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ON GAMMA-RINGS WITH (σ,τ)-SKEW-COMMUTING AND (σ,τ)-SKEW-CENTRALIZING MAPPINGS

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Abstract. Let M be a 2-torsion free Γ-ring with left identity e. Let $D : M \times M \to M$ be a symmetric bi-additive mapping and $d(x) = D(x,x)$. Let σ and τ be an endomorphism and an epimorphism of M, respectively. We prove the following:

(i) if d is (σ,τ)-skew-commuting on M, then $D = 0$;
(ii) if d is (τ,τ)-skew-centralizing on M, then d is (τ,τ)-commuting on M;
(iii) if M is a 3-torsion free Γ-ring satisfying $x\alpha y\beta z = x\beta y\alpha z$ for all $x,y,z \in M$ and $\alpha,\beta \in \Gamma$, then 2-(σ,τ)-commutingness of d on M implies its (σ,τ)-commutingness.

1. Introduction

Yong-Soo Jung and Ick-Soon Chang [4] worked on (σ,τ)-skew commuting and (σ,τ)-skew centralizing maps of rings with left identity. Many authors (see, e.g. Bresar [3], Vukman [10] and references there in) investigated and studied skew-centralizing and skew-commuting mappings in classical ring theories. Bell and Lucier [2] studied skew-commuting and skew-centralizing additive maps by the existence of a left identity element instead of the condition of primeness of a ring and obtained some fruitful results concerning these. The study of permuting tri-derivations in prime and semiprime Γ-rings has been investigated by Duran Ozden and M. Ali Ozturk [5]. Symmetric bi-derivations and generalized symmetric bi-derivations have been studied in [6] and [7] by the authors Ozturk et al. and Ozturk and Sapanci, respectively. In [8], Ozturk worked on permuting tri-derivations in prime and semi-prime rings and developed some fruitful results in ring theories. M. A. Ozturk et al. [9] worked on symmetric

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bi-derivations on prime Γ-rings. They obtained some remarkable results on prime Γ-rings. In this paper, we study symmetric bi-additive maps with the generalized skew-commuting and skew-centralizing mappings of the trace, that is (σ,τ)-skew-commuting and (σ,τ)-skew-centralizing ones, in Γ-rings with left identity.

2. Preliminaries

Let M and Γ be additive abelian groups. Then, M is called a Γ-ring in the sense of Barnes [1] if there is a mapping $M\times\Gamma\times M\to M$ for all $a,b,c\in M$, $\alpha,\beta\in\Gamma$, such that the following conditions are satisfied:

(i) $(a+b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\alpha(b+c) = a\alpha b + a\alpha c$,
(ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$.

Every ring is a Γ-ring and many notions on the ring theory are generalized to Γ-rings. Let M be a Γ-ring. A subring I of M is an additive subgroup which is also a Γ-ring. A right ideal of M is a subring I such that $I\Gamma M\subseteq I$. Similarly, a left ideal can be defined. If I is both a right and a left ideal then we say that I is an ideal. In this paper, we shall take the following assumption

$$x\alpha y\beta z = x\beta y\alpha z, \tag{2.1}$$

for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$. Throughout this paper, M will represent a Γ-ring, and $Z(M)$ will be its center. Let $x,y\in M$ and $\alpha\in\Gamma$, the commutator $x\alpha y - y\alpha x$ will be denoted by $[x,y]$. It is easy to see that

$$[x\beta y, z]_\alpha = x\beta[y,z]_\alpha + [x,z]_\alpha\beta y + x[\beta,\alpha]_z y$$

and

$$[x, y\beta z]_\alpha = y\beta[x,z]_\alpha + [x,y]_\alpha\beta z + y[\beta,\alpha]_x z,$$

for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$. The assumption (2.1) reduces the above identities respectively to

$$[x\beta y, z]_\alpha = x\beta[y,z]_\alpha + [x,z]_\alpha\beta y$$

and

$$[x, y\beta z]_\alpha = y\beta[x,z]_\alpha + [x,y]_\alpha\beta z,$$

for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$.

Let σ, τ be additive mappings from M into M and let $x,y\in M$. For convenience, the products $y\alpha x + x\alpha y$, $y\alpha\sigma(x)+\tau(x)\alpha y$ and $y\alpha\sigma(x)-\tau(x)\alpha y$ are denoted by $\langle y,x\rangle_\alpha$, $\langle y,x\rangle_\alpha^{(\sigma,\tau)}$ and $[y,x]_\alpha^{(\sigma,\tau)}$, respectively.

3. Main Results

We begin with the following results.

Theorem 3.1. *Let M be a 2-torsion-free Γ-ring with left identity e. Let $\sigma : M\to M$ be an endomorphism and $\tau : M\to M$ be an epimorphism. Let $D : M\times M\to M$ be a symmetric bi-additive mapping and d the trace of D. If d is (σ,τ)-skew-commuting on M, then we have $D = 0$.*

Proof. We are given that

$$\langle d(x),x\rangle_\alpha^{(\sigma,\tau)} = d(x)\alpha\sigma(x)+\tau(x)\alpha d(x)=0, \tag{3.1}$$

for all $x\in M$ and $\alpha\in\Gamma$. First, observe that if τ is onto, then $\tau(e)$ is also a left identity of M. This along with (3.1) gives

$$\langle d(e),e\rangle_\alpha^{(\sigma,\tau)} = d(e)\alpha\sigma(e)+\tau(e)\alpha d(e)=d(e)\alpha\sigma(e)+d(e)=0, \tag{3.2}$$

for all $x\in M$ and $\alpha\in\Gamma$. Then multiplying (3.2) from right-hand side by $\alpha\sigma(e)$ we obtain $2d(e)\alpha\sigma(e)=0$, and it implies that $d(e)\alpha\sigma(e)=0$. Hence, from (3.2) we get $d(e)=0$. Let us replace x by $x+e$ in (3.1). Then we have

$$\langle d(x+e),x+e\rangle_\alpha^{(\sigma,\tau)} = d(x+e)\alpha\sigma(x+e)+\tau(x+e)\alpha d(x+e)=0,$$

for all $x\in M$ and $\alpha\in\Gamma$. We obtain

$$\langle d(x),e\rangle_\alpha^{(\sigma,\tau)}+2\langle D(x,e),x\rangle_\alpha^{(\sigma,\tau)}+2\langle D(x,e),e\rangle_\alpha^{(\sigma,\tau)}=0, \tag{3.3}$$

for all $x\in M$ and $\alpha\in\Gamma$. Substituting $-x$ for x in (3.3) and comparing (3.3) with the result, we get

$$\langle d(-x),e\rangle_\alpha^{(\sigma,\tau)}+2\langle D(-x,e),-x\rangle_\alpha^{(\sigma,\tau)}+2\langle D(-x,e),e\rangle_\alpha^{(\sigma,\tau)}=0,$$

for all $x\in M$ and $\alpha\in\Gamma$. Then

$$\langle d(x),e\rangle_\alpha^{(\sigma,\tau)} = D(x,e)\alpha\sigma(e)+D(x,e)=0, \tag{3.4}$$

for all $x\in M$ and $\alpha\in\Gamma$, since d is an even function and M is 2-torsion free. Right multiplication (3.4) by $\alpha\sigma(e)$ gives

$$2D(x,e)\alpha\sigma(e)=0=D(x,e)\alpha\sigma(e),$$

and so, by (3.4), we have $D(x,e)=0$, for all $x\in M$. Therefore we arrive at

$$d(x+e)=d(x)+d(e)+2D(x,e)=d(x),$$

for all $x\in M$. Since d is (σ,τ)-skew-commuting on M, the relation

$$d(x+e)\alpha\sigma(x+e)+\tau(x+e)\alpha d(x+e)=0$$

becomes

$$d(x)\alpha\sigma(x)+d(x)\alpha\sigma(e)+\tau(x)\alpha d(x)+\tau(e)\alpha d(x)=0,$$

and thus we obtain

$$d(x)\alpha\sigma(e)+d(x)=0, \tag{3.5}$$

for all $x\in M$ and $\alpha\in\Gamma$. Right-multiplying by $\alpha\sigma(e)$ in (3.5), we get $2d(x)\alpha\sigma(e)=0=d(x)\alpha\sigma(e)$, and hence the relation (3.5) implies that $d(x)=0$ for all $x\in M$, which gives the conclusion. ☐

The next result is to improve the above result.

Corollary 3.1. *Let M be a 2-torsion-free Γ-ring with left identity e. Let $\sigma: M\to M$ be endomorphisms and $\tau: M\to M$ be epimorphisms. If f is an additive map on M such that the mapping $x\mapsto\langle f(x),x\rangle_\alpha^{(\sigma,\tau)}$ is (σ,τ)-skew-commuting on M, then $f=0$.*

Proof. Define a mapping $D : M \times M \to M$ by

$$D(x, y) = \langle f(x), y\rangle_{\alpha}^{(\sigma,\tau)} + \langle f(y), x\rangle_{\alpha}^{(\sigma,\tau)},$$

for all $x, y \in M$ and $\alpha \in \Gamma$, and a mapping $d : M \to M$ by $d(x) = D(x, x)$, for all $x \in M$, it is obvious that D is symmetric and bi-additive, and that d is the trace of D. The hypothesis that the mapping $x \mapsto \langle f(x), x\rangle_{\alpha}^{(\sigma,\tau)}$ is (σ, τ)-skew- commuting on M is equivalent to the fact that d is (σ, τ)-skew-commuting on M, and so the theorem asserts us that $d = 0$, that is, f is (σ, τ)-skew-commuting on M, from which it follows that

$$f(e)\alpha\sigma(e) + \tau(e)\alpha f(e) = f(e)\alpha\sigma(e) + f(e) = 0, \tag{3.6}$$

for all $\alpha \in \Gamma$, and right-multiplying by $\alpha\sigma(e)$ gives $2f(e)\alpha\sigma(e) = 0 = f(e)\alpha\sigma(e)$. By (3.6), since M is a 2-torsion free Γ-ring we get $f(e) = 0$ and so $f(x + e) = f(x)$ for all $x \in M$. The condition that $f(x + e)\alpha\sigma(x + e) + \tau(x + e)\alpha f(x + e) = 0$ now makes $f(x)\alpha\sigma(x) + f(x)\alpha\sigma(e) + \tau(x)\alpha f(x) + f(x) = 0$, and it follows that

$$f(x)\alpha\sigma(e) + f(x) = 0, \tag{3.7}$$

for all $x, y \in M$ and $\alpha \in \Gamma$. Right-multiplying by $\alpha\sigma(e)$, we get $2f(x)\alpha\sigma(e) = 0 = f(x)\alpha\sigma(e)$, so by (3.7) we have $f(x) = 0$, for all $x \in M$. □

We continue our investigation with the next result.

Theorem 3.2. *Let M be a 2-torsion-free Γ-ring with left identity e. Let $\tau : M \to M$ be an epimorphism. Let $D : M \times M \to M$ be a symmetric bi-additive mapping and d the trace of D. If d is (τ, τ)-skew-centralizing on M, then d is (τ, τ)-commuting on M.*

Proof. Suppose that

$$\langle d(x), x\rangle_{\alpha}^{(\tau,\tau)} = d(x)\alpha\tau(x) + \tau(x)\alpha d(x) \in Z(M), \tag{3.8}$$

for all $x \in M$ and $\alpha \in \Gamma$. Since $\tau(e)$ is a left identity of M by the ontoness of τ, the supposition implies

$$d(e)\alpha\tau(e) + \tau(e)\alpha d(e) = d(e)\alpha\tau(e) + d(e) \in Z(M). \tag{3.9}$$

Commuting with $\tau(e)$ we get $d(e) = d(e)\alpha\tau(e)$, and it along with (3.9) gives $2d(e) \in Z(M)$, hence $d(e) \in Z(M)$. Let us replace x by $x + e$ in (3.8). Then we have

$$\begin{aligned} d(x)\alpha\tau(e) + 2\tau(x)\alpha d(e) + 2D(x, e)\alpha\tau(x) + 2D(x, e)\alpha\tau(e) + d(x) \\ + 2\tau(x)\alpha D(x, e) + 2D(x, e) \in Z(M), \end{aligned} \tag{3.10}$$

for all $x \in M$ and $\alpha \in \Gamma$. Substituting $-x$ for x in (3.10) and comparing (3.10) with the result, we obtain

$$\begin{aligned} d(-x)\alpha\tau(e) + 2\tau(-x)\alpha d(e) + 2D(-x, e)\alpha\tau(-x) + 2D(-x, e)\alpha\tau(e) + d(-x) \\ + 2\tau(-x)\alpha D(-x, e) + 2D(-x, e) \in Z(M). \end{aligned}$$

We get

$$\tau(x)\alpha d(e)+D(x,e)\alpha\tau(e)+D(x,e)\in Z(M), \tag{3.11}$$

for all $x\in M$ and $\alpha\in\Gamma$, because of d is even and M is 2-torsion free.

Since $d(e)\in Z(M)$ and e is a left identity of M, commuting with $\tau(e)$ in (3.11) gives

$$[D(x,e),\tau(e)]_\alpha=0, \tag{3.12}$$

for all $x\in M$ and $\alpha\in\Gamma$. Thus, from (3.12) we conclude that $D(x,e)=D(x,e)\alpha\tau(e)$, for all $x\in M$ and $\alpha\in\Gamma$. Now, we can rewrite (3.11) as follows

$$\tau(x)\alpha d(e)+2D(x,e)\in Z(M), \tag{3.13}$$

and commuting with $\tau(x)$ in (3.13) gives

$$2[D(x,e),\tau(x)]_\alpha=0=[D(x,e),\tau(x)]_\alpha,$$

for all $x\in M$ and $\alpha\in\Gamma$. Due to the ontoness of τ we obtain $D(x,e)\in Z(M)$, for all $x\in M$ and $\alpha\in\Gamma$.

In view of $D(x,e)=D(x,e)\alpha\tau(e)$ and $D(x,e)\in Z(M)$, the relation (3.10) can be rewritten in the form

$$d(x)\alpha\tau(e)+d(x)+2\tau(x)\alpha d(e)+4\tau(x)\alpha D(x,e)\in Z(M), \tag{3.14}$$

for all $x\in M$ and $\alpha\in\Gamma$. Commuting with $\tau(e)$ in (3.14) and then using the fact that $[y,\tau(e)]_\alpha\beta z=0$, for all $y,z\in M$ and $\alpha,\beta\in\Gamma$, yields

$$[d(x),\tau(e)]_\alpha\beta\tau(e)+[d(x),\tau(e)]_\alpha=0, \tag{3.15}$$

for all $x\in M$ and $\alpha,\beta\in\Gamma$, and right-multiplying by $\beta\tau(e)$ gives

$$2[d(x),\tau(e)]_\alpha\beta\tau(e)=0=[d(x,\tau(e)]_\alpha\beta\tau(e)$$

and so it follows from (3.15) that $d(x)=d(x)\alpha\tau(e)$, for all $x\in M$ and $\alpha\in\Gamma$. Consequently, we see that the relation (3.14) becomes

$$d(x)+\tau(x)\alpha d(e)+2\tau(x)\alpha D(x,e)\in Z(M), \tag{3.16}$$

since M is 2-torsion free. Commuting with $\tau(x)$ in (3.16), we have $[d(x),\tau(x)]_\alpha=0$, for all $x\in M$ and $\alpha\in\Gamma$, which completes the proof. □

Let $\sigma,\tau:M\to M$ be endomorphisms. We define a mapping $f:M\to M$ to be 2-(σ,τ)-skew-commuting (respectively, 2-(σ,τ)-skew-centralizing) on the subset S if $\langle f(x),x\beta x\rangle_\alpha^{(\sigma,\tau)}=0$ (respectively, $\langle f(x),x\beta x\rangle_\alpha^{(\sigma,\tau)}\in Z(M)$), for all $x\in S$ and $\alpha,\beta\in\Gamma$, and f is said to be 2-(σ,τ)-commuting on S if $[f(x),x\beta x]_\alpha^{(\sigma,\tau)}=0$, for all $x\in S$ and $\alpha,\beta\in\Gamma$. Of course, when $\sigma=\tau=1$ (the identity map on M), f is simply called 2-skew-commuting, 2-skew-centralizing and 2-commuting on S, respectively. Here we extend the results on (σ,τ)-skew-commuting maps to 2-(σ,τ)-skew-commuting ones.

Theorem 3.3. *Let M be a 2,3-torsion-free Γ-ring with left identity e. Let $\sigma : M \to M$ be an endomorphism and $\tau : M \to M$ be an epimorphism. Let $D : M \times M \to M$ be a symmetric bi-additive mapping and d the trace of D. If d is 2-(σ, τ)-skew-commuting on M, then we have $D = 0$.*

Proof. Assume that

$$\langle d(x), x\beta x\rangle_\alpha^{(\sigma,\tau)} = 0, \tag{3.17}$$

for all $x \in M$ and $\alpha, \beta \in \Gamma$. Note that $d(e) = 0$ by the same argument used in the proof of Theorem 3.1. Let t be any positive integer. Replacing x by $x + te$ in (3.17) and using $d(x + te) = d(x) + t^2 d(e) + 2tD(x, e)$, for all $x \in M$ and $\alpha \in \Gamma$, we obtain

$$\langle d(x + te), (x + te)\beta(x + te)\rangle_\alpha^{(\sigma,\tau)} = 0,$$

for all $x \in M$ and $\alpha \in \Gamma$. Then

$$\langle d(x) + 2tD(x, e), x\beta x + te\beta x + tx\beta e + t^2 e\beta e\rangle_\alpha^{(\sigma,\tau)} = 0,$$

for all $x \in M$ and $\alpha \in \Gamma$. Hence

$$\begin{aligned}\langle d(x), x\beta x\rangle_\alpha^{(\sigma,\tau)} &+ t\langle d(x), e\beta x\rangle_\alpha^{(\sigma,\tau)} + t\langle d(x), x\beta e\rangle_\alpha^{(\sigma,\tau)} \\ &+ t^2\langle d(x), e\beta e\rangle_\alpha^{(\sigma,\tau)} \\ &+ 2t\langle D(x, e), x\beta x\rangle_\alpha^{(\sigma,\tau)} + 2t^2\langle D(x, e), e\beta x\rangle_\alpha^{(\sigma,\tau)} \\ &+ 2t^2\langle D(x, e), x\beta e\rangle_\alpha^{(\sigma,\tau)} + 2t^3\langle D(x, e), e\beta e\rangle_\alpha^{(\sigma,\tau)} = 0,\end{aligned}$$

for all $x \in M$ and $\alpha \in \Gamma$. Since t is arbitrary and the coefficient determinant $\neq 0$, and also M is 2,3 torsion free,we have

$$\begin{aligned}\langle D(x, e), x\beta x\rangle_\alpha^{(\sigma,\tau)} + \langle d(x), e\beta x\rangle_\alpha^{(\sigma,\tau)} + \langle d(x), x\beta e_\alpha^{(\sigma,\tau)} &= 0, \\ \langle D(x, e), e\beta x\rangle_\alpha^{(\sigma,\tau)} + \langle D(x, e), x\beta e\rangle_\alpha^{(\sigma,\tau)} &= 0, \\ \langle D(x, e), e\beta x\rangle_\alpha^{(\sigma,\tau)} &= 0.\end{aligned}$$

In particular, for all $x \in M$ and $\alpha, \beta \in \Gamma$, we have

$$\langle D(x, e), e\beta e\rangle_\alpha^{(\sigma,\tau)} = 0. \tag{3.18}$$

By (3.18), we obtain that

$$2\{D(x, e)\alpha\sigma(e) + \tau(e)\alpha D(x, e)\} = 0 = D(x, e)\alpha\sigma(e) + D(x, e), \tag{3.19}$$

for all $x \in M$ and $\alpha \in \Gamma$; and right-multiplying by $\alpha\sigma(e)$ and using (3.19), we get $D(x, e) = 0$, for all $x \in M$. Hence this forces (3.19) to

$$\langle d(x), e\beta e\rangle_\alpha^{(\sigma,\tau)} = d(x)\alpha\sigma(e) + \tau(e)\alpha d(x) = d(x)\alpha\sigma(e) + d(x) = 0, \tag{3.20}$$

for all $x \in M$ and $\alpha \in \Gamma$. Multiplying by $\alpha\sigma(e)$ on the right and utilizing (3.20), we conclude that $d(x) = 0$ for all $x \in M$. This completes the proof. □

Corollary 3.2. *Let M be a 2,3-torsion-free Γ-ring with left identity e. Let $\sigma : M \to M$ be an endomorphism and $\tau : M \to M$ be an epimorphism such that σ is (τ,τ)-commuting on M. If f is an additive map on M which is 2-(σ,τ)-skew-centralizing on M, then f is (τ,τ)-commuting on M.*

Proof. Since $f(x)\alpha\sigma(x)\beta\sigma(x) + \tau(x)\beta\tau(x)\alpha f(x) \in Z(M)$, for all for all $x \in M$ and $\alpha, \beta \in \Gamma$, we have

$$[f(x)\alpha\sigma(x)\beta\sigma(x) + \tau(x)\beta\tau(x)\alpha f(x), \tau(x)]_\gamma = 0,$$

for all $x \in M$ and $\alpha, \beta, \gamma \in \Gamma$, hence

$$[f(x),\tau(x)]_\gamma\alpha\sigma(x)\beta\sigma(x) + f(x)\alpha[\sigma(x)\beta\sigma(x),\tau(x)]_\gamma + \tau(x)\beta\tau(x)\alpha[f(x),\tau(x)]_\gamma = 0,$$

which reduces to

$$[f(x),\tau(x)]_\gamma\alpha\beta\sigma(x)\beta\sigma(x) + \tau(x)\beta\tau(x)\alpha[f(x),\tau(x)]_\gamma = 0, \tag{3.21}$$

for all $x \in M$ and $\alpha, \beta, \gamma \in \Gamma$, because σ is (τ,τ)-commuting on M, i.e., $[\sigma(x),\tau(x)]_\gamma = 0$, for all $x \in M$ and $\alpha, \beta, \gamma \in \Gamma$. We introduce the mapping $D : M \times M \to M$ by

$$D(x,y) = [f(x),\tau(y)]_\gamma + [f(y),\tau(x)]_\gamma,$$

for all $x, y \in M$ and $\gamma \in \Gamma$; and the mapping $d : M \to M$ by $d(x) = D(x,x)$, for all $x \in M$, it is obvious that D is symmetric and bi-additive, and that d is the trace of D. Now the relation (3.21) is equivalent to the fact that d is 2-(σ,τ)-skew-commuting, and so it follows from Theorem 3.3 that $d(x) = 2[f(x),\tau(x)]_\gamma = 0$, for all $x \in M$ and $\gamma \in \Gamma$. Since M is 2-torsion-free, we obtain the conclusion of the theorem. ☐

Theorem 3.4. *Let M be a 2,3-torsion-free Γ-ring satisfying the condition* (3.1) *with left identity e. Let $\sigma : M \to M$ be an endomorphism and $\tau : M \to M$ be an epimorphism. Let $D : M \times M \to M$ be a symmetric bi-additive mapping and d the trace of D. If d is 2-(σ,τ)-commuting on M, then d is (σ,τ)-commuting on M.*

Proof. Let us define a mapping $h : M \to M$ by $h(x) = [d(x),x]_\alpha^{(\sigma,\tau)}$ for all $x \in M$ and $\alpha \in \Gamma$. Our assumption can now be written in the form

$$\langle h(x),x\rangle_\alpha^{(\sigma,\tau)} = [d(x),x\beta x]_\alpha^{(\sigma,\tau)} = 0, \quad \text{for all } x \in M,\ \alpha,\beta \in \Gamma. \tag{3.22}$$

Since $\tau(e)$ is also a left identity of M by the ontoness of τ, it follows that

$$h(e)\alpha\sigma(e) + \tau(e)\alpha h(e) = h(e)\alpha\sigma(e) + h(e) = 0, \quad \text{for all } x \in M,\ \alpha \in \Gamma, \tag{3.23}$$

and right-multiplying by $\alpha\sigma(e)$ gives $2h(e)\alpha\sigma(e) = 0 = h(e)\alpha\sigma(e)$. Hence, by (3.23), we get $h(e) = [d(e),e]_\alpha^{(\sigma,\tau)} = 0$. Note that h is odd and for all $x \in M$ and $\alpha \in \Gamma$,

$$h(x+e) = h(x) + [d(e),x]_\alpha^{(\sigma,\tau)} + 2[D(x,e),e]_\alpha^{(\sigma,\tau)} + [d(x),e]_\alpha^{(\sigma,\tau)} + 2[D(x,e),x]_\alpha^{(\sigma,\tau)}. \tag{3.24}$$

We claim that $h(x+e)=h(x)$ $x\in M$ and $\alpha\in\Gamma$. Replacing x by $x+e$ in (3.22) and using (3.24), we have, $x\in M$ and $\alpha\in\Gamma$

$$
\begin{aligned}
0 &= \langle h(x+e), x+e\rangle_\beta^{(\sigma,\tau)} \\
&= h(x)\alpha\sigma(e) + [d(e),x]_\alpha^{(\sigma,\tau)}\beta\sigma(x) + [d(e),x]_\alpha^{(\sigma,\tau)}\beta\sigma(e) \\
&\quad + 2[D(x,e),e]_\alpha^{(\sigma,\tau)}\beta\sigma(x) + 2[D(x,e),e]_\alpha^{(\sigma,\tau)}\beta\sigma(e) \\
&\quad + [d(x),e]_\alpha^{(\sigma,\tau)}\beta\sigma(x) + [d(x),e]_\alpha^{(\sigma,\tau)}\beta\sigma(e) \\
&\quad + 2[D(x,e),x]_\alpha^{(\sigma,\tau)}\beta\sigma(x) + 2[D(x,e),x]_\alpha^{(\sigma,\tau)}\beta\sigma(e) + h(x) \\
&\quad + \tau(x)\beta[d(e),x]_\alpha^{(\sigma,\tau)} + [d(e),x]_\alpha^{(\sigma,\tau)} + 2\tau(x)\beta[D(x,e),e]_\alpha^{(\sigma,\tau)} \\
&\quad + 2[D(x,e),e]_\alpha^{(\sigma,\tau)} + \tau(x)\beta[d(x),e]_\alpha^{(\sigma,\tau)} + [d(x),e]_\alpha^{(\sigma,\tau)} \\
&\quad + 2\tau(x)\beta[D(x,e),x]_\alpha^{(\sigma,\tau)} + 2[D(x,e),x]_\alpha^{(\sigma,\tau)}.
\end{aligned} \tag{3.25}
$$

Substituting $-x$ for x in (3.25) and comparing (3.25) with the result, we get, $x\in M$ and $\alpha\in\Gamma$

$$
\begin{aligned}
&[d(e),x]_\alpha^{(\sigma,\tau)}\beta\sigma(x) \\
&+ 2[D(x,e),e]_\alpha^{(\sigma,\tau)}\beta\sigma(x) + [d(x),e]_\alpha^{(\sigma,\tau)}\beta\sigma(e) \\
&+ 2[D(x,e),x]_\alpha^{(\sigma,\tau)}\beta\sigma(e) + \tau(x)\beta[d(e),x]_\alpha^{(\sigma,\tau)} \\
&+ 2\tau(x)\beta[D(x,e),e]_\alpha^{(\sigma,\tau)} + [d(x),e]_\alpha^{(\sigma,\tau)} + 2[D(x,e),x]_\alpha^{(\sigma,\tau)} = 0;
\end{aligned} \tag{3.26}
$$

and right multiplication of (3.26) by $\beta\sigma(e)$ gives,

$$
\begin{aligned}
0 &= [d(e),x]_\alpha^{(\sigma,\tau)}\beta\sigma(x)\beta\sigma(e) + 2[D(x,e),e]_\alpha^{(\sigma,\tau)}\beta\sigma(x)\beta\sigma(e) \\
&\quad + 2[d(x),e]_\alpha^{(\sigma,\tau)}\beta\sigma(e) + 4[D(x,e),x]_\alpha^{(\sigma,\tau)}\beta\sigma(e) \\
&\quad + \tau(x)\beta[d(e),x]_\alpha^{(\sigma,\tau)}\beta\sigma(e) + 2\tau(x)\beta[D(x,e),e]_\alpha^{(\sigma,\tau)}\beta\sigma(e).
\end{aligned} \tag{3.27}
$$

Let us put $x+e$ instead of x in (3.27) and utilize (3.27). Then we obtain

$$6[d(e),x]_\alpha^{(\sigma,\tau)}\beta\sigma(e) + 12[D(x,e),e]_\alpha^{(\sigma,\tau)}\beta\sigma(e) = 0;$$

and so

$$[d(e),x]_\alpha^{(\sigma,\tau)}\beta\sigma(e) + 2[D(x,e),e]_\alpha^{(\sigma,\tau)}\beta\sigma(e) = 0; \tag{3.28}$$

and the relation (3.28) yields

$$
\begin{aligned}
&[d(e),x]_\alpha^{(\sigma,\tau)}\beta\sigma(x) + 2[D(x,e),e]_\alpha^{(\sigma,\tau)}\beta\sigma(x) \\
&= [d(e),x]_\alpha^{(\sigma,\tau)}\beta\sigma(e\gamma x) + 2[D(x,e),e]_\alpha^{(\sigma,\tau)}\beta\sigma(e\gamma x) \\
&= \left\{[d(e),x]_\alpha^{(\sigma,\tau)}\beta\sigma(e) + 2[D(x,e),e]_\alpha^{(\sigma,\tau)}\beta\sigma(e)\right\}\gamma\sigma(x) = 0.
\end{aligned} \tag{3.29}
$$

Hence the relation (3.27) becomes

$$2[d(x),e]_\alpha^{(\sigma,\tau)}\beta\sigma(e) + 4[D(x,e),x]_\alpha^{(\sigma,\tau)}\beta\sigma(e) = 0;$$

which gives

$$[d(x), e]_\alpha^{(\sigma,\tau)}\beta\sigma(e) + 2[D(x, e), x]_\alpha^{(\sigma,\tau)}\beta\sigma(e) = 0, \tag{3.30}$$

for all $x \in M$ and $\alpha, \beta \in \Gamma$. According to (3.29) and (3.30), we therefore can be written (3.26) in the form

$$\tau(x)\beta[d(e), x]_\alpha^{(\sigma,\tau)} + 2\tau(x)\beta[D(x, e), e]_\alpha^{(\sigma,\tau)} + [d(x), e]_\alpha^{(\sigma,\tau)} + 2[D(x, e), x]_\alpha^{(\sigma,\tau)} = 0, \tag{3.31}$$

for all $x \in M$ and $\alpha, \beta \in \Gamma$. Finally, replacing x by $x + e$ in (3.31) and applying (3.31) to the result, we obtain

$$3[d(e), x]_\alpha^{(\sigma,\tau)} + 6[D(x, e), e]_\alpha^{(\sigma,\tau)} = 0;$$

which implies that

$$[d(e), x]_\alpha^{(\sigma,\tau)} + 2[D(x, e), e]_\alpha^{(\sigma,\tau)} = 0, \tag{3.32}$$

for all $x \in M$ and $\alpha \in \Gamma$; and the relation (3.31) with (3.32) yields

$$[d(x), e]_\alpha^{(\sigma,\tau)} + 2[D(x, e), x]_\alpha^{(\sigma,\tau)} = 0, \tag{3.33}$$

for all $x \in M$ and $\alpha \in \Gamma$. By applying (3.33) and (3.24), we now obtain that $h(x + e) = h(x)$, for all $x \in M$ and $\alpha \in \Gamma$, as claimed. Since $\langle h(x), x\rangle_\alpha^{(\sigma,\tau)} = 0$ for all $x \in M$ and $\alpha \in \Gamma$, the relation $h(x + e)\alpha\sigma(x + e) + \tau(x + e)\alpha h(x + e) = 0$ becomes $h(x)\alpha(\sigma(x) + \sigma(e)) + (\tau(x) + \tau(e))\alpha h(x) = 0$, and it follows that

$$h(x)\alpha\sigma(e) + h(x) = 0, \tag{3.34}$$

for all $x \in M$ and $\alpha \in \Gamma$. Right-multiplying by $\alpha\sigma(e)$ in (3.34), we get $2h(x)\alpha\sigma(e) = 0 = h(x)\alpha\sigma(e)$, and hence the relation (3.34) yields $h(x) = 0$, for all $x \in M$ and $\alpha \in \Gamma$ which gives the conclusion. □

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