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GEODESIC E-INVEX SETS AND GEODESIC E-PREINVEX FUNCTIONS ON RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we first introduce two new classes of sets and functions called geodesic E-invex sets and geodesic E-preinvex functions on a Riemannian manifold, respectively. Moreover, we present the definition and properties of geodesic E-quasi-preinvex functions on Riemannian manifolds. Finally, we investigate the properties and characterizations of these two classes of sets and functions.

1. Introduction

Convexity plays an important and significant role in optimization theory. This concept in the linear topological vector spaces relies on the possibility of connecting any two points of the space by the line segment between them. Since convexity is often not enjoyed by the real problems, various approaches have been proposed by several reseachers in order to extend the validity of results to the larger classes of optimization. An important and significant generalization of convexity is invexity, which was introduced by Hanson [8] in 1981. Hanson's initial results inspired a great deal of subsequent work which has greatly expanded the roles and applications of invexity in nonlinear optimization and other branches of pure and applied sciences. Ben-Isreal and Mond [5] introduced a new generalization of convex sets and convex functions that called by Craven [6] the invex sets and preinvex functions, respectively, see also [3].

In general, a manifold is not a linear space, but the extension of concepts and techniques from linear spaces to Riemannian manifolds are natural and applicable. Rapcsak [18] and Udriste [19] proposed a generalization of convexity, called geodesic

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convexity, and extended many results of convex analysis and optimization theory to Riemannian manifolds. In this setting, the linear space has been replaced by a Riemannian manifold and the line segment by a geodesic. For more details, we refer the reader to [10–12, 15, 17, 18] and the references therein.

The notion of invex functions on Riemannian manifolds was introduced in [16]. However, its generalization has been investigated by Mititelu [13]. The concept of geodesic invex sets, geodesic invex functions and geodesic preinvex functions on a Riemannian Manifold with respect to the particular mappings have been introduced in [4].

In this paper, we first discuss various concepts, definitions and properties of functions defined on a Riemannian manifold. The notion of invexity and its generalization on Riemannian manifolds are presented in Section 2. In Section 3, we first define the concept of geodesic E-invex sets and geodesic E-preinvex functions on a Riemannian manifold. Next, we investigate their properties and characterizations. The class of geodesic E-quasi-preinvex functions are introduced in Section 4, and we give their characterizations.

2. Preliminaries

We first recall some definitions and known results about η-invex sets and geodesic η-preinvex functions on Riemannian manifolds, which will be used throughout the paper.

Let M be an n-dimensional differentiable manifold, and let T_pM be the tangent space to M at the point $p \in M$. Suppose that at each point $p \in M$, a positive inner product $g_p(x, y)$ on T_pM is given ($x, y \in T_pM$). Recall that [12], a C^∞ mapping $g : p \to g_p$, which assigns a positive inner product g_p on T_pM to each point $p \in M$, is called a Riemannian metric. A manifold M equipped with the Riemannian metric g is called a Riemannian manifold. We denote by TM the tangent space to M.

Suppose that (M, g) is a complete n-dimensional Riemannian manifold with Riemannian connection ∇ (see [12]). Let x, y be two points in M, and $\gamma_{x,y} : [0, 1] \to M$ be a geodesic joining the points x and y, i.e., $\gamma_{x,y}(0) = y$, $\gamma_{x,y}(1) = x$.

Let us recall that [12] the length of a piecewise C^1 curve $\gamma : [a, b] \to M$ is defined by

$$L(\gamma) := \int_a^b \|\gamma'(t)\| dt.$$

For any two points $p, q \in M$, we define [12]

$$d(p, q) := \inf\{L(\gamma) : \gamma \text{ is a piecewise } C^1 \text{ curve joining } p \text{ and } q\}.$$

Then, d is a distance which induces the original topology on M. We know that on every Riemannian manifold there exists exactly one covariant derivation called Levi-Civita connection, denoted by ∇_{XY} for any vector fields $X, Y \in M$. We also recall that a geodesic is a C^∞ smooth path γ whose tangent is parallel along the path γ, i.e., γ satisfies the equation $\nabla_{d\gamma(t)/d(t)} d\gamma(t)/d(t)$=0. Any path γ joining p and $q \in M$ such that $L(\gamma) = d(p, q)$ is a geodesic and is called a minimal geodesic.

Definition 2.1 ([9])**.** A subset A of $\mathbb{R}^n$ is called η-invex with respect to the function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ if $x, y \in A$, $\lambda \in [0, 1]$, then $y + \lambda\eta(x, y) \in A$.

It is obvious that Definition 2.1 is a generalization of the notion of a convex set (with $\eta(x, y) := x - y$). Note that any set in $\mathbb{R}^n$ is invex with respect to $\eta(x, y) \equiv 0$, for all $x, y \in \mathbb{R}^n$.

In 1987, Hanson and Mond [9] introduced the notion of preinvex functions. The following definition of a preinvex function has been given by Jeyakumar [19].

Definition 2.2 ([19])**.** Let f be a real valued function defined on an η-invex set $A \subseteq \mathbb{R}^n$. Then, f is said to be preinvex with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ if

$$f[y + \lambda\eta(x, y)] \leq \lambda f(x) + (1 - \lambda)f(y), \quad \text{for all } x, y \in A,\ \lambda \in [0, 1].$$

In the sequel, we consider the function $E : \mathbb{R}^n \to \mathbb{R}^n$.

Definition 2.3. ([7, Definition 2.2]). A subset A of $\mathbb{R}^n$ is said to be E-invex with respect to a given mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ if

$$E(y) + \lambda\eta(E(x), E(y)) \in A, \quad \text{for all } x, y \in A,\ \lambda \in [0, 1].$$

Definition 2.4. ([7, Definition 2.3]). Let $A \subseteq \mathbb{R}^n$ be an E-invex set with respect to a given mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be E-preinvex on A with respect to η if

$$f(E(y) + \lambda\eta(E(x), E(y)) \leq \lambda f(E(x)) + (1 - \lambda)f(E(y)), \quad \text{for all } x, y \in A,\ \lambda \in [0, 1].$$

The concept of geodesic invex sets and the invexity of a function f defined on an open geodesic invex subset of a Riemannian manifold were given in [4].

Definition 2.5. ([4, Definition 3.1]). Let M be a Riemannian manifold and $\eta : M \times M \to TM$ be a function such that $\eta(x, y) \in T_yM$ for each $x, y \in M$. A nonempty subset S of M is said to be geodesic invex with respect to η if for each $x, y \in S$ there exists exactly one geodesic $\alpha_{x,y} : [0, 1] \to M$ such that

$$\alpha_{x,y}(0) = y, \quad \alpha'_{x,y}(0) = \eta(x, y), \quad \alpha_{x,y}(t) \in S, \quad \text{for all } t \in [0, 1].$$

Recall that a subset S of a Riemannian manifold is called geodesic convex if any two points $x, y \in S$ can be joined by exactly one geodesic of length $d(x, y)$, which belongs entirely to S.

Definition 2.6. ([4, Definition 3.3]). Let M be a Riemannian manifold and $\eta : M \times M \to TM$ be a function such that $\eta(x, y) \in T_yM$ for each $x, y \in M$. Let $S \subseteq M$ be a geodesic invex set with respect to η. We say that a function $f : S \to \mathbb{R}$ is geodesic η-preinvex if

$$f(\alpha_{x,y}(t)) \leq tf(x) + (1 - t)f(y), \quad \text{for all } t \in [0, 1],\ x, y \in S,$$

where $\alpha_{x,y}$ is the unique geodesic which defined by Definition 2.5. If the inequality is strict, then we say that f is a strictly geodesic η-preinvex function.

3. Geodesic E-Invex Sets and Geodesic E-Preinvex Functions

The definition of a preinvex function on $\mathbb{R}^n$ was given in [20], see also [3, 14, 21] for the properties of preinvex functions. Fulga and Preda [7] introduced the class of E-preinvex and E-quasi-preinvex functions defined on $\mathbb{R}^n$. In [4, 10, 11], this notion has been extended for Reimannian manifolds.

Throughout the paper, let $E : M \to M$ and $\eta : M \times M \to TM$ be fixed mappings. We now introduce the concept of geodesic E-invex sets and geodesic E-preinvex functions on a Riemannian manifold as follows.

Definition 3.1. Let M be a Riemannian manifold and $\eta : M \times M \to TM$ be a function such that $\eta(x, y) \in T_yM$ for each $x, y \in M$. A nonempty subset S of M is said to be geodesic E-invex with respect to η if for each $x, y \in S$ there exists exactly one geodesic $\alpha_{E(x),E(y)} : [0, 1] \to M$ such that

$$\begin{aligned}&\alpha_{E(x),E(y)}(0) = E(y), \quad \alpha'_{E(x),E(y)}(0) = \eta(E(x), E(y)),\\ &\alpha_{E(x),E(y)}(t) \in S, \quad \text{for all } t \in [0, 1].\end{aligned}$$

Note that, in the special case, let $M := \mathbb{R}^n$, $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a function. Consider $\alpha_{x,y} : [0, 1] \to \mathbb{R}^n$ is defined by $\alpha_{x,y}(t) := y + t\eta(x, y)$ for all $t \in [0, 1]$. Then

$$\alpha_{x,y}(0) = y, \quad \alpha'_{x,y}(0) = \lim_{t\to 0} \frac{\alpha_{x,y}(t) - \alpha_{x,y}(0)}{t} = \lim_{t\to 0} \frac{y + t\eta(x, y) - y}{t} = \eta(x, y),$$

and $\alpha_{x,y}(t) \in S$ for all $t \in [0, 1]$ because S is invex with respect to η. Therefore, the definition of geodesic invexity and geodesic E-invexity coincide in $\mathbb{R}^n$.

Definition 3.2. Let M be a Riemannian manifold and $S \subseteq M$ be a geodesic E-invex set with respect to $\eta : M \times M \to TM$. A function $f : S \to \mathbb{R}$ is said to be geodesic E-preinvex with respect to η if

$$f(\alpha_{E(x),E(y)}(t)) \leq tf(E(x)) + (1 - t)f(E(y)), \quad \text{for all } t \in [0, 1],\ x, y \in S,$$

where $\alpha_{E(x),E(y)}$ is the unique geodesic which defined by Definition 3.1. If the inequality is strict, then we say that f is strictly geodesic E-preinvex with respect to η.

Let $M := \mathbb{R}^n$ and $S \subseteq \mathbb{R}^n$ be a geodesic invex set with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Consider $\alpha_{x,y}(t) = y + t\eta(x, y)$ for all $t \in [0, 1]$. Then $f(\alpha_{x,y}(t)) = f(y + t\eta(x, y)) \leq tf(x) + (1 - t)f(y)$, i.e., the definition of geodesic preinvex and geodesic E-preinvex coincide for a function $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ whenever $M = \mathbb{R}^n$.

From now on, for simplicity, we will call geodesic E-invex set with respect to η, geodesic E-quasi-preinvex set with respect to η, geodesic E-preinvex function with respect to η and geodesic E-quasi-preinvex function with respect to η by geodesic E-invex set, geodesic E-quasi-preinvex set, geodesic E-preinvex function and gedesic E-quasi-preinvex function, respectively.

We now give some results related to geodesic E-convex sets on Riemannian manifolds (see also [1]).

Proposition 3.1. *Every geodesic invex set $A \subseteq M$ is geodesic E-invex.*

Proof. The proof is obvious by taking the mapping $E : M \to M$ as the identity map. □

Proposition 3.2. *Let A be a subset of M. If A is a geodesic E-invex set, then $E(A) \subseteq A$.*

Proof. Since A is geodesic E-invex set, then for each $x, y \in A$ there exists exactly one geodesic $\alpha_{E(x),E(y)} : [0,1] \to M$ such that $\alpha_{E(x),E(y)}(0) = E(y)$, $\alpha'_{E(x),E(y)}(0) = \eta(E(x), E(y))$ and $\alpha_{E(x),E(y)}(t) \in A$ for all $t \in [0,1]$. Put $t := 0$, then $E(y) = \alpha_{E(x),E(y)}(0) \in A$, so, $E(A) \subseteq A$. □

Proposition 3.3. *Let $E(A)$ be an invex set. If $E(A) \subseteq A$, then A is a geodesic E-invex set.*

Proof. Let $x, y \in A$ be arbitrary. Then $E(x), E(y) \in E(A)$. Since $E(A)$ is invex with respect to η, thus there exists exactly one geodesic $\alpha_{E(x),E(y)} : [0,1] \to M$ such that $\alpha_{E(x),E(y)}(0) = E(y)$, $\alpha'_{E(x),E(y)}(0) = \eta(E(x), E(y))$ and $\alpha_{E(x),E(y)}(t) \in E(A) \subseteq A$ for all $t \in [0,1]$, hence, A is a geodesic E-invex set. □

Proposition 3.4. *If $\{A_i\}_{i\in I}$ is an arbitrary collection of geodesic E-invex subsets of M with respect to the mapping $E : M \to M$, then $\cap_{i\in I} A_i$ is a geodesic E-invex subset of M.*

Proof. Let $\{A_i\}_{i\in I}$ be a collection of geodesic E-invex subsets of M with respect to the mapping $E : M \to M$. If $\cap_{i\in I} A_i = \emptyset$, we are done. Let $x, y \in \cap_{i\in I} A_i$ be arbitrary. Then $x, y \in A_i$ for all $i \in I$. By the geodesic E-invexity of A_i, there exists exactly one geodesic $\alpha_{E(x),E(y)} : [0,1] \to M$ such that $\alpha_{E(x),E(y)}(0) = E(y)$, $\alpha'_{E(x),E(y)}(0) = \eta(E(x), E(y))$ and $\alpha_{E(x),E(y)}(t) \in A_i$ for all $t \in [0,1]$ and all $i \in I$, which implies that $\alpha_{E(x),E(y)}(t) \in \cap_{i\in I} A_i$ for all $t \in [0,1]$, and hence, $\cap_{i\in I} A_i$ is a geodesic E-invex set. □

Lemma 3.1. *Let $A \subseteq M$ be a geodesic E_1-invex and E_2-invex set. Then A is a geodesic $E_1 \circ E_2$-invex and $E_2 \circ E_1$-invex set, where $E_1, E_2 : M \to M$ are arbitrary mappings.*

Proof. By the hypothesis, since $A \subseteq M$ is a geodesic E_1-invex and E_2-invex set, then for each $x, y \in A$ there exist exactly one geodesic $\alpha_{E_1(x),E_1(y)} : [0,1] \to M$ such that $\alpha_{E_1(x),E_1(y)}(0) = E_1(y)$, $\alpha'_{E_1(x),E_1(y)}(0) = \eta(E_1(x), E_1(y))$, $\alpha_{E_1(x),E_1(y)}(t) \in A$, and exactly one geodesic $\alpha_{E_2(x),E_2(y)} : [0,1] \to M$ such that $\alpha_{E_2(x),E_2(y)}(0) = E_2(y)$, $\alpha'_{E_2(x),E_2(y)}(0) = \eta(E_2(x), E_2(y))$ and $\alpha_{E_2(x),E_2(y)}(t) \in A$ for all $t \in [0,1]$. Now, let $x, y \in A$ be arbitrary. Put $x_0 := E_2(x)$ and $y_0 := E_2(y)$. Thus, in view of Proposition 3.2, we conclude that $x_0, y_0 \in A$. Therefore,

$$\alpha_{E_1\circ E_2(x),E_1\circ E_2(y)}(0) = \alpha_{E_1(x_0),E_1(y_0)}(0) = E_1(y_0) = E_1 \circ E_2(y)$$

and

$$\alpha'_{E_1\circ E_2(x),E_1\circ E_2(y)}(0)=\alpha'_{E_1(x_0),E_1(y_0)}=\eta(E_1(x_0),E_1(y_0))=\eta(E_1\circ E_2(x),E_1\circ E_2(y))$$

and

$$\alpha_{E_1\circ E_2(x),E_1\circ E_2(y)}(t)=\alpha_{E_1(x_0),E_1(y_0)}(t)\in A,\quad \text{for all } t\in[0,1],$$

so, $A\subseteq M$ is a geodesic $E_1\circ E_2$-invex set. Similarly, $A\subseteq M$ is a geodesic $E_2\circ E_1$-invex set. □

Theorem 3.1. *Let $A\subseteq M$ be a geodesic invex set with respect to the function $\eta: M\times M\to TM$ and $f:A\to\mathbb{R}$ be a geodesic η-preinvex function. If $g: I\subseteq\mathbb{R}\to M$ is an increasing (strictly increasing) convex function such that $ran(f)\subseteq I$, then $g\circ f$ is geodesic (strictly geodesic) η-preinvex function on A.*

Proof. Since f is a geodesic η-preinvex functin, we have $f(\alpha_{x,y}(t))\le tf(x)+(1-t)f(y)$ for all $x,y\in A$ and all $t\in[0,1]$, where $\alpha_{x,y}$ is the unique geodesic which defined by Definition 2.5. Since g is an increasing convex function, we get

$$\begin{aligned} g[f(\alpha_{x,y}(t))] &\le g[(1-t)f(y)+tf(x)]\\ &\le (1-t)g(f(y))+tg(f(x))\\ &=(1-t)(g\circ f)(y)+t(g\circ f)(x), \end{aligned}$$

which shows that $g\circ f$ is a geodesic η-preinvex function on A. Similarly, we can show that $g\circ f$ is a strictly geodesic η-preinvex function if g is a strictly increasing convex function. □

Theorem 3.2. *Let $A\subseteq M$ be a geodesic E-invex set, and let $f_i: A\to\mathbb{R}$, $i=1,\dots,p$ be a geodesic E-preinvex function. Then, $f:=\sum_{i=1}^{p}\lambda_i f_i$ is a geodesic E-preinvex function on A with respect to the function η, where $\lambda_i\in\mathbb{R}$ with $\lambda_i\ge 0$, $i=1,\dots,p$.*

Proof. By the hypothesis, for each $i=1,\dots,p$, one has

$$f_i(\alpha_{E(x),E(y)}(t))\le(1-t)f_i(E(y))+tf_i(E(x)),$$

where $\alpha_{E(x),E(y)}$ is the unique geodesic which defined by Definition 2.5. It follows that

$$\lambda_i f_i(\alpha_{E(x),E(y)}(t))\le(1-t)\lambda_i f_i(E(y))+t\lambda_i f_i(E(x)),$$

and hence

$$\sum_{i=1}^{p}\lambda_i f_i(\alpha_{E(x),E(y)}(t))\le(1-t)\sum_{i=1}^{p}\lambda_i f_i(E(y))+t\sum_{i=1}^{p}\lambda_i f_i(E(x)),$$

which completes the proof. □

Proposition 3.5. *Let M be a Riemannian manifold and $A\subseteq M$ be a geodesic E-invex set. Assume that $E: M\to M$ is an idempotent mapping (i.e., $E^2=E$). Suppose that $f\circ E: A\to\mathbb{R}$ is a geodesic E-preinvex function. Then the following holds.*

(i) Every lower level set of $f\circ E$ which defined by $S(f\circ E,\lambda):=\{x\in A:(f\circ E)(x)\le\lambda\}$, $\lambda\in\mathbb{R}$, is a geodesic E-invex set with respect to the function $\eta: M\times M\to TM$.

(ii) The solution set K of the following optimization problem:

$$(P) \qquad \min(f \circ E)(x) \text{ subject to } x \in A,$$

is a geodesic E-invex set.

Moreover, if f is a strictly geodesic E-preinvex function, then K contains at most one point.

Proof. (i) Let $x, y \in S(f \circ E, \lambda) \subseteq A$ be arbitrary. Since A is a geodesic E-invex set with respect to the function η, then there exists exactly one geodesic $\alpha_{E(x),E(y)} : [0,1] \to M$ such that $\alpha_{E(x),E(y)}(0) = E(y)$, $\alpha'_{E(x),E(y)}(0) = \eta(E(x), E(y))$ and $\alpha_{E(x),E(y)}(t) \in A$ for all $t \in [0,1]$. By the geodesic E-preinvexity of $f \circ E$, we have

$$\begin{aligned}(f \circ E)(\alpha_{E(x),E(y)}(t)) &\leq tf(E(E(x))) + (1-t)f(E(E(y))) \\ &= t(f \circ E^2)(x) + (1-t)(f \circ E^2)(y) \\ &= tf(E(x)) + (1-t)f(E(y)) \\ &\leq t\lambda + (1-t)\lambda \\ &= \lambda, \quad \text{for all } t \in [0,1].\end{aligned}$$

Therefore, $\alpha_{E(x),E(y)}(t) \in S(f \circ E, \lambda)$ for all $t \in [0,1]$, and so, $S(f \circ E, \lambda)$ is a geodesic E-invex set with respect to the function η.

(ii) Put $\alpha := \inf_{x \in A}(f \circ E)(x)$. Then, clearly $K = \cap_{\lambda > \alpha} S(f \circ E, \lambda)$, i.e., K is an intersection of geodesic E-invex sets, and so in view of Proposition 3.4, it is a geodesic E-invex set.

Now, suppose that f is a strictly geodesic E-preinvex function. If $K = \emptyset$, we are done. Assume that $K \neq \emptyset$. We claim that K has only one point. Assume if possible that there exist $x, y \in K$ such that $x \neq y$. Then, by the geodesic E-invexity of K with respect to the function η, there exists exactly one geodesic $\beta_{E(x),E(y)} : [0,1] \to M$ such that

$$\beta_{E(x),E(y)}(0) = E(y), \quad \beta'_{E(x),E(y)}(0) = \eta(E(x), E(y)),$$

and $\beta_{E(x),E(y)}(t) \in K$ for all $t \in [0,1]$. Since f is a strictly E-preinvex function, thus

$$\begin{aligned}\alpha &= f(\beta_{E(x),E(y)}(t)) \\ &< tf(E(x)) + (1-t)f(E(y)) \\ &\leq t\alpha + (1-t)\alpha \\ &= \alpha, \quad \text{for all } t \in [0,1],\end{aligned}$$

which is a contradiction. □

4. Generalized Geodesic E-preinvex Functions

In [16], it has been introduced the notion of η-quasi-preinvex functions on an invex set. In [2], this notion extended to geodesic η-quasi-preinvexity on a geodesic invex set by replacing the line segments with geodesics. In this section, we extend this concept and define geodesic E-quasi-preinvex functions. Moreover, some properties and characterizations of this class of functions are presented.

Definition 4.1. Let $A \subseteq M$ be a nonempty geodesic E-invex set with respect to $\eta : M \times M \to TM$. A function $f : A \to \mathbb{R}$ is said to be

(i) geodesic E-quasi-preinvex if

$$f(\alpha_{E(x),E(y)}(t)) \leq \max\{f(E(x)), f(E(y))\},$$

for all $x, y \in A$ and all $t \in [0, 1]$;

(ii) strictly geodesic E-quasi-preinvex if for all $x, y \in A$ with $E(x) \neq E(y)$ and all $t \in (0, 1)$, $f(\alpha_{E(x),E(y)}(t)) < \max\{f(E(x)), f(E(y))\}$.

Theorem 4.1. *Let $A \subseteq M$ be a geodesic E-invex set and let $\{f_i\}_{i\in I}$ be a collection of real valued functions defined on A such that $\sup_{i\in I} f_i(x)$ is finite for each $x \in A$. Let $f : A \to \mathbb{R}$ be defined by $f(x) := \sup_{i\in I} f_i(x)$ for each $x \in A$.*

(i) If $f_i : A \to \mathbb{R}$, $i \in I$, is a geodesic E-preinvex function on A with respect to the function $\eta : M \times M \to TM$, then the function f is geodesic E-preinvex on A.

(ii) If $f_i : A \to \mathbb{R}$, $i \in I$, is a geodesic E-quasi-preinvex function on A, then the function f is geodesic E-quasi-preinvex on A.

Proof. (i) Let $f_i : A \to \mathbb{R}$, $i \in I$, be a geodesic E-preinvex function on A. Then, for each $x, y \in A$ and each $t \in [0, 1]$, we have

$$f_i(\alpha_{E(x),E(y)}(t)) \leq (1-t)f_i(E(x)) + tf_i(E(y)), \quad \text{for all } i \in I,$$

and so

$$\begin{aligned} f(\alpha_{E(x),E(y)}(t)) &= \sup_{i\in I} f_i(\alpha_{E(x),E(y)}(t)) \\ &\leq \sup_{i\in I}[(1-t)f_i(E(x)) + tf_i(E(y))] \\ &\leq (1-t)\sup_{i\in I} f_i(E(x)) + t\sup_{i\in I} f_i(E(y)) \\ &= (1-t)f(E(x)) + tf(E(y)). \end{aligned}$$

So, f is a geodesic E-preinvex function on A.

(ii) Suppose that $f_i : A \to \mathbb{R}$, $i \in I$, is a geodesic E-quasi-preinvex function on A. Therefore, by Definition 4.1, for each $x, y \in A$ and each $t \in [0, 1]$, one has

$$\begin{aligned} f(\alpha_{E(x),E(y)}(t)) &= \sup_{i\in I} f_i(\alpha_{E(x),E(y)}(t)) \\ &\leq \sup_{i\in I} \max\{f_i(E(x)), f_i(E(y))\} \\ &\leq \max\{\sup_{i\in I} f_i(E(x)), \sup_{i\in I} f_i(E(y))\} \\ &= \max\{f(E(x)), f(E(y))\}, \end{aligned}$$

and hence, f is a geodesic E-quasi-preinvex function on A. □

Let $A \subseteq M$ be a nonempty geodesic E-invex set. It follows from Proposition 3.2 that $E(A) \subseteq A$. Hence, for any function $f : A \to \mathbb{R}$, define the restriction $\tilde{f}$ of f to $E(A)$ by $\tilde{f}(\tilde{x}) := f(\tilde{x})$ for all $\tilde{x} \in E(A)$.

Theorem 4.2. *Let $A \subseteq M$ be a geodesic E-invex set and let $f : A \to \mathbb{R}$ be a geodesic E-quasi-preinvex function on A. Then the restriction $\tilde{f} : C \to \mathbb{R}$ of f to any nonempty invex subset C of $E(A)$ is a geodesic η-quasi-preinvex function on C.*

Proof. Let $x, y \in C \subseteq E(A)$ be arbitrary. Then there exist $x_1, y_1 \in A$ such that $x = E(x_1)$ and $y = E(y_1)$. Since C is an invex set, there exists exactly one geodesic $\alpha_{E(x),E(y)} : [0,1] \to M$ such that $\alpha_{x,y}(0) = y$, $\alpha'_{x,y}(0) = \eta(x,y)$ and $\alpha_{x,y}(t) \in C$ for all $t \in [0,1]$. But, $\alpha_{E(x_1),E(y_1)}(t) = \alpha_{x,y}(t) \in C$ for all $t \in [0,1]$. Therefore, since f is a geodesic E-quasi-preinvex function on A, we conclude that

$$\begin{aligned}\tilde{f}(\alpha_{x,y}(t)) &= f(\alpha_{E(x_1),E(y_1)}(t)) \\ &\leq \max\{f(E(x_1)), f(E(y_1))\} \\ &= \max\{f(x), f(y)\} \\ &= \max\{\tilde{f}(x), \tilde{f}(y)\},\end{aligned}$$

which completes the proof. □

Theorem 4.3. *Let $A \subseteq M$ be a geodesic E-invex set, $f : A \to \mathbb{R}$ be a real valued function and $E(A)$ be an invex set. Then, f is geodesic E-quasi-preinvex on A if and only if its restriction $\tilde{f}$ to $E(A)$ is geodesic E-quasi-preinvex on $E(A)$.*

Proof. Let $x, y \in A$ be arbitrary. So, $E(x), E(y) \in E(A)$. By the hypothesis, $E(A)$ is an invex set. Therefore, by the definition, we have $\alpha_{E(x),E(y)}(t) \in E(A)$ for all $t \in [0,1]$, where $\alpha_{E(x),E(y)}$ is the unique geodesic function corresponding to $E(A)$. Since $E(A) \subseteq A$ (because A is a geodesic E-invex set and using Proposition 3.4), it follows that

$$\alpha_{E(x),E(y)}(t) \in A, \quad \text{for all } t \in [0,1], x, y \in A. \tag{4.1}$$

Now, suppose that f is a geodesic E-quasi-preinvex function on A. Then

$$\begin{aligned}\tilde{f}(\alpha_{E(x),E(y)}(t)) &= f(\alpha_{E(x),E(y)}(t)) \\ &\leq \max\{f(E(x)), f(E(y))\} \\ &= \max\{\tilde{f}(E(x)), \tilde{f}(E(y))\},\end{aligned}$$

i.e., $\tilde{f}$ is geodesic E-quasi-preinvex on $E(A)$.

Conversely, assume that $\tilde{f}$ is a geodesic E-quasi-preinvex function on $E(A)$. Then, by (4.1), for each $x, y \in A$ and each $t \in [0,1]$, one has

$$\begin{aligned}f(\alpha_{E(x),E(y)}(t)) &= \tilde{f}(\alpha_{E(x),E(y)}(t)) \\ &\leqslant \max\{\tilde{f}(E(x)), \tilde{f}(E(y))\} \\ &= \max\{f(E(x)), f(E(y))\},\end{aligned}$$

and the proof is complete. □

An analogous result to Theorem 4.2 for the geodesic E-preinvex functions is presented as follows. The proof is similar to the one of Theorem 4.2.

Theorem 4.4. *Let $A \subseteq M$ be a geodesic E-invex set and $f : A \to \mathbb{R}$ be a geodesic E-preinvex function on A. Then, the restriction $\tilde{f} : C \to \mathbb{R}$ of f to any nonempty invex subset C of $E(A)$ is a geodesic invex function.*

An analogous result to Theorem 4.3 for the geodesic E-preinvex functions is presented as follows. The proof is similar to the one of Theorem 4.3.

Theorem 4.5. *Let $A \subseteq M$ be a geodesic E-invex set, $f : A \to \mathbb{R}$ be a real valued function and $E(A)$ be an invex set. Then, f is a geodesic E-preinvex function on A if and only if its restriction $\tilde{f}$ to $E(A)$ is a geodesic E-preinvex function on $E(A)$.*

We now characterize geodesic E-quasi-preinvex functions in terms of their lower level sets. For any real number $r \in \mathbb{R}$, the lower level set of the function $f \circ E : A \to \mathbb{R}$ is defined by $L_r(f \circ E) := \{x \in A : (f \circ E)(x) = f(E(x)) \leqslant r\}$. Moreover, the lower level set of the function $\tilde{f} : E(A) \to \mathbb{R}$ is defined by $L_r(\tilde{f}) := \{\tilde{x} \in E(A) : \tilde{f}(\tilde{x}) = f(\tilde{x}) \leqslant r\}$.

Theorem 4.6. *Let $E(A)$ be an invex set and $f : A \to \mathbb{R}$ be a real valued function. A function f is geodesic E-quasi-preinvex if and only if the lower level set $L_r(\tilde{f})$ is an invex set for each $r \in \mathbb{R}$.*

Proof. Suppose that f is a geodesic E-quasi-preinvex function. Since $E(A)$ is an invex set, for each $x, y \in A$, we have $E(x), E(y) \in E(A)$ and $\alpha_{E(x),E(y)}(t) \in E(A) \subseteq A$, where $\alpha_{E(x),E(y)}$ is the unique geodesic which defined by Definition 2.5. Let $r \in \mathbb{R}$ and $E(x), E(y) \in L_r(\tilde{f})$ be arbitrary. Put $\tilde{x} := E(x)$ and $\tilde{y} := E(y)$. Then, $\tilde{x}, \tilde{y} \in L_r(\tilde{f})$, and so, $f(\tilde{x}) \leqslant r$ and $f(\tilde{y}) \leqslant r$. Thus,

$$\tilde{f}(\alpha_{\tilde{x},\tilde{y}}(t)) = f(\alpha_{E(x),E(y)}(t)) \leqslant \max\{f(E(x)), f(E(y))\} = \max\{f(\tilde{x}), f(\tilde{y})\} \leqslant r,$$

which shows that $\alpha_{\tilde{x},\tilde{y}}(t) \in L_r(\tilde{f})$ for all $t \in [0,1]$. Moreover, one has $\alpha_{\tilde{x},\tilde{y}}(0) = \alpha_{E(x),E(y)}(0) = E(y) = \tilde{y}$ and $\alpha'_{\tilde{x},\tilde{y}}(0) = \alpha'_{E(x),E(y)}(0) = \eta(E(x), E(y)) = \eta(\tilde{x}, \tilde{y})$ because $E(A)$ is an invex set. Hence, $L_r(\tilde{f})$ is an invex set.

Conversely, assume that $L_r(\tilde{f})$ is an invex set for each $r \in \mathbb{R}$. Let $x, y \in A$ and $t \in [0,1]$ be arbitrary. Take $r := \max\{f(E(x)), f(E(y))\}$ and $\tilde{x} := E(x)$, $\tilde{y} := E(y)$. Therefore, $\tilde{f}(\tilde{x}) = f(E(x)) \leq r$ and $\tilde{f}(\tilde{y}) = f(E(y)) \leq r$ because $E(x), E(y) \in E(A)$. This implies that $\tilde{x}, \tilde{y} \in L_r(\tilde{f})$. Since, by the hypothesis, $L_r(\tilde{f})$ is an invex set, so there exists exactly one geodesic $\alpha_{\tilde{x},\tilde{y}} : [0,1] \to M$ such that $\alpha_{\tilde{x},\tilde{y}}(0) = \tilde{y}$, $\alpha'_{\tilde{x},\tilde{y}}(0) = \eta(\tilde{x}, \tilde{y})$ and $\alpha_{\tilde{x},\tilde{y}}(t) \in L_r(\tilde{f})$ for all $t \in [0,1]$. Then, since $L_r(\tilde{f}) \subseteq E(A)$, it follows that

$$f(\alpha_{E(x),E(y)}(t)) = f(\alpha_{\tilde{x},\tilde{y}}(t)) = \tilde{f}(\alpha_{\tilde{x},\tilde{y}}(t)) \leqslant r = \max\{f(E(x)), f(E(y))\},$$

and so, f is a geodesic E-quasi-preinvex function. □

The geodesic E-quasi-preinvexity preserves under nondecreasing functions.

Theorem 4.7. *Let $A \subseteq M$ be a nonempty geodesic E-invex set and let $f : A \to \mathbb{R}$ be a geodesic E-quasi-preinvex function. Suppose that $\Phi : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function. Then $\Phi \circ f$ is a geodesic E-quasi-preinvex function on A.*

Proof. Since the function $f : A \to \mathbb{R}$ is geodesic E-quasi-preinvex and $\Phi : \mathbb{R} \to \mathbb{R}$ is a nondecreasing function, then, for all $x, y \in A$ and all $t \in [0, 1]$, it follows that

$$\begin{aligned}(\Phi \circ f)(\alpha_{E(x),E(y)}(t)) &= \Phi(f(\alpha_{E(x),E(y)}(t))) \\ &\leq \Phi\{\max\{f(E(x)), f(E(y))\} \\ &\leq \max\{\Phi(f(E(x)), \Phi(f(E(y)))\} \\ &= \max\{(\Phi \circ f)(E(x)), (\Phi \circ f)(E(y))\},\end{aligned}$$

which shows that $\Phi \circ f$ is a geodesic E-quasi-preinvex function on A. □

Theorem 4.8. *If the function $f : A \to \mathbb{R}$ is geodesic E-preinvex on A, then f is a geodesic E-quasi-preinvex function on A.*

Proof. Let f be geodesic E-preinvex on A. Then, for all $x, y \in A$ and all $t \in [0, 1]$, we have

$$\begin{aligned}f(\alpha_{E(x),E(y)}(t)) &\leq (1-t)f(E(y)) + tf(E(x)) \\ &\leq (1-t)\max\{f(E(x)), f(E(y))\}, \\ &\quad + t\max\{f(E(x)), f(E(y))\} \\ &= \max\{f(E(x)), f(E(y))\},\end{aligned}$$

and hence, f is a geodesic E-quasi-preinvex function on A. □

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