-i}{3})$, thus by Lemma 2.1, $v_j = v_{3k}$ or $v_j = v_{3k+2}$ (so $i$ is at most $\frac{3k}{2} + 1$). Since $r(v_{3k-5}|W') = r(v_{3k-4}|W') = (k - 1, k - 1 + \frac{2-i}{3}, 2)$ if $j = 3k$, and $r(v_{3k-4}|W') = r(v_{3k-3}|W') = (k - 1, k - 1 + \frac{2-i}{3}, 2)$ if $j = 3k + 2$, the set $W$ cannot resolve the graph $C_n(2,3)$.

It remains to consider the case $p = -1$ (if $i \equiv 2 \ (\text{mod} \ 3)$). Then
\[ r(v_{3k-5}|W') = r(v_{3k-4}|W') = (k - 1, k - 2 + \frac{2-i}{3}), \]
\[ r(v_{3k-2}|W') = r(v_{3k-1}|W') = (k, k - 1 + \frac{2-i}{3}), \]
\[ r(v_{3k}|W') = r(v_{3k+1}|W') = (k, k + \frac{2-i}{3}), \]
\[ r(v_{6k-4}|W') = r(v_{6k-3}|W') = (1, \frac{i+1}{3} + 1). \]
Let us distinguish 3 subcases.

Case 3a: $j \equiv 0 \pmod{3}$.

Then $d(v_j, v_{3k-2}) = d(v_j, v_{3k-1}) = k - \frac{j}{3}$ if $j < 3k - 3$, $d(v_j, v_{3k-5}) = d(v_j, v_{3k-4}) = \frac{j}{3} - k + 2$ if $j > 3k - 3$, and $d(v_{3k-3}, v_{6k-4}) = d(v_{3k-3}, v_{6k-3}) = k$, which means that $W$ cannot resolve the graph $C_n(2, 3)$.

Case 3b: $j \equiv 1 \pmod{3}$.

If $j \leq 3k - 2$, then $d(v_j, v_{3k}) = d(v_j, v_{3k+1}) = k + \frac{j}{3}$, and if $j \geq 3k + 1$, then $d(v_j, v_{3k-2}) = d(v_j, v_{3k-1}) = \frac{j}{3} - k + 1$.

Case 3c: $j \equiv 2 \pmod{3}$.

If $j < 3k - 1$, then $d(v_j, v_{3k}) = d(v_j, v_{3k+1}) = k + \frac{2j}{3}$, and if $j > 3k + 2$, then $d(v_j, v_{3k}) = d(v_j, v_{3k+1}) = \frac{j}{3} - k$.

Let $j = 3k - 1$. Then $r(v_2|W) = r(v_3|W) = \left(1, \frac{i-2}{3}, k-1\right)$ if $i \geq 5$, and $r(v_3|W) = r(v_{6k-4}|W) = (1, 2, k-1)$ if $i = 2$.

Let $j = 3k + 2$. Then $r(v_2|W) = r(v_3|W) = \left(1, \frac{i-2}{3}, k\right)$ if $i \geq 5$, and $r(v_5|W) = r(v_{6k-2}|W) = (2, 1, k-1)$ if $i = 2$. Hence, $C_n(2, 3)$ cannot be resolved by three vertices, which means that $\dim(C_n(2, 3)) \geq 4$.

\end{proof}

3. Resolving Sets of $C_n(2, 3)$

In this section we present resolving sets, which yield upper bound on the metric dimension of $C_n(2, 3)$. We show that there exists an infinite set of graphs $C_n(2, 3)$ containing a resolving set, which consists of 3 vertices.

\begin{theorem}
Let $n \equiv 4 \pmod{6}$ where $n \geq 22$. Then $\dim(C_n(2, 3)) \leq 3$.
\end{theorem}

\begin{proof}
Let $n = 6k + 4$ where $k \geq 3$. We show that $W = \{v_0, v_4, v_8\}$ is a resolving set of $C_n(2, 3)$. We give representations of distances of all vertices in $V(C_n(2, 3)) \setminus W$ with respect to $W$:

\begin{align*}
  r(v_{3i-2}|W) &= (i, i - 2, i - 3), & 4 \leq i \leq k + 1, \\
  r(v_{3i-1}|W) &= (i, i - 1, i - 3), & 4 \leq i \leq k + 1, \\
  r(v_{3i}|W) &= (i, i - 1, i - 2), & 4 \leq i \leq k + 1, \\
  r(v_{6k-3i+4}|W) &= (i, i + 2, i + 3), & 1 \leq i \leq k - 2, \\
  r(v_{6k-3i+5}|W) &= (i, i + 1, i + 3), & 1 \leq i \leq k - 2, \\
  r(v_{6k-3i+6}|W) &= (i, i + 1, i + 2), & 2 \leq i \leq k - 1, \\
  r(v_{i}|W) &= (2, 1, 3), & r(v_{2}|W) = (1, 1, 2), \\
  r(v_{3}|W) &= (1, 2, 2), & r(v_{5}|W) = (2, 2, 1), \\
  r(v_{6}|W) &= (2, 1, 1), & r(v_{7}|W) = (3, 1, 2), \\
  r(v_{9}|W) &= (3, 2, 2), & r(v_{6k+3}|W) = (2, 2, 3), \\
  r(v_{3k+4}|W) &= (k, k, k - 1), & r(v_{3k+5}|W) = (k, k + 1, k - 1),
\end{align*}

\end{proof}
Theorem 3.2. Let \( n \equiv q \pmod{6} \) where \( n \geq 15 \) and \( q = 2, 3, 5 \). Then \[
\dim(C_n(2,3)) \leq 4.
\]

Proof. Let \( n = 6k + q \) where \( n \geq 15 \) and \( q = 2, 3, 5 \). Let us show that \( W = \{v_0, v_1, v_2, v_3, v_4, v_5\} \) is a resolving set of \( C_n(2,3) \). First we give representations of distances of the vertices \( v_i \) for \( 3 \leq i \leq 3k+1 \) and \( 3k + q + 1 \leq i \leq 6k + q - 1 \) with respect to \( W' = \{v_0, v_1, v_2\} \subset W \) (see Table 1).

**Table 1.** Representations of distances of the vertices \( v_i \) for \( 3 \leq i \leq 3k+1 \) and \( 3k + q + 1 \leq i \leq 6k + q - 1 \) with respect to \( W' \)

<table>
<thead>
<tr>
<th>Representation</th>
<th>( v_0 )</th>
<th>( v_1 )</th>
<th>( v_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_3 )</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( v_{3i-2} ) ((2 \leq i \leq k + 1))</td>
<td>( i )</td>
<td>( i - 1 )</td>
<td>( i - 1 )</td>
</tr>
<tr>
<td>( v_{3i-1} ) ((2 \leq i \leq k))</td>
<td>( i )</td>
<td>( i )</td>
<td>( i - 1 )</td>
</tr>
<tr>
<td>( v_{3i} ) ((2 \leq i \leq k))</td>
<td>( i )</td>
<td>( i )</td>
<td>( i )</td>
</tr>
<tr>
<td>( v_{6k-3i+q} ) ((1 \leq i \leq k - 1))</td>
<td>( i )</td>
<td>( i + 1 )</td>
<td>( i + 1 )</td>
</tr>
<tr>
<td>( v_{6k-3i+q+1} ) ((1 \leq i \leq k))</td>
<td>( i )</td>
<td>( i )</td>
<td>( i + 1 )</td>
</tr>
<tr>
<td>( v_{6k-3i+q+2} ) ((2 \leq i \leq k))</td>
<td>( i )</td>
<td>( i )</td>
<td>( i )</td>
</tr>
<tr>
<td>( v_{6k+i-q-1} )</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

The only vertices, which have the same representations are the following pairs: \( v_3, v_{6k+q-2} \); \( v_4, v_{6k+q-1} \) and \( v_{3i}, v_{6k-3i+q+2} \) for \( 2 \leq i \leq k \). The vertex \( v_6 \) resolves all these pairs, since

\[
\begin{align*}
    d(v_6, v_3) &= 1, & d(v_6, v_{6k+q-2}) &= 3, \\
    d(v_6, v_4) &= 1, & d(v_6, v_{6k+q-1}) &= 3, \\
    d(v_6, v_{3i}) &= i - 2, \text{ for } 2 \leq i \leq k, \\
    d(v_6, v_{6k-3i+q+2}) &= i + 2, \text{ for } 2 \leq i \leq k - 1, \\
    d(v_6, v_{3k+q+2}) &= \begin{cases} i, & \text{if } q = 2 \text{ or } 3, \\
                       i + 1, & \text{if } q = 5. \end{cases}
\end{align*}
\]

Let us present representations of the vertices in \( V(C_n(2,3)) \setminus W' \), which are not given in Table 1.
If \( n = 6k + 2 \), then
\[
r(v_{3k+2}|W') = (k, k + 1, k),
\]
and if \( n = 6k + 3 \), we have
\[
r(v_{3k+2}|W') = (k + 1, k + 1, k),
nr(v_{3k+3}|W') = (k, k + 1, k + 1).
\]
Since these representations are different from the representation of any vertex in Table 1, \( W \) is a resolving set of \( C_n(2, 3) \) if \( q = 2 \) or \( 3 \).

If \( n = 6k + 5 \), we obtain
\[
r(v_{3k+2}|W') = (k + 1, k + 1, k),
nr(v_{3k+3}|W') = (k + 1, k + 1, k + 1),
nr(v_{3k+5}|W') = (k, k + 1, k + 1).
\]
Since \( d(v_6, v_{3k+3}) = k - 1 \) and \( d(v_6, v_{3k+4}) = k \), we have \( \dim(C_n(2, 3)) \leq 4 \) for \( q = 5 \) too.

**Theorem 3.3.** Let \( n \equiv 0 \pmod{6} \) where \( n \geq 12 \). Then \( \dim(C_n(2, 3)) \leq 4 \).

**Proof.** Let \( n = 6k \) where \( k \geq 2 \), and let \( W' = \{v_0, v_1, v_2\} \). We consider representations of distances of the vertices \( v_{3i-2}, v_{3i-1}, v_{3i} \) and \( v_{6k-3i+q+2} \) with respect to \( W' \) given in Table 1 for \( 2 \leq i \leq k \) and \( q = 0 \). Similarly, consider representations of the vertices \( v_{6k-3i+q} \) and \( v_{6k-3i+q+1} \) given in Table 1 for \( 1 \leq i \leq k - 1 \) and \( q = 0 \). It remains to give the representation of \( v_{3k+1} \), which is \( r(v_{3k+1}|W') = (k, k, k) \).

Let us present all vertices of \( C_n(2, 3) \), which have the same representations of distances with respect to \( W' \):
\[
r(v_3|W') = r(v_{6k-2}|W') = (1, 1, 2),
nr(v_4|W') = r(v_{6k-1}|W') = (2, 1, 1),
r(v_3|W') = r(v_{6k-3i+2}|W') = (i, i, i), \text{ for } 2 \leq i \leq k - 1,
nr(v_3k|W') = r(v_{3k+1}|W') = r(v_{3k+2}|W') = (k, k, k).
\]

We show that the vertex \( v_{3k} \) resolves all these vertices.
\[
d(v_{3k}, v_3) = k - 1, \quad d(v_{3k}, v_{6k-2}) = k,
d(v_{3k}, v_4) = k - 1, \quad d(v_{3k}, v_{6k-1}) = k,
d(v_{3k}, v_{3i}) = k - i, \quad d(v_{3k}, v_{6k-3i+2}) = k - i + 1, \text{ for } 2 \leq i \leq k - 1,
d(v_{3k}, v_{3k+1}) = 2, \quad d(v_{3k}, v_{6k+2}) = 1.
\]
Thus \( W = \{v_0, v_1, v_2, v_{3k}\} \) is a resolving set of the graph \( C_n(2, 3) \), which implies that \( \dim(C_n(2, 3)) \leq 4 \).

**Theorem 3.4.** Let \( n \equiv 1 \pmod{6} \) where \( n \geq 13 \). Then \( \dim(C_n(2, 3)) \leq 4 \).
Table 2. Representations of distances of all vertices in $V(C_n(2, 3)) \setminus W$ with respect to $W$

<table>
<thead>
<tr>
<th>Representation</th>
<th>$v_0$</th>
<th>$v_1$</th>
<th>$v_{3k-1}$</th>
<th>$v_{3k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_2$</td>
<td>1</td>
<td>2</td>
<td>$k - 1$</td>
<td>$k$</td>
</tr>
<tr>
<td>$v_{3i-2}$ ($2 \leq i \leq k - 1$)</td>
<td>$i$</td>
<td>$i - 1$</td>
<td>$k - i + 1$</td>
<td>$k - i + 1$</td>
</tr>
<tr>
<td>$v_{3i-1}$ ($2 \leq i \leq k - 1$)</td>
<td>$i$</td>
<td>$i$</td>
<td>$k - i$</td>
<td>$k - i + 1$</td>
</tr>
<tr>
<td>$v_{3i}$ ($1 \leq i \leq k - 1$)</td>
<td>$i$</td>
<td>$i$</td>
<td>$k - i$</td>
<td>$k - i$</td>
</tr>
<tr>
<td>$v_{3k-2}$</td>
<td>$k$</td>
<td>$k - 1$</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$v_{3k+1}$</td>
<td>$k$</td>
<td>$k$</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$v_{6k-3i+1}$ ($1 \leq i \leq k - 1$)</td>
<td>$i$</td>
<td>$i + 1$</td>
<td>$k - i + 1$</td>
<td>$k - i + 1$</td>
</tr>
<tr>
<td>$v_{6k-3i+2}$ ($1 \leq i \leq k$)</td>
<td>$i$</td>
<td>$i$</td>
<td>$k - i + 1$</td>
<td>$k - i + 1$</td>
</tr>
<tr>
<td>$v_{6k-3i+3}$ ($2 \leq i \leq k$)</td>
<td>$i$</td>
<td>$i$</td>
<td>$k - i + 2$</td>
<td>$k - i + 1$</td>
</tr>
</tbody>
</table>

Proof. Let $n = 6k + 1$ where $k \geq 2$. We show that $W = \{v_0, v_1, v_{3k-1}, v_{3k}\}$ is a resolving set of $C_n(2, 3)$. Representations of distances of all vertices in $V(C_n(2, 3)) \setminus W$ with respect to $W$ are given in Table 2.

Any two vertices have different representations, hence $W$ is a resolving set of $C_n(2, 3)$ and $\dim(C_n(2, 3)) \leq 4$. □

4. Conclusion

In Section 3 we presented resolving sets of $C_n(2, 3)$ except for a few small values of $n$. Resolving sets for those values of $n$, which are not included in our theorems, are given in Table 3.

Table 3. Resolving sets of $C_n(2, 3)$ for $n = 7, 8, 9, 10, 11, 14$ and $16$

<table>
<thead>
<tr>
<th>$C_n(2, 3)$</th>
<th>Resolving set</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 7$</td>
<td>${v_0, v_1, v_2}$</td>
</tr>
<tr>
<td>$n = 8$</td>
<td>${v_0, v_1, v_4}$</td>
</tr>
<tr>
<td>$n = 9$</td>
<td>${v_0, v_1, v_2, v_4}$</td>
</tr>
<tr>
<td>$n = 10$</td>
<td>${v_0, v_1, v_2, v_3, v_4}$</td>
</tr>
<tr>
<td>$n = 11$</td>
<td>${v_0, v_1, v_3, v_4}$</td>
</tr>
<tr>
<td>$n = 14$</td>
<td>${v_0, v_1, v_2, v_6}$</td>
</tr>
<tr>
<td>$n = 16$</td>
<td>${v_0, v_1, v_2, v_6}$</td>
</tr>
</tbody>
</table>

We carefully checked that these resolving sets are the smallest ones (this can be checked also by computer programs). Note that $n = 10$ is the only case, such that $\dim(C_n(2, 3)) > 4$. The case $n = 14$ could be included in the proof of Theorem 3.2, but we would have to consider a short part of the proof of Theorem 3.2 separately for this case.

From our theorems presented in Sections 2 and 3 we obtain Table 4. Our results yield exact values of the metric dimension of $C_n(2, 3)$ if $n \equiv q \pmod{6}$ where $q = 0, 1, 4, 5$. 
Table 4. Lower and upper bounds on $\dim(C_n(2, 3))$

<table>
<thead>
<tr>
<th>$\dim(C_n(2, 3))$</th>
<th>Lower bound for $n \geq 26$</th>
<th>Upper bound for $n \geq 17$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \equiv 0 \pmod{6}$</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$n \equiv 1 \pmod{6}$</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$n \equiv 2 \pmod{6}$</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$n \equiv 3 \pmod{6}$</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$n \equiv 4 \pmod{6}$</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>$n \equiv 5 \pmod{6}$</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

We have

$\dim(C_n(2, 3)) = 3$, for $n \equiv 4 \pmod{6}$, where $n \geq 22$,

$\dim(C_n(2, 3)) = 4$, for $n \equiv q \pmod{6}$, where $n \geq 29$ and $q = 0, 1, 5$.

It would be interesting to know exact values of $\dim(C_n(2, 3))$ also for $n \equiv 2$ or 3 (mod 6). We conjecture that all resolving sets presented in this paper are the smallest ones, thus we close this section by presenting the following conjecture.

Conjecture 4.1. Let $n \equiv 2$ or 3 (mod 6), where $n \geq 9$. Then $\dim(C_n(2, 3)) = 4$.

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References

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