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# ON COMMUTATIVITY DEGREE OF CROSSED MODULES

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Abstract. In this paper, we define and study the notion of commutativity degree of finite crossed modules. We shall state some results concerning commutativity degree of crossed modules and obtain some upper and lower bounds for commutativity degree of finite crossed modules. Finally we show that, if two crossed modules are isoclinic, then they have the same commutativity degree.

## 1. Introduction

In 1968, Erdös and Turán [3], introduced the concept of commutativity degree of groups, when they worked on symmetric groups. Let $G$ be a finite group, the commutativity degree of $G$, denoted by $d(G)$ is defined as

$$d(G) = \frac{|\{(x, y) \in G \times G : xy = yx\}|}{|G|^2}.$$

Note that $d(G) > 0$ and $d(G) = 1$ if and only if $G$ is abelian. In 1973, Gustafson [5] obtained an upper bound for $d(G)$, when $G$ is a non-abelian finite group. Few years later, Rusin [14] computed the value of $d(G)$, when $G' \subseteq Z(G)$ and $G' \cap Z(G)$ is trivial and classified all finite groups $G$ for which $d(G)$ is greater than $\frac{11}{32}$. In 1995, all finite groups $G$, where $d(G) \geq \frac{1}{2}$ are classified, up to isoclinism, by Lescot [7]. Furthermore, Lescot [8] has also classified, up to isomorphism, all finite groups whose commutativity degrees lie in the interval $[\frac{1}{2}, 1]$. In 2006, Barry et al. [1] have shown that if $G$ is a finite group with odd order and $d(G) \geq \frac{11}{75}$, then $G$ is supersolvable. In addition, it had been proved that if $d(G) > \frac{1}{3}$, then $G$ is supersolvable. In 2007, Erfanian et al. [4] studied relative commutativity degree $d(H, G)$, the probability that

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elements of a given subgroup $H$ of a finite group $G$ commute with elements of $G$. In 2008, Pournaki and Sobhani [11] studied $d_g(G)$, the probability that the commutator of an arbitrarily chosen pair of elements in a finite group $G$ equals a given element $g$. In 2018, Sepehrizadeh and Rismanchian [16] introduced and studied the concept of characteristic degree of a subgroup in a finite group and determined the upper and lower bounds for this probability.

A crossed module $(T, G, \delta)$ is a group homomorphism $\delta : T \to G$ together with an action of $G$ on $T$ satisfying certain conditions. This notion is an algebraic model for homotopy 3-types was already introduced by Whitehead [17] in 1948. In [10], [13] and [15] the concepts of isoclinism and $n$-isoclinism have been generalized for crossed modules (see also [6]). In 2019, Yavari and Salemkar [18] presented a generalized crossed module and investigated the category of generalized crossed modules. Also, in [2] and [12] the notions of stem cover and universal central extension have been extended for lie crossed modules.

In this paper, we generalize the concept of commutativity degree for the finite crossed modules and show that two isoclinic crossed modules have the same commutativity degree.

## 2. Definitions and Preliminaries on Crossed Modules

In this section, we state some basic definitions, notions and elementary results.

A crossed module $(T, G, \delta)$ is a pair of groups $T$ and $G$ together with an action of $G$ on $T$ and a homomorphism $\delta : T \to G$ called the boundary map, satisfying the following axioms:

*i)* $\delta({}^{g}t) = g\delta(t)g^{-1}$ for all $g \in G$, $t \in T$;

*ii)* ${}^{\delta(t)}s = tst^{-1}$ for all $t, s \in T$.

We will denote such a crossed module by $T \xrightarrow{\delta} G$. A crossed module $(T, G, \delta)$ is said to be finite, if the groups $T$ and $G$ are both finite. A crossed module $(S, H, \delta')$ is a subcrossed module of $(T, G, \delta)$, when

*i)* $S$ is a subgroup of $T$ and $H$ is a subgroup of $G$;

*ii)* $\delta' = \delta|_S$, the restriction of $\delta$ to $S$;

*iii)* the action of $H$ on $S$ is induced by the action of $G$ on $T$.

In this case, we write $(S, H, \delta') \leqslant (T, G, \delta)$. A subcrossed module $(S, H, \delta)$ of $(T, G, \delta)$ is a normal subcrossed module, if

*i)* $H$ is a normal subgroup of $G$;

*ii)* ${}^{g}s \in S$ for all $g \in G$, $s \in S$;

*iii)* ${}^{h}tt^{-1} \in S$ for all $h \in H$, $t \in T$.

This is denoted by $(S, H, \delta) \trianglelefteq (T, G, \delta)$.

Let $(S, H, \delta)$ be a normal subcrossed module of $(T, G, \delta)$. Consider the triple $(\frac{T}{S}, \frac{G}{H}, \bar{\delta})$, where $\bar{\delta} : \frac{T}{S} \to \frac{G}{H}$ is induced by $\delta$. There is the action of $\frac{G}{H}$ on $\frac{T}{S}$ given by ${}^{gH}(tS) = ({}^{g}t)S$. It is called the quotient crossed module of $(T, G, \delta)$ by $(S, H, \delta)$ and denoted by $\frac{(T,G,\delta)}{(S,H,\delta)}$.

Let $(T, G, \delta)$ be a crossed module. The center of $(T, G, \delta)$ is the crossed module $Z(T, G, \delta) : T^G \to St_G(T) \cap Z(G)$, where $T^G = \{t \in T : {}^g t = t \text{ for all } g \in G\}$ and $St_G(T) = \{g \in G : {}^g t = t \text{ for all } t \in T\}$. A crossed module $(T, G, \delta)$ is abelian, if $(T, G, \delta) = Z(T, G, \delta)$. In addition, the commutator subcrossed module $[(T, G, \delta), (T, G, \delta)]$ of $(T, G, \delta)$ is $[(T, G, \delta), (T, G, \delta)] : D_G(T) \to [G, G]$, where $D_G(T)$ is the subgroup generated by $\{{}^g t t^{-1} : t \in T,\ g \in G\}$ and $[G, G]$ is the commutator subgroup of $G$. A crossed module $(T, G, \delta)$ is faithful, if the action of $G$ on $T$ is faithful, that is $St_G(T) = 1$.

If $(S, H, \delta')$ and $(R, K, \delta'')$ are two crossed modules, then consider the triple $(S \times R, H \times K, \delta' \times \delta'')$, where $S \times R$ and $H \times K$ are direct products of groups and $\delta' \times \delta'' : S \times R \to H \times K$ is defined by $(\delta' \times \delta'')(s, r) = (\delta'(s), \delta''(r))$ for all $(s, r) \in S \times R$. There is a componentwise action of $H \times K$ on $S \times R$, induced by the actions of two crossed modules. The crossed module $(S \times R, H \times K, \delta' \times \delta'')$ is called the direct product of $(S, H, \delta')$ and $(R, K, \delta'')$ and denoted by $(S, H, \delta') \times (R, K, \delta'')$.

Let $(T, G, \delta)$ and $(T', G', \delta')$ be crossed modules. A crossed module morphism $\langle \alpha, \phi \rangle : (T, G, \delta) \to (T', G', \delta')$ is a pair of homomorphism $\alpha : T \to T'$, $\phi : G \to G'$ such that

*i)* $\delta'(\alpha(t)) = \phi(\delta(t))$ for all $t \in T$;

*ii)* $\alpha({}^g t) = {}^{\phi(g)}\alpha(t)$ for all $t \in T$, $g \in G$.

If $\langle \alpha, \phi \rangle : (S, H, \delta') \to (T, G, \delta)$ is a crossed module morphism such that $\alpha$ and $\phi$ are both group isomorphisms, then $\langle \alpha, \phi \rangle$ is called an isomorphism.

**Lemma 2.1** ([9])**.** *The $(T, G, \delta)$ is abelian if and only if $G$ is abelian and the action of the crossed module is trivial.*

*Remark* 2.1. Let $(T, G, \delta)$ be a crossed module. We denote $\frac{(T,G,\delta)}{Z(T,G,\delta)}$ by $\bar{T} \xrightarrow{\bar{\delta}} \bar{G}$, where $\bar{T} = \frac{T}{T^G}$ and $\bar{G} = \frac{G}{St_G(T) \cap Z(G)}$, for shortness.

**Lemma 2.2** ([10])**.** *Let $(T, G, \delta)$ be a crossed module. Define the maps $c_1 : \bar{T} \times \bar{G} \to D_G(T)$, where $(tT^G, g(St_G(T) \cap Z(G))) \mapsto {}^g t t^{-1}$ and $c_0 : \bar{G} \times \bar{G} \to [G, G]$, where $(g(St_G(T) \cap Z(G)), g'(St_G(T) \cap Z(G))) \mapsto [g, g']$ for all $t \in T$, $g, g' \in G$. Then the maps $c_1$ and $c_0$ are well-defined.*

**Definition 2.1** ([15])**.** The crossed modules $(T_1, G_1, \delta_1)$ and $(T_2, G_2, \delta_2)$ are isoclinic, if there exist isomorphisms

$$(\eta_1, \eta_0) : (\bar{T_1}, \bar{G_1}, \bar{\delta_1}) \to (\bar{T_2}, \bar{G_2}, \bar{\delta_2})$$

and

$$(\epsilon_1, \epsilon_0) : (D_{G_1}(T_1) \to [G_1, G_1]) \to (D_{G_2}(T_2) \to [G_2, G_2])$$

such that the diagrams

$$\begin{array}{ccc}
\bar{T_1} \times \bar{G_1} & \xrightarrow{c_1} & D_{G_1}(T_1) \\
\Big\downarrow{\scriptstyle \eta_1 \times \eta_0} & & \Big\downarrow{\scriptstyle \epsilon_1} \\
\bar{T_2} \times \bar{G_2} & \xrightarrow{c_1'} & D_{G_1}(T_1)
\end{array}$$

and

$$\begin{array}{ccc}
\bar{G_1} \times \bar{G_1} & \xrightarrow{c_0} & [G_1, G_1] \\
\Big\downarrow{\scriptstyle \eta_0 \times \eta_0} & & \Big\downarrow{\scriptstyle \epsilon_0} \\
\bar{G_2} \times \bar{G_2} & \xrightarrow{c_0'} & [G_2, G_2]
\end{array}$$

are commutative, where $(c_1, c_0)$ and $(c_1', c_0')$ are commutator maps of crossed modules $(T_1, G_1, \delta_1)$ and $(T_2, G_2, \delta_2)$, that introduced in Lemma 2.2. The pair $((\eta_1, \eta_0), (\epsilon_1, \epsilon_0))$ will be called an isoclinism from $(T_1, G_1, \delta_1)$ to $(T_2, G_2, \delta_2)$ and this situation will be denoted by $((\eta_1, \eta_0), (\epsilon_1, \epsilon_0)) : (T_1, G_1, \delta_1) \sim (T_2, G_2, \delta_2)$.

## 3. Commutativity Degree of Crossed Modules

In the section, we generalize the notion of commutativity degree for crossed modules and state the main results of this paper.

**Definition 3.1.** Let $(T, G, \delta)$ be a finite crossed module. The commutativity degree $d(T, G, \delta)$ of $(T, G, \delta)$ is defined by

$$d(T, G, \delta) = \frac{|\{(x, y) \in G \times G : xy = yx, x, y \in St_G(T)\}|}{|G|^2}.$$

Let $(T, G, \delta)$ be a finite crossed module, then we set $cs(G) = \{(x, y) \in G \times G : xy = yx \text{ and } x, y \in St_G(T)\}$. Now, the commutativity degree $(T, G, \delta)$ is $d(T, G, \delta) = \frac{|cs(G)|}{|G \times G|}$. If $d(T, G, \delta)$ is abelian and the action of $G$ on $T$ is trivial, then $|cs(G)| = |G \times G|$ and $d(T, G, \delta) = 1$ and vice-versa. Therefore, $(T, G, \delta)$ is abelian if and only if $d(T, G, \delta) = 1$. In addition, if the action of $G$ on $T$ is faithful, then $d(T, G, \delta) = \frac{1}{|G|^2}$.

**Proposition 3.1.** *Let $(S, H, \delta')$ and $(R, K, \delta'')$ be two crossed modules and $(T, G, \delta)$ $= (S, H, \delta') \times (R, K, \delta'')$. Then $d(T, G, \delta) = d(S, H, \delta') \times d(R, K, \delta'')$.*

*Proof.* By the definition of commutativity degree, we have

$$\begin{aligned}
d(T, G, \delta) =& \frac{1}{|G|^2} |\{((h_1, k_1), (h_2, k_2)) \in G^2 : (h_1, k_1)(h_2, k_2) = (h_2, k_2)(h_1, k_1) \\
& \text{and } (h_1, k_1), (h_2, k_2) \in St_G(T)\}| \\
=& \frac{1}{|G|^2} |\{((h_1, k_1), (h_2, k_2)) \in G^2 : (h_1 h_2, k_1 k_2) = (h_2 h_1, k_2 k_1) \\
& \text{and } (h_1, k_1), (h_2, k_2) \in St_G(T)\}| \\
=& \left( \frac{1}{|H|^2} |\{(h_1, h_2) \in H^2 : h_1 h_2 = h_2 h_1 \text{ and } h_1, h_2 \in St_H(S)\}| \right)
\end{aligned}$$

$$\times\left(\frac{1}{|K|^2}|\{(k_1,k_2)\in K^2 : k_1k_2=k_2k_1 \text{ and } k_1,k_2\in St_K(R)\}|\right)$$
$$=d(S,H,\delta')\times d(R,K,\delta''). \qquad \square$$

**Theorem 3.1.** *Let $(T,G,\delta)$ be a crossed module. Then $d(T,G,\delta)\leq \frac{K(G)}{|G|}$, where $K(G)$ is the number of conjugacy classes of $G$.*

*Proof.* Let $r$ be the number of conjugacy classes of $G$ and $C_1,C_2,C_3,\dots,C_r$ be the conjugacy classes of $G$. For $i\in\{1,2,3,\dots,r\}$, let $x_i\in C_i$. If $y\in C_i$, then $y=x_i^g$ for some $g\in G$. Thus, $C_G(y)=C_G(x_i^g)=C_G(x_i)^g$ and $|C_G(y)|=|C_G(x_i)|$. Now

$$\begin{aligned}|G|^2d(T,G,\delta)=&|\{(x,y)\in G\times G : xy=yx \text{ and } x,y\in St_G(T)\}|\\ =&|cs(G)|\leq\sum_{x\in G}|C_G(x)|=\sum_{i=1}^{r}\sum_{x\in C_i}|C_G(x)|\\ =&\sum_{i=1}^{r}[G:C_G(x_i)]|C_G(x_i)|=|G|r=|G|K(G).\end{aligned}$$

Therefore, $d(T,G,\delta)\leq\frac{K(G)}{|G|}$. $\square$

**Corollary 3.1.** *If $(T,G,\delta)$ is a crossed module and the action of $G$ on $T$ is trivial, then $d(T,G,\delta)=\frac{K(G)}{|G|}$.*

**Corollary 3.2.** *Let $(T,G,\delta)$ be a crossed module. If the action of $G$ on $T$ is trivial, then $\frac{1}{|G'|}\leq d(T,G,\delta)$.*

*Proof.* Since $[G:G']$ count irreducible characters of degree one, $[G:G']<K(G)$. Then $\frac{|G|}{|G||G'|}\leq\frac{K(G)}{|G|}=d(T,G,\delta)$ so $\frac{1}{|G'|}\leq d(T,G,\delta)$. $\square$

**Theorem 3.2.** *Let $(T,G,\delta)$ be a crossed module. Then $d(T,G,\delta)\leq\frac{1}{4}(1+\frac{3}{|G'|})$.*

*Proof.* Let $l$ be the number of non-equivalent irreducible representation of degree $1,n_2,\dots,n_l$. Consider the degree equation

$$|G|=[G:G']+\sum_{i=[G:G']+1}^{K(G)}(n_i)^2,$$

for each $n_i\geq 2$. Hence, $|G|\geq[G:G']+4(K(G)-[G:G'])$. Solving for $K(G)$ yield $K(G)\leq\frac{1}{4}(|G|+3[G:G'])$. Therefore, $d(T,G,\delta)\leq\frac{K(G)}{|G|}\leq\frac{1}{4}(1+\frac{3}{|G'|})$. $\square$

**Theorem 3.3.** *Let $(T,G,\delta)$ be a crossed module. If $G$ is a non-abelian finite group, then $d(T,G,\delta)\leq\frac{5}{8}$.*

*Proof.* Consider the class equation $|G|=|Z(G)|+\sum_{i=|Z(G)|+1}^{K(G)}|[x_i]|$, where for each $i$, $|[x_i]|\geq 2$. So $|G|\geq|Z(G)|+2(K(G)-|Z(G)|)\geq|Z(G)\cap St_G(T)|+2(K(G)-|Z(G)|)$.

Since $G$ is not abelian, $\frac{G}{Z(G)}$ is not cyclic. Then $|\frac{G}{Z(G)}| \geq 4$, hence

$$\left|\frac{G}{Z(G) \cap St_G(T)}\right| \geq \left|\frac{G}{Z(G)}\right| \geq 4$$

and $|Z(G) \cap St_G(T)| \leq \frac{|G|}{4}$. Therefore, $K(G) \leq \frac{1}{2}\left(|G| + \frac{|G|}{4}\right) = \frac{5|G|}{8}$. Then $d(T, G, \delta) \leq \frac{K(G)}{|G|} \leq \frac{5}{8}$. □

**Theorem 3.4.** *Let $(T, G, \delta)$ be a crossed module. If $G$ be a nonablian $p$-group, then $d(T, G, \delta) \leq \frac{p^2+p-1}{p^3}$.*

*Proof.* Let $p$ be the prime number, $|G| = p^n$ and $|Z(G)| = P^m$. If $|G/Z(G)| = 1$ or $|G/Z(G)| = p$, then $G/Z(G)$ is cyclic and $G$ will be abelian. Thus, $m \leq n - 2$ and we have

$$\begin{aligned}
|G|^2 d(T, G, \delta) \leq & |G| K(G) \\
= & \sum_{x \in G} |C_G(x)| \\
= & \sum_{x \in Z(G)} |C_G(x)| + \sum_{x \in G \setminus Z(G)} |C_G(x)| \\
\leq & p^{m+n} + p^{n-1}(p^n - p^m) \\
= & p^{m+n} + p^{n-1}(|G| - |Z(G)|) \\
\leq & p^{2n-3}(p^2 + p - 1).
\end{aligned}$$

Hence, $d(T, G, \delta) \leq \dfrac{p^2 + p - 1}{p^3}$. □

The following corollary obtains from previous theorem.

**Corollary 3.3.** *Let $(T, G, \delta)$ be a crossed module and the action of $G$ on $T$ is trivial. If $p$ is a prime number and $G$ is non-abelian with $|G| = p^3$, then $d(T, G, \delta) = \frac{p^2+p-1}{p^3}$.*

*Proof.* Let $|G| = p^3$. By Corollary 3.1 and Theorem 3.4,

$$\begin{aligned}
p^6 d(T, G, \delta) = & |G|^2 d(T, G, \delta) \\
= & |G| K(G) \\
= & \sum_{x \in G} |C_G(x)| \\
= & \sum_{x \in Z(G)} |C_G(x)| + \sum_{x \in G \setminus Z(G)} |C_G(x)| \\
= & p^4 + p^2(|G| - |Z(G)|) \\
= & p^3(p^2 + p - 1).
\end{aligned}$$

Hence, $d(T, G, \delta) = \frac{p^2+p-1}{p^3}$. □

**Proposition 3.2.** *Let $(T, G, \delta)$ be a crossed module and the action of $G$ on $T$ be trivial. If $|\frac{G}{Z(G)}| = p^k$, then $d(T, G, \delta) \geq \frac{p^{k-1}+p^k-1}{p^{2k-1}}$.*

*Proof.* Let $|\frac{G}{Z(G)}| = p^k$ and $x \in G$ such that $x \notin Z(G)$. Then, since $x \in C_G(x)$ and $x \notin Z(G)$, we have $C_G(x) \subsetneq G$. Also $Z(G) \subsetneq C_G(x)$. Thus $|Z(G)| < |C_G(x)| < |G|$ and then $p|Z(G)| \leq |C_G(x)| \leq p^{k-1}|Z(G)|$, where $|C_G(x)|$ dividing $|G|$. So $|[x]| = [G : C_G(x)]$ and $p^{k-1} \geq |[x]| \geq p$. From the class equation we have $|G| = |Z(G)| + \sum_{x \in G} |[x]| \leq |Z(G)| + p^{k-1}(K(G) - |Z(G)|) \leq |Z(G)| + p^{k-1}(K(G) - |Z(G) \cap St_G(T)|)$, therefore

$$K(G) \geq \frac{|G| + (p^{k-1})|Z(G) \cap St_G(T)| - |Z(G)|}{p^{k-1}}.$$

Now solving for $d(T, G, \delta)$ and Corollary 3.1 yield $d(T, G, \delta) \geq \frac{p^k+p^{k-1}-1}{p^{2k-1}}$. □

*Example* 3.1. Let $D_{pq} =< a, b : a^p = b^q = e, bab^{-1} = a^r >$ such that $p$ is prime, $q|p-1$ and $r$ has order $q$ mod $p$. This type of group is called a generalized dihedral group. Conjugacy classes type are $[e]$, $[a^u]$ and $[b^w]$ so that no classes are 1, $\frac{p-1}{q}$ and $q-1$, respectively and $Z(D_{pq}) = \{e\}$. Consider the map $i : D_{pq} \to D_{pq}$. If the action of $D_{pq}$ on $D_{pq}$ is conjugacy, then $St_{D_{pq}}(D_{pq}) = Z(D_{pq})$ and $d(D_{pq}, D_{pq}, i) = \frac{|Z(D_{pq})|^2}{|D_{pq}|^2} = \frac{1}{(pq)^2}$. If the action of $D_{pq}$ on $D_{pq}$ is trivial, then

$$d(D_{pq}, D_{pq}, i) = \frac{K(D_{pq})}{|D_{pq}|} = \frac{1 + \frac{p-1}{q} + q - 1}{pq} = \frac{q^2 + p - 1}{pq^2}.$$

If the action of $D_{pq}$ on $D_{pq}$ is faithful, then $d(D_{pq}, D_{pq}, i) = \frac{1}{|D_{pq}|^2} = \frac{1}{(pq)^2}$.

**Lemma 3.1.** *Let $(T_1, G_1, \delta_1)$ and $(T_2, G_2, \delta_2)$ be two crossed modules and $\varphi = \langle \alpha, \beta \rangle : (T_1, G_1, \delta_1) \to (T_2, G_2, \delta_2)$ be an isomorphism. Then $\varphi$ induced isomorphisms*

$$\varphi_1 : \frac{G_1}{St_{G_1}(T_1) \cap Z(G_1)} \to \frac{G_2}{St_{G_2}(T_2) \cap Z(G_2)},$$

*where*

$$\varphi_1(g_1(St_{G_1}(T_1) \cap Z(G_1))) = \beta(g_1)(St_{G_2}(T_2) \cap Z(G_2)),$$

*$\varphi_2 : \frac{T_1}{T_1^{G_1}} \to \frac{T_2}{T_2^{G_2}}$, where $\varphi_2(t_1 T_1^{G_1}) = \alpha(t_1) T_2^{G_2}$, $\varphi_3 : [G_1, G_1] \to [G_2, G_2]$, where $\varphi_3[g_1, g_1] = [g_2, g_2]$ or $\varphi_3(g_1) = \beta(g_1)$, $\varphi_4 : D_{G_1}(T_1) \to D_{G_2}(T_2)$, where $\varphi_4({}^{g_1}t_1 t_1^{-1}) = {}^{\beta(g_1)}\alpha(t_1)\alpha(t_1{}^{-1})$. In addition the following diagrams commute:*

$$\begin{array}{ccc} \bar{G}_1 \times \bar{G}_1 & \xrightarrow{\varphi_1 \times \varphi_1} & \bar{G}_2 \times \bar{G}_2 \\ \downarrow{\scriptstyle c_0} & & \downarrow{\scriptstyle c_0'} \\ G_1' & \xrightarrow{\varphi_3} & G_2' \end{array}$$

*and*

$$\begin{array}{ccc}
\bar{T_1}\times\bar{G_1} & \xrightarrow{\varphi_2\times\varphi_1} & \bar{T_2}\times\bar{G_2} \\
\Big\downarrow{\scriptstyle c_1} & & \Big\downarrow{\scriptstyle c_1'} \\
D_{G_1}(T_1) & \xrightarrow{\varphi_4} & D_{G_1}(T_1).
\end{array}$$

*Proof.* Let $(g_1,g_1')\in G_1\times G_1$. Then $c_0'(\varphi_1\times\varphi_1)(\bar{g_1},\bar{g_1'})=c_0'(\varphi_1(\bar{g_1}),\varphi_1(\bar{g_1'}))=c_0'(\overline{\beta(g_1)},\overline{\beta(g_1')})=[\beta(g_1),\beta(g_1')]=\beta[g_1,g_1']=\varphi_3[g_1,g_1']=\varphi_3c_0(\bar{g_1},\bar{g_1'})$. Now, let $(\bar{t_1},\bar{g_1})\in\bar{T_1}\times\bar{G_1}$. Then

$$\begin{aligned}
c_1'(\varphi_2\times\varphi_1)(\bar{t_1},\bar{g_1}) =&c_1'(\varphi_2(\bar{t_1}),\varphi_1(\bar{g_1}))\\
=&c_1'(\alpha(t_1)T_2^{G_2},\beta(g_1)(St_{G_2}(T_2)\cap Z(G_2)))\\
=&{}^{\beta(g_1)}\alpha(t_1)\alpha(t_1)^{-1}=\varphi_4({}^{g_1}t_1t_1^{-1})\\
=&\varphi_4c_1(\bar{t_1},\bar{g_1}).
\end{aligned}$$

□

**Theorem 3.5.** *Let $(T_1,G_1,\delta_1)$,$(T_2,G_2,\delta_2)$ be two isoclinic finite crossed modules. Then $d(T_1,G_1,\delta_1)=d(T_2,G_2,\delta_2)$.*

*Proof.* Consider the following relation:

$$\begin{aligned}
\frac{|G_1|^2d(T_1,G_1,\delta_1)}{|St_{G_1}(T_1)\cap Z(G_1)|^2} =&\frac{|G_1|^2}{|St_{G_1}(T_1)\cap Z(G_1)|^2}\\
&\times\frac{|\{(x,y)\in G_1\times G_1: xy=yx \text{ and } x,y\in St_{G_1}(T_1)\}|}{|G_1|^2}\\
=&\left|\frac{1}{St_{G_1}(T_1)\cap Z(G_1)}\right|^2\\
&\times|\{(x,y)\in G_1\times G_1: xy=yx \text{ and } x,y\in St_{G_1}(T_1)\}|\\
=&\left|\frac{1}{St_{G_1}(T_1)\cap Z(G_1)}\right|^2\\
&\times|\{(x,y)\in G_1\times G_1: [x,y]=1 \text{ and } x,y\in St_{G_1}(T_1)\}|\\
=&\left|\frac{1}{St_{G_1}(T_1)\cap Z(G_1)}\right|^2|\{(x,y)\in G_1\times G_1: x,y\in St_{G_1}(T_1)\\
&\text{and } c_0(x(St_{G_1}(T_1)\cap Z(G_1)),y(St_{G_1}(T_1)\cap Z(G_1)))=1\}|\\
=&\left|\left\{(\alpha,\beta)\in\left(\frac{G_1}{St_{G_1}(T_1)\cap Z(G_1)}\right)^2: c_0(\alpha,\beta)=1\right\}\right|\\
=&\left|\left\{(\alpha,\beta)\in\left(\frac{G_1}{St_{G_1}(T_1)\cap Z(G_1)}\right)^2: \varphi_3c_0(\alpha,\beta)=1\right\}\right|\\
=&\left|\left\{(\alpha,\beta)\in\left(\frac{G_1}{St_{G_1}(T_1)\cap Z(G_1)}\right)^2: c_0'(\varphi_1,\varphi_1)(\alpha,\beta)=1\right\}\right|
\end{aligned}$$

$$= \left| \left\{ (\alpha, \beta) \in \left( \frac{G_1}{St_{G_1}(T_1) \cap Z(G_1)} \right)^2 : c_0'(\varphi_1(\alpha), \varphi_1(\beta)) = 1 \right\} \right|$$
$$= \left| \left\{ (\gamma, \sigma) \in \left( \frac{G_2}{St_{G_2}(T_2) \cap Z(G_2)} \right)^2 : c_0'(\gamma, \sigma) = 1 \right\} \right|.$$

By the above reasoning applied to $(T_2, G_2, \delta_2)$ in place of $(T_1, G_1, \delta_1)$, this expression equals to $\left|\frac{G_2}{St_{G_2}(T_2)\cap Z(G_2)}\right|^2 d(T_2, G_2, \delta_2)$. That is $\left|\frac{G_1}{St_{G_1}(T_1)\cap Z(G_1)}\right|^2 d(T_1, G_1, \delta_1) = \left|\frac{G_2}{St_{G_2}(T_2)\cap Z(G_2)}\right|^2 d(T_2, G_2, \delta_2)$. But $\frac{G_1}{St_{G_1}(T_1)\cap Z(G_1)}$ and $\frac{G_2}{St_{G_2}(T_2)\cap Z(G_2)}$ are isomorphic, hence $\left|\frac{G_1}{St_{G_1}(T_1)\cap Z(G_1)}\right| = \left|\frac{G_2}{St_{G_2}(T_2)\cap Z(G_2)}\right|$. Now the equality $d(T_1, G_1, \delta_1) = d(T_2, G_2, \delta_2)$ follows. □

**Corollary 3.4.** *Let $(T_2, G_2, \delta_2)$ be a subcrossed module of crossed module $(T_1, G_1, \delta_1)$ and $(T_1, G_1, \delta_1) = (T_2, G_2, \delta_2)Z(T_1, G_1, \delta_1)$, where $T_1 = G_2 T_1^{G_1}$ and $G_1 = G_2(St_{G_1}(T_1) \cap Z(G_1))$. Then $d(T_1, G_1, \delta_1) = d(T_2, G_2, \delta_2)$.*

*Proof.* By Proposition 4 of [10], $(T_1, G_1, \delta_1)$ and $(T_2, G_2, \delta_2)$ are isoclinic, therefore $d(T_1, G_1, \delta_1) = d(T_2, G_2, \delta_2)$. □

## 4. Conclusion

In this paper, we extended the concept of commutativity degree in group theory to finite crossed modules and derived some properties of this new concept. All our previous results show that the notion of commutativity degree, which was introduced in this paper, can be used to classify finite crossed modules. It is clear that this study which started here, can be successfully extended to calculating commutativity degree of some specific crossed modules. This will surely be the subject of further research.

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