REFINING SOME INEQUALITIES FOR FRAMES WITH SPECHT'S RATIO

FAHIMEH SULTANZADEH[1], MAHMOUD HASSANI[1], MOHSEN ERFANIAN OMIDVAR[1], AND RAJAB ALI KAMYABI GOL[2]

ABSTRACT. We give a new lower bound in some inequalities for frames in a Hilbert space. If $\{f_i\}_{i \in I}$ is a Parseval frame for the Hilbert space $\mathbb{H}$ with frame operator $Sf = \sum_{i \in I} \langle f, f_i \rangle f_i$, then, for every $J \subset I$ and $f \in \mathbb{H}$, we have

$$\left(\frac{1 + 2\alpha}{2 + 2\alpha} \right) \|f\|^2 \leq \sum_{i \in J} |\langle f, f_i \rangle|^2 + \left\| \sum_{i \in J^c} \langle f, f_i \rangle f_i \right\|^2,$$

where $\alpha = \inf \left\{ R \left(\frac{\|S_{J^c} f\|}{\|S_J f\|} \right) : f \in \mathbb{H}, J \subset I \right\}$ with Specht's ratio R. Also we obtain some improvements of the inequalities for general frames and alternate dual frames under suitable conditions. Our results refine the remarkable results obtained by Balan et al. and Gavruta.

1. INTRODUCTION

Frame theory was introduced by Duffin and Schaeffer [6] in 1952. Frames are an essential tool for many emerging applications. Their main advantage is the fact that frames can be designed to be redundant while still providing reconstruction formulas. Due to their numerical stability, tight frames and, in particular, Parseval frames are of increasing interest in applications (see [5, 7, 13]). Let $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ be a separable Hilbert space. We denote by $L(\mathbb{H})$ the algebra of all linear operators on $\mathbb{H}$. The space $\ell^2(I)$ is the set of $\{a_i\}_{i \in I}$ such that $a_i \in \mathbb{C}$ and $\sum_{i \in I} |a_i|^2 < \infty$ when I is a finite or countable set. A frame for $\mathbb{H}$ is a family of vectors $F = \{f_i\}_{i \in I}$ in $\mathbb{H}$ which satisfies

(1.1) $$A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \quad \text{for every } f \in \mathbb{H},$$

for positive constants $0 < A \leq B$. The bounded, invertible, and positive linear operator $S : \mathbb{H} \to \mathbb{H}$ defined by

$$Sf = \sum_{i \in I} \langle f, f_i \rangle f_i, \quad f \in \mathbb{H},$$

is known as the frame operator associated to F. It allows reconstruction of each vector $f \in \mathbb{H}$ in terms of the family F as follows:

$$f = \sum_{i \in I} \langle f, S^{-1} f_i \rangle f_i = \sum_{i \in I} \langle f, f_i \rangle S^{-1} f_i.$$

If F is a Parseval frame, that is, $S = id$, then the reconstruction formula resembles the Fourier series of f associated to an orthonormal basis $B = \{b_j\}_{j \in J}$ of $\mathbb{H}$:

$$f = \sum_{j \in J} \langle f, b_j \rangle b_j,$$

but the frame coefficients $\{\langle f, f_i \rangle\}_{i \in I}$ given by $F = \{f_i\}_{i \in I}$ allow us to reconstruct f even when some of these coefficients are corrupted (see [6]).

Balan et al. [1] and Gavruta [9] established several identities and inequalities for frames in Hilbert spaces. Furuichi [8] refined Young inequalities with Specht's ratio and introduced their properties. In this paper, first we use this improved inequality in some inequalities for Parseval frames and get new inequalities. Thereafter we give improvements for general frames. However our main focus will be on Parseval frames because of their importance in applications, particularly for signal processing. Finally we give improvements for alternative dual frames too.

2. Notation and Preliminary Results

2.1. **Frames in Hilbert space.** In the definition of frame for Hilbert spaces, the optimal constants (maximal for A and minimal for B) are known as the upper and lower frame bounds, respectively. If $A = B$, then this frame is called an A-tight frame, and if $A = B = 1$, then it is called a Parseval frame. If a family of vectors $F = \{f_i\}_{i \in I}$ satisfies the upper bound condition (1.1), we call F a Bessel family. Associated with each frame $F = \{f_i\}_{i \in I}$, there are three linear and bounded operators:

$T : \ell^2(I) \to \mathbb{H}$, $Tx = \sum_{i \in I} \langle x, e_i \rangle f_i$ (synthesis operator);

$T^* : \mathbb{H} \to \ell^2(I)$, $T^*(f) = \{\langle f, f_i \rangle\}_{i \in I}$, (analysis operator);

$S : \mathbb{H} \to \mathbb{H}$, $Sf = TT^* f = \sum_{i \in I} \langle f, f_i \rangle f_i$ (frame operator),

where $\{e_i\}_{i \in I}$ is the standard orthonormal basis of $\ell^2(I)$. The inequalities (1.1) imply that S is a (positive) self-adjoint invertible operator, and it allows reconstruction of each vector $f \in \mathbb{H}$ in terms of the family F as follows:

$$f = \sum_{i \in I} \langle f, S^{-1} f_i \rangle f_i = \sum_{i \in I} \langle f, f_i \rangle S^{-1} f_i.$$

Then the family $\{\tilde{f}_i\}_{i \in I}$, where $\tilde{f}_i = S^{-1} f_i$, $i \in I$, is also a frame for $\mathbb{H}$ called the canonical dual frame of the $F = \{f_i\}_{i \in I}$.

In general, the Bessel family $\{g_i\}_{i \in I}$ is called an alternative dual of the frame $F = \{f_i\}_{i \in I}$ if the following formula holds:

$$f = \sum_{i \in I} \langle f, g_i \rangle f_i, \quad \text{for all } f \in \mathbb{H}.$$

If $\{f_i\}_{i \in I}$ is a frame for $\mathbb{H}$ for every $J \subset I$ we define the operator

$$S_J f = \sum_{i \in J} \langle f, f_i \rangle f_i$$

and denote $J^c = I \setminus J$. It follows that $S = S_J + S_{J^c}$. By this definition, it is clear that if $J_1 \subseteq J_2$, then $\|S_{J_1} f\| \leq \|S_{J_2} f\|$.

For more details, we refer the reader to [2–4, 10, 12]. In [1], Balan et al. proved the following identity for Parseval frames:

$$(2.1) \qquad \sum_{i \in J} |\langle f, f_i \rangle|^2 - \left\| \sum_{i \in J} \langle f, f_i \rangle f_i \right\|^2 = \sum_{i \in J^c} |\langle f, f_i \rangle|^2 - \left\| \sum_{i \in J^c} \langle f, f_i \rangle f_i \right\|^2.$$

Moreover, in [1] the following inequality was obtained

$$(2.2) \qquad \frac{3}{4} \|f\|^2 \leq \sum_{i \in J} |\langle f, f_i \rangle|^2 + \left\| \sum_{i \in J^c} \langle f, f_i \rangle f_i \right\|^2.$$

See [9, 11] for further details. In fact, identity (2.1) was obtained as a particular case from the following result for general frames:

$$(2.3) \qquad \sum_{i \in J} |\langle f, f_i \rangle|^2 + \sum_{i \in I} |\langle S_{J^c} f, \tilde{f}_i \rangle|^2 = \sum_{i \in J^c} |\langle f, f_i \rangle|^2 + \sum_{i \in I} |\langle S_J f, \tilde{f}_i \rangle|^2.$$

Inequality (2.2) leads us to introduce, for a Parseval frame, the numbers

$$v_+(F; J) = \sup_{f \neq 0} \frac{\| \sum_{i \in J} \langle f, f_i \rangle f_i \|^2 + \sum_{i \in J^c} |\langle f, f_i \rangle|^2}{\|f\|^2},$$

$$v_-(F; J) = \inf_{f \neq 0} \frac{\| \sum_{i \in J} \langle f, f_i \rangle f_i \|^2 + \sum_{i \in J^c} |\langle f, f_i \rangle|^2}{\|f\|^2}.$$

Recall that $v_+(F; J)$ is called the upper index of F relative to J and $v_-(F; J)$ is called the lower index of F relative to J.

Gavruta [9] presented basic properties of these indexes.

2.2. Improved Young inequality with Specht's ratio.
The well-known Young inequality says that $(1 - v)a + vb \geq a^{1-v} b^v$ for positive numbers a, b and $v \in [0, 1]$. A refinement of this inequality is given in the following proposition.

Lemma 2.1 ([8]). *Specht's ratio*

$$R(h) \equiv \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}, \quad h \neq 1, h > 0,$$

has the following properties:

(i) $R(1) = 1$ and $R(h) = R(1/h) > 1$ for $h > 0$;

(ii) $R(h)$ *is a monotone increasing function on* $(1, \infty)$*;*
(iii) $R(h)$ *is a monotone decreasing function on* $(0, 1)$*.*

Proposition 2.1 ([8])**.** *For* $a, b > 0$ *and* $v \in [0, 1]$*, it follows that*

$$(1 - v)a + vb \geq R\left(\left(\frac{b}{a}\right)^r\right) a^{1-v}b^v,$$

where $r \equiv \min\{v, 1 - v\}$ *and* $R(\cdot)$ *is Specht's ratio.*

2.3. **Operator Theory.** We first state a basic result from operator theory. The following lemma appeared in [1].

Lemma 2.2. *If* $S, T \in L(\mathbb{H})$ *satisfy* $S + T = id$*, then* $S - T = S^2 - T^2$*.*

Proof. The proof follows from

$$S - T = S - (id - S) = 2S - id = S^2 - (id - 2S + S^2) = S^2 - (id - S)^2 = S^2 - T^2.$$

$\square$

3. Main Results

In this section, first we improve the left-hand-side of inequality (2.2) with Specht's ratio. Thereafter in Lemma 3.1 improvements for self-adjoint operators are given, which we apply for general frames. Finally in Theorems 3.3 and 3.4, we present the results for alternate dual frames.

Theorem 3.1. *If* $\{f_i\}_{i \in I}$ *is a Parseval frame for the Hilbert space* $\mathbb{H}$ *with frame operator* S*, then, for every* $\emptyset \neq J \subset I$ *and* $f \in \mathbb{H}$*, it follows that*

$$(3.1) \qquad \left(\frac{1 + 2\alpha}{2 + 2\alpha}\right) \|f\|^2 \leq \sum_{i \in J} |\langle f, f_i \rangle|^2 + \left\| \sum_{i \in J^c} \langle f, f_i \rangle f_i \right\|^2,$$

where $\alpha = \inf\left\{ R\left(\frac{\|S_{J^c}f\|}{\|S_J f\|}\right) : f \in \mathbb{H}, J \subset I \right\}$*.*

Proof. Since

$$\|f\|^2 = \|S_J f + S_{J^c} f\|^2 \leq \|S_J f\|^2 + \|S_{J^c} f\|^2 + 2\|S_J f\|\|S_{J^c} f\|,$$

by letting $v = \frac{1}{2}$, $a = \|S_J f\|$ and $b = \|S_{J^c} f\|$ in Proposition 2.1, we have

$$\|f\|^2 \leq (\|S_J f\|^2 + \|S_{J^c} f\|^2) \left(1 + \frac{1}{R\left(\frac{\|S_{J^c}f\|}{\|S_J f\|}\right)}\right).$$

Put $\alpha = \inf\left\{ R\left(\frac{\|S_{J^c}f\|}{\|S_J f\|}\right) : f \in \mathbb{H}, J \subset I \right\}$, then

$$\left\langle \left(\frac{\alpha}{1 + \alpha}\right) id\, f, f \right\rangle \leq \langle (S_J^2 + S_{J^c}^2)f, f \rangle.$$

This implies that

$$\left(\frac{\alpha}{1 + \alpha}\right) id \leq S_J^2 + S_{J^c}^2.$$

So,
$$\left(\frac{\alpha}{1+\alpha}+1\right)id \leq S_J + S_{J^c}^2 + S_{J^c} + S_J^2.$$

Now, by applying Lemma 2.2, it follows that
$$\left(\frac{1+2\alpha}{2+2\alpha}\right)id \leq S_J + S_{J^c}^2.$$

Hence,
$$\left(\frac{1+2\alpha}{2+2\alpha}\right)\|f\|^2 \leq \langle S_J f, f\rangle + \langle S_{J^c}f, S_{J^c}f\rangle = \sum_{i\in J}|\langle f, f_i\rangle|^2 + \left\|\sum_{i\in J^c}\langle f, f_i\rangle f_i\right\|^2. \qquad \square$$

Note that for $\alpha = 1$, inequality (3.1) is the same as inequality (2.2) and that for $\alpha > 1$, (3.1) is an improvement of (2.2).

Corollary 3.1. *Let $F = \{f_i\}_{i\in I}$ be a Parseval frame and let $J \subset I$. Then*
$$\frac{1+2\alpha}{2+2\alpha} \leq v_-(F; J) \leq v_+(F; J) \leq 1,$$

where $\alpha = \inf\left\{R\left(\frac{\|S_{J^c}f\|}{\|S_J f\|}\right) : f \in \mathbb{H}, J \subset I\right\}$.

Proof. By using Theorem 3.1 and the hypothesis that F is a Parseval frame, we have
$$\left(\frac{1+2\alpha}{2+2\alpha}\right)\|f\|^2 \leq \sum_{i\in J}|\langle f, f_i\rangle|^2 + \|\sum_{i\in J^c}\langle f, f_i\rangle f_i\|^2 \leq \|f\|^2.$$

So,
$$\frac{1+2\alpha}{2+2\alpha} \leq \frac{\sum_{i\in J}|\langle f, f_i\rangle|^2 + \|\sum_{i\in J^c}\langle f, f_i\rangle f_i\|^2}{\|f\|^2} \leq 1.$$

Hence,
$$\frac{1+2\alpha}{2+2\alpha} \leq v_-(F; J) \leq v_+(F; J) \leq 1. \qquad \square$$

In the following lemma, we give an improvement of the inequality proved in [9, Theorem 2.1], under some conditions. Then we present Theorem 3.2 for general frames.

Lemma 3.1. *Let $T_1, T_2 \in L(\mathbb{H})$ be self-adjoint operators satisfying $T_1 + T_2 = id$, such that $T_1 \geq \frac{k}{k+1}id$, where $k \in \mathbb{N}$ and $k > 3$. Then*

(3.2) $$\frac{k}{k+1}\|f\|^2 \leq \langle T_1 f, f\rangle + \|T_2 f\|^2 = \langle T_2 f, f\rangle + \|T_1 f\|^2, \quad for \ f \in \mathbb{H}.$$

Proof. From our assumptions, we have

$$
\begin{aligned}
\langle T_2 f, f \rangle + \|T_1 f\|^2 &= \langle (id - T_1)f, f \rangle + \langle T_1^2 f, f \rangle \\
&= \langle (T_1^2 - T_1 + id)f, f \rangle \\
&= \langle T_1 f, f \rangle + \langle (id - T_1)^2 f, f \rangle \\
&= \langle T_1 f, f \rangle + \|T_2 f\|^2,
\end{aligned}
$$

and also

$$
\langle T_1 f, f \rangle + \|T_2 f\|^2 \geq \left\langle \frac{k}{(k+1)} f, f \right\rangle + \|T_2 f\|^2 \geq \frac{k}{(k+1)} \|f\|^2. \qquad \square
$$

Remark 3.1. Notice that for $k > 3$, from the inequality $\frac{3}{4} < \frac{k}{k+1}$, it follows that inequality (3.2) is an improvement of the inequality proved in [9, Theorem 2.1].

For $k = 3, 4, 5, 6, 7, \ldots$, the correspondence values of $\frac{k}{k+1}$ or $1 - \frac{1}{k+1}$ are $0.75 < 0.8 < 0.83 < 0.86 < 0.87 < \cdots$, respectively. Hence, by increasing k, we see that $1 - \frac{1}{k+1}$ is rapidly approaching to 1. Therefore, inequality (3.2) is better in application and we use it for frames.

Theorem 3.2. *Let $\{f_i\}_{i \in I}$ be a frame for the Hilbert space $\mathbb{H}$ with frame operator S and canonical dual frame $\{\tilde{f}_i\}_{i \in I}$. For every $\emptyset \neq J \subset I$, if $S^{-\frac{1}{2}} S_J S^{-\frac{1}{2}} \geq \frac{k}{k+1} id$, where $k \in \mathbb{N}$ and $k \geq 3$, then*

$$
\begin{aligned}
(3.3) \qquad \frac{k}{k+1} \sum_{i \in I} |\langle f, f_i \rangle|^2 &\leq \sum_{i \in J} |\langle f, f_i \rangle|^2 + \sum_{i \in I} |\langle S_{J^c} f, \tilde{f}_i \rangle|^2 \\
&= \sum_{i \in J^c} |\langle f, f_i \rangle|^2 + \sum_{i \in I} |\langle S_J f, \tilde{f}_i \rangle|^2, \quad \text{for } f \in \mathbb{H}.
\end{aligned}
$$

Proof. For every $J \subset I$, we have $S_J + S_{J^c} = S$, and hence $S^{-\frac{1}{2}} S_J S^{-\frac{1}{2}} + S^{-\frac{1}{2}} S_{J^c} S^{-\frac{1}{2}} = id$. By our assumptions and taking $T_1 = S^{-\frac{1}{2}} S_J S^{-\frac{1}{2}}$, $T_2 = S^{-\frac{1}{2}} S_{J^c} S^{-\frac{1}{2}}$ and $S^{\frac{1}{2}} f$ instead of f in Lemma 3.1, we get

$$
\frac{k}{k+1} \|S^{\frac{1}{2}} f\|^2 \leq \langle S^{-\frac{1}{2}} S_J f, S^{\frac{1}{2}} f \rangle + \|S^{-\frac{1}{2}} S_{J^c} f\|^2 = \langle S^{-\frac{1}{2}} S_{J^c} f, S^{\frac{1}{2}} f \rangle + \|S^{-\frac{1}{2}} S_J f\|^2,
$$

or equivalently,

$$
\frac{k}{k+1} \langle Sf, f \rangle \leq \langle S_J f, f \rangle + \langle S^{-1} S_{J^c} f, S_{J^c} f \rangle = \langle S_{J^c} f, f \rangle + \langle S^{-1} S_J f, S_J f \rangle.
$$

Therefore,

$$
\begin{aligned}
\frac{k}{k+1} \sum_{i \in I} |\langle f, f_i \rangle|^2 &\leq \sum_{i \in J} |\langle f, f_i \rangle|^2 + \sum_{i \in I} |\langle S_{J^c} f, \tilde{f}_i \rangle|^2 \\
&= \sum_{i \in J^c} |\langle f, f_i \rangle|^2 + \sum_{i \in I} |\langle S_J f, \tilde{f}_i \rangle|^2. \qquad \square
\end{aligned}
$$

For $k = 3$ inequality (3.3) is the same as the inequality proved in [9, Theorem 2.2] and for $k > 3$ it is an improvement for it.

In the following, we give an improvement for alternate dual frames. We first improve an inequality given in [9] for operators under conditions.

Lemma 3.2. *If $T_1, T_2 \in L(\mathbb{H})$ satisfy $T_1 + T_2 = id$ and $\operatorname{Re} T_1 \geq \frac{k}{k+1} id$, where $k \in \mathbb{N}$ and $k \geq 3$, then*

$$(3.4) \qquad \frac{k}{k+1} id \leq T_1^* T_1 + \frac{1}{2}(T_2^* + T_2) = T_2^* T_2 + \frac{1}{2}(T_1^* + T_1).$$

Proof. From our assumptions, we have

$$
\begin{aligned}
T_1^* T_1 + \frac{1}{2}(T_2^* + T_2) &= T_1^* T_1 + \frac{1}{2}(id - T_1^* + id - T_1) \\
&= T_1^* T_1 - \frac{1}{2}(T_1^* + T_1) + id \\
&= (id - T_1^*)(id - T_1) + \frac{1}{2}(T_1^* + T_1) \\
&= T_2^* T_2 + \frac{1}{2}(T_1^* + T_1).
\end{aligned}
$$

And also $T_2^* T_2 + \frac{1}{2}(T_1^* + T_1) = T_2^* T_2 + \operatorname{Re} T_1 \geq \frac{k}{k+1} id$. $\qquad \square$

Note that, for $k = 3$, inequality (3.4) is the same as the inequality proved in [9, Theorem 3.1] and for every $k > 3$, inequality (3.4) is its improvement.

Theorem 3.3. *Let $\{f_i\}_{i \in I}$ be a frame for the Hilbert space $\mathbb{H}$ and let $\{g_i\}_{i \in I}$ be an alternate dual frame of $\{f_i\}_{i \in I}$. For every $J \subset I$ and $f \in \mathbb{H}$, if $\operatorname{Re}\langle(\sum_{i \in J}\langle f, g_i \rangle f_i), f \rangle \geq \frac{k}{k+1}\langle f, f \rangle$, where $k \in \mathbb{N}$ and $k \geq 3$, then*

$$
(3.5) \qquad
\begin{aligned}
\frac{k}{k+1}\|f\|^2 &\leq \operatorname{Re} \sum_{i \in J}\langle f, g_i \rangle \overline{\langle f, f_i \rangle} + \|\sum_{i \in J^c}\langle f, g_i \rangle f_i\|^2 \\
&= \operatorname{Re} \sum_{i \in J^c}\langle f, g_i \rangle \overline{\langle f, f_i \rangle} + \|\sum_{i \in J}\langle f, g_i \rangle f_i\|^2, \quad \text{for } f \in \mathbb{H}.
\end{aligned}
$$

Proof. For every $J \subset I$, define the bounded linear operator Z_J on $\mathbb{H}$ by

$$Z_J f := \sum_{i \in J}\langle f, g_i \rangle f_i.$$

By the Cauchy–Schwartz inequality, it follows that this series converges unconditionally. Since $Z_J + Z_{J^c} = id$, by Lemma 3.1, for every $f \in \mathbb{H}$, we have

$$
\begin{aligned}
\frac{k}{k+1}\langle f, f \rangle &\leq \langle Z_J^* Z_J f, f \rangle + \frac{1}{2}\langle (Z_{J^c}^* + Z_{J^c})f, f \rangle \\
&= \langle Z_{J^c}^* Z_{J^c} f, f \rangle + \frac{1}{2}\langle (Z_J^* + Z_J)f, f \rangle
\end{aligned}
$$

or

$$\frac{k}{k+1}\|f\|^2 \le \|K_J f\|^2 + \frac{1}{2}(\overline{\langle Z_{J^c}f, f\rangle} + \langle Z_{J^c}f, f\rangle)$$
$$= \|Z_{J^c}f\|^2 + \frac{1}{2}(\overline{\langle Z_J f, f\rangle} + \langle Z_J f, f\rangle).$$

Hence,

$$\frac{k}{k+1}\|f\|^2 \le \|\sum_{i \in J}\langle f, g_i\rangle f_i\|^2 + \mathrm{Re}\langle\sum_{i \in J^c}\langle f, g_i\rangle f_i, f\rangle$$
$$= \|\sum_{i \in J^c}\langle f, g_i\rangle f_i\|^2 + \mathrm{Re}\langle\sum_{i \in J}\langle f, g_i\rangle f_i, f\rangle,$$

and the proof is completed. $\qquad\square$

Note that, for $k = 3$, inequality (3.5) is the same as the inequality proved in [9, Theorem 3.2] and for every $k > 3$, inequality (3.5) is its improvement. Finally, we give a more general result.

Theorem 3.4. *Let $\{f_i\}_{i \in I}$ be a frame for the Hilbert space $\mathbb{H}$ and let $\{g_i\}_{i \in I}$ be an alternate dual frame of $\{f_i\}_{i \in I}$. For every $f \in \mathbb{H}$, if $\mathrm{Re}\langle(\sum_{i \in J}\langle f, g_i\rangle f_i), f\rangle \ge \frac{k}{k+1}\langle f, f\rangle$, where $k \in \mathbb{N}$ and $k \ge 3$, then for any bounded sequence $\{w_i\}_{i \in I}$, we have*

$$\frac{k}{k+1}\|f\|^2 \le \mathrm{Re}\sum_{i \in I} w_i\langle f, g_i\rangle\overline{\langle f, f_i\rangle} + \left\|\sum_{i \in I}(1 - w_i)\langle f, g_i\rangle f_i\right\|^2$$
$$= \mathrm{Re}\sum_{i \in I}(1 - w_i)\langle f, g_i\rangle\overline{\langle f, f_i\rangle} + \left\|\sum_{i \in I} w_i\langle f, g_i\rangle f_i\right\|^2, \quad \textit{for } f \in \mathbb{H}.$$

Proof. In Lemma 3.2, we put

$$T_1 f = \sum_{i \in I} w_i\langle f, g_i\rangle f_i, \quad T_2 f = \sum_{i \in I}(1 - w_i)\langle f, g_i\rangle f_i.$$

The result is obtained from Theorem 3.3, if we take $J \subset I$ and

$$w_i = \begin{cases} 1, & \text{for } i \in J, \\ 0, & \text{for } i \in J^c. \end{cases}$$

$\qquad\square$

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¹Department of Mathematics,
Mashhad Branch, Islamic Azad University,
Mashhad, Iran
Email address: `fsultanzadeh@gmail.com`
Email address: `mhassanimath@gmail.com`
Email address: `math.erfanian@gmail.com`

²Department of Pure Mathematics,
Ferdowsi University of Mashhad,
Mashhad, Iran
Email address: `kamyabi@um.ac.ir`