Kragujevac Journal of Mathematics
Volume 47(1) (2023), Pages 39–55.

(FUZZY) FILTERS OF SHEFFER STROKE BL-ALGEBRAS

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Abstract. In this study, some (fuzzy) filters of a Sheffer stroke BL-algebra and its properties are presented. To show a relationship between a filter and a fuzzy filter of Sheffer stroke BL-algebra, we prove that f is a fuzzy (ultra) filter of C if and only if f_p is either empty or a (ultra) filter of C for each $p \in [0, 1]$, and it is satisfied for $p = f(1)$ and for the characteristic function of a nonempty subset of a Sheffer stroke BL-algebra.

1. Introduction

The idea of fuzzy set theory as well as fuzzy logic was propounded by Lotfi Zadeh ([20, 21]). The interest in foundations of fuzzy logic has been rapidly proceeding recently and many new algebras playing the role of the structures of truth values have been introduced.

The most important task of artificial intelligence is to make computers which simulate human behaviors. The classical logic deals with certain information while nonclassical logic such as many valued logics and fuzzy logic engages in uncertainty, or fuzziness and randomness. Since fuzziness and randomness are closely related to human's intelligence and behaviors, the fuzzy theory using in many various areas from science to technology plays an important role in improving artificial intelligence.

Filters have fundamental importance in algebra and play significant role in studying fuzzy logics. From logical point of view, they correspond to sets of provable formulas. Besides, they have a variety of some applications in logic and topology. Different approaches of fuzzy filters have been investigated by many authors ([6, 9, 10, 18]).

Petr Hájek introduced the axiom system of basic logic (BL) for fuzzy propositional logic and defined the class of BL-algebras [5]. He presented filters and prime filters

Key words and phrases. (Sheffer stroke) BL-algebra, Sheffer stroke, filter.
2010 *Mathematics Subject Classification.* Primary: 03G25. Secondary: 03B60, 03G10.
Received: March 19, 2020.
Accepted: June 21, 2020.

on this algebraic structure and gave the completeness proof of basic logic by using these prime filters [5]. Since the filters and fuzzy filters have an important place in the logical algebra theory, Boolean, positive implicative, maximal, prime, proper filters and implicative deductive systems on BL-algebras are researched ([6, 8]). In recent times, Liu and Li studied on fuzzy filters on a BL-algebra ([9, 10]), and Saeid et al. analyzed some kinds of filters, open problems on fuzzy filters and double complemeted elements of BL-algebras ([1–3]). Also, Xueling et al. generalized fuzzy filters of BL-algebras [18] and Zhan et al. examined on their some types [17]. Moreover, Yin and Zhan researched new types of fuzzy filters in these algebras [19]. Indeed, Haveshki and Eslami investigated n-fold filters of BL-algebras [7] and Motamed et al. studied on n-fold obstinate filters [12] and radicals of filters in BL-algebras [13]. Besides, H. M. Sheffer introduced Sheffer operation [16], and then McCune et al. showed that every Boolean function or axiom may be restated by this operation [11]. Since the Sheffer operation is a commutative, it satisfies that many algebraic structures have more useful axiom system. Recently, Sheffer stroke Hilbert algebras and filters a strong Sheffer stroke non-associative MV-algebras are studied (for details [14] and [15], respectively).

We first give basic definitions and notions related to a Sheffer stroke BL-algebra, and present new properties. Then some kind of (fuzzy) filters are defined and exemplified. Besides, we prove that f is a fuzzy (ultra) filter of a Sheffer stroke BL-algebra if and only if $f_p = \{c_1 \in C : p \leq f(c_1)\} \neq \emptyset$ is its (ultra) filter for any $p \in (0, 1]$, and it is satisfied for $p = f(1)$ and for the characteristic function χ_P of P in which P is the nonempty subset of a Sheffer stroke BL-algebra.

2. Preliminaries

In this section, we give fundamental definitions and notions about Sheffer stroke BL-algebras, BL-algebras, filters and fuzzy filters of BL-algebras.

Definition 2.1 ([4])**.** Let $\mathcal{C} = \langle C, | \rangle$ be a groupoid. The operation $|$ is said to be a *Sheffer stroke* if it satisfies the following conditions:

$(S1)$ $c_1|c_2 = c_2|c_1$;
$(S2)$ $(c_1|c_1)|(c_1|c_2) = c_1$;
$(S3)$ $c_1|((c_2|c_3)|(c_2|c_3)) = ((c_1|c_2)|(c_1|c_2))|c_3$;
$(S4)$ $(c_1|((c_1|c_1)|(c_2|c_2)))|(c_1|((c_1|c_1)|(c_2|c_2))) = c_1$.

Definition 2.2. A Sheffer stroke BL-algebra is an algebra $(C, \vee, \wedge, |, 0, 1)$ of type $(2, 2, 2, 0, 0)$ satisfying the following conditions:

$(sBL-1)$ $(C, \vee, \wedge, 0, 1)$ is a bounded lattice;
$(sBL-2)$ $(C, |)$ is a groupoid with the Sheffer stroke;
$(sBL-3)$ $c_1 \wedge c_2 = (c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2)))$;
$(sBL-4)$ $(c_1|(c_2|c_2)) \vee (c_2|(c_1|c_1)) = 1$,

for all $c_1, c_2 \in C$.

$1 = 0|0$ is the greatest element and $0 = 1|1$ is the least element of C.

Example 2.1. Consider a structure $(C, \vee, \wedge, |, 0, 1)$ with the following Hasse diagram (see Figure 1), where $C = \{0, u, v, 1\}$.

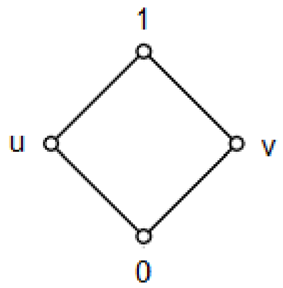

FIGURE 1.

The binary operations $|$, $\vee$ and $\wedge$ on C have Cayley tables as follow in Table 1, 2 and 3.

TABLE 1. The table of a Sheffer stroke $|$

$\|$	0	u	v	1
0	1	1	1	1
u	1	v	1	v
v	1	1	u	u
1	1	v	u	0

TABLE 2. The table of $\vee$

$\vee$	0	u	v	1
0	0	u	v	1
u	u	u	1	1
v	v	1	v	1
1	1	1	1	1

TABLE 3. The table of $\wedge$

$\wedge$	0	u	v	1
0	0	0	0	0
u	0	u	0	u
v	0	0	v	v
1	0	u	v	1

Then this structure is a Sheffer stroke BL-algebra.

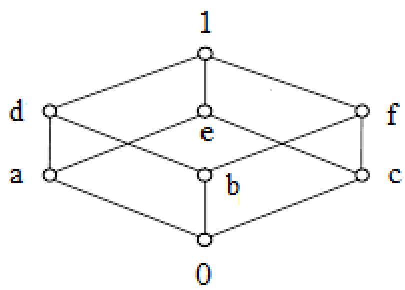

FIGURE 2.

Example 2.2. Consider a structure $(C, \vee, \wedge, |, 0, 1)$ with the following Hasse diagram (see Figure 2), where $C = \{0, a, b, c, d, e, f, 1\}$.

The binary operations $|$, $\vee$ and $\wedge$ on C have Cayley tables as follow in Table 4, 5 and 6. Then this structure is a Sheffer stroke BL-algebra.

TABLE 4. The table of a Sheffer stroke $|$

$\|$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	1	f	1	1	f	f	1	f
b	1	1	e	1	e	1	e	e
c	1	1	1	d	1	d	d	d
d	1	f	e	1	c	f	e	c
e	1	f	1	d	f	b	d	b
f	1	1	e	d	e	d	a	a
1	1	f	e	d	c	b	a	1

TABLE 5. The table of $\vee$

$\vee$	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	a	d	e	d	e	1	1
b	b	d	b	f	d	1	f	1
c	c	e	f	c	1	e	f	1
d	d	d	d	1	d	1	1	1
e	e	e	1	e	1	e	1	1
f	f	1	f	f	1	1	f	1
1	1	1	1	1	1	1	1	1

Proposition 2.1. *In any Sheffer stroke BL-algebra C, the following features hold, for all $c_1, c_2, c_3 \in C$:*

Table 6. The table of $\wedge$

$\wedge$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	0	0	a	a	0	a
b	0	0	b	0	b	0	b	b
c	0	0	0	c	0	c	c	c
d	0	a	b	0	d	a	b	d
e	0	a	0	c	a	e	c	e
f	0	0	b	c	b	c	f	f
1	0	a	b	c	d	e	f	1

(1) $c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) = c_2|((c_1|(c_3|c_3))|(c_1|(c_3|c_3)))$;
(2) $c_1|(c_1|c_1) = 1$;
(3) $1|(c_1|c_1) = c_1$;
(4) $c_1|(1|1) = 1$;
(5) $(c_1|1)|(c_1|1) = c_1$;
(6) $(c_1|c_2)|(c_1|c_2) \leq c_3 \Leftrightarrow c_1 \leq c_2|(c_3|c_3)$;
(7) $c_1 \leq c_2$ *if and only if* $c_1|(c_2|c_2) = 1$;
(8) $c_1 \leq c_2|(c_1|c_1)$;
(9) $c_1 \leq (c_1|c_2)|c_2$;
(10) (*a*) $(c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2))) \leq c_1$;
(*b*) $(c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2))) \leq c_2$;
(11) *if* $c_1 \leq c_2$, *then*
(*i*) $c_3|(c_1|c_1) \leq c_3|(c_2|c_2)$;
(*ii*) $(c_1|c_3)|(c_1|c_3) \leq (c_2|c_3)|(c_2|c_3)$;
(*iii*) $c_2|(c_3|c_3) \leq c_1|(c_3|c_3)$;
(12) $c_1|(c_2|c_2) \leq (c_3|(c_1|c_1))|((c_3|(c_2|c_2))|(c_3|(c_2|c_2)))$;
(13) $c_1|(c_2|c_2) \leq (c_2|(c_3|c_3))|((c_1|(c_3|c_3))|(c_1|(c_3|c_3)))$;
(14) $((c_1 \vee c_2)|c_3)|((c_1 \vee c_2)|c_3) = ((c_1|c_3)|(c_1|c_3)) \vee ((c_2|c_3)|(c_2|c_3))$;
(15) $c_1 \vee c_2 = ((c_1|(c_2|c_2))|(c_2|c_2)) \wedge ((c_2|(c_1|c_1))|(c_1|c_1))$.

Proof. (1) It follows from $(S1)$ and $(S3)$.

(2) We get $c_1|(c_1|c_1) = (c_1|(c_1|c_1)) \vee (c_1|(c_1|c_1)) = 1$ from $(sBL-1)$ and $(sBL-4)$.

(3) We have $1|(c_1|c_1) = (c_1|(c_1|c_1))|(c_1|c_1) = c_1$ from (2), $(S1)$ and $(S2)$.

(4) It is obtained from (3), $(S1)$ and $(S2)$ that $c_1|(1|1) = (1|(c_1|c_1))|(1|1) = 1$.

(5) It follows from $(S1)$, $(S2)$ and (3) that

$$(c_1|1)|(c_1|1) = (1|((c_1|c_1)|(c_1|c_1)))|(1|((c_1|c_1)|(c_1|c_1))) = (c_1|c_1)|(c_1|c_1) = c_1.$$

(6) $(\Rightarrow)$ Let $(c_1|c_2)|(c_1|c_2) \leq c_3$. Then it follows from $(sBL-1)$, $(sBL-3)$, $(S1)$ and $(S3)$ that

$$\begin{aligned}(c_1|c_2)|(c_1|c_2) &= ((c_1|c_2)|(c_1|c_2)) \wedge c_3\\ &= (((c_1|c_2)|(c_1|c_2))|(((c_1|c_2)|(c_1|c_2))|(c_3|c_3)))\end{aligned}$$

$$
\begin{aligned}
&|(((c_1|c_2)|(c_1|c_2))|(((c_1|c_2)|(c_1|c_2))|(c_3|c_3)))\\
&= (c_2|((c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))\\
&|(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))))|\\
&(c_2|((c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))\\
&|(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))))))) \qquad (*_1),
\end{aligned}
$$

for all $c_1, c_2, c_3 \in C$. Thus, we have

$$
\begin{aligned}
c_1 \wedge (c_2|(c_3|c_3)) =&(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))\\
&|(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))))) \qquad (sBL-3)\\
=&(c_1|(((c_1|c_2)|(c_1|c_2))|(c_3|c_3)))\\
&|(c_1|(((c_1|c_2)|(c_1|c_2))|(c_3|c_3))) \qquad (S3)\\
=&(c_1|(((c_2|((c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))\\
&|(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))))\\
&|(c_2|((c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))\\
&|(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))))))))\\
&|(c_3|c_3)))|(c_1|(((c_2|((c_1|(c_1|((c_2|(c_3|c_3))\\
&|(c_2|(c_3|c_3)))))|(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))))\\
&|(c_2|((c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))|\\
&(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))))))))|(c_3|c_3))) \qquad (*_1)\\
=&(c_1|(((c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))\\
&|(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))))))|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))\\
&|(c_1|(((c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))\\
&|(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))))))\\
&|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))))) \quad ((S1) \text{ and } (S3))\\
=&(c_1|((c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))))|((c_1|((c_2|(c_3|c_3))\\
&|(c_2|(c_3|c_3))))|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))))\\
&|(c_1|((c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))))|((c_1|((c_2|(c_3|c_3))\\
&|(c_2|(c_3|c_3))))|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))))))) \qquad ((S1) \text{ and } (S3))\\
=&(c_1|1)|(c_1|1) \qquad (2)\\
=&c_1 \qquad (5),
\end{aligned}
$$

i.e., $c_1 \leq c_2|(c_3|c_3)$ from $(sBL-1)$.

($\Leftarrow$) Let $c_1 \leq c_2|(c_3|c_3)$. Then we obtain from $(sBL-1)$ and $(sBL-3)$ that

$$
\begin{aligned}
c_1 &= c_1 \wedge (c_2|(c_3|c_3))\\
&= (c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))|(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))))) \qquad (*_2),
\end{aligned}
$$

for all $c_1, c_2, c_3 \in C$. Thus, it follows

$$\begin{aligned}((c_1|c_2)|(c_1|c_2)) \wedge c_3 =&(((c_1|c_2)|(c_1|c_2))|(((c_1|c_2)|(c_1|c_2))|(c_3|c_3)))\\ &|(((c_1|c_2)|(c_1|c_2))|(((c_1|c_2)|(c_1|c_2))|(c_3|c_3))) \qquad (sBL-3)\\ =&(c_2|((c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))\\ &|(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))))\\ &|(c_2|((c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))))\\ &|(c_1|(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))))))) \qquad ((S1) \text{ and } (S3))\\ =&(c_1|c_2)|(c_1|c_2) \qquad ((*_2) \text{ and } (S1))\end{aligned}$$

i.e., $(c_1|c_2)|(c_1|c_2) \leq C_3$ from $(sBL-1)$.

(7) Let $c_1 \leq c_2$. Then we obtain from (5) and $(S1)$ that $c_1 = (1|c_1)|(1|c_1) \leq c_2$. So, it follows from (6) that $1 \leq c_1|(c_2|c_2)$. Since it is known $c_1|(c_2|c_2) \leq 1$ for all $c_1, c_2 \in A$, we have $c_1|(c_2|c_2) = 1$.

Conversely, let $c_1|(c_2|c_2) = 1$. Because it is known $c_1 \leq 1$ for all $c_1 \in C$, we get $c_1 \leq 1 = c_1|(c_2|c_2)$ by the hypothesis. Thus, it follows from (6) and $(S2)$ that $c_1 = (c_1|c_1)|(c_1|c_1) \leq c_2$.

(8) Since it is known that $c_2 \leq 1$ for all $c_2 \in C$, we have

$$\begin{aligned}c_2 \leq 1 &\Leftrightarrow c_2 \leq c_1|(c_1|c_1) \qquad (2)\\ &\Leftrightarrow (c_1|c_2)|(c_1|c_2) \leq c_1 \qquad ((6) \text{ and } (S1))\\ &\Leftrightarrow c_1 \leq c_2|(c_1|c_1) \qquad (6).\end{aligned}$$

(9) For all $c_1, c_2 \in C$, it follows from (6), $(S2)$ and $(S1)$, respectively, that $(c_1|c_2)|(c_1|c_2) \leq (c_1|c_2)|(c_1|c_2) \Leftrightarrow c_1 \leq (c_1|c_2)|c_2$.

(10)

(a) Because $c_1 \leq c_1$ for all $c_1 \in C$, we get from $(S2)$, $(S1)$ and (6), respectively, that

$$c_1 \leq c_1 \Leftrightarrow c_1 \leq (c_1|(c_2|c_2))|(c_1|c_1) \Leftrightarrow (c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2))) \leq c_1.$$

(b) Since $c_1|(c_2|c_2) \leq c_1|(c_2|c_2)$ for all $c_1, c_2 \in C$, it is obtained from (6) and $(S1)$ that

$$c_1|(c_2|c_2) \leq c_1|(c_2|c_2) \Leftrightarrow (c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2))) \leq c_2.$$

(11) Let $c_1 \leq c_2$.

(i) We have $(c_3|(c_3|(c_1|c_1)))|(c_3|(c_3|(c_1|c_1))) \leq c_1 \leq c_2$ from 10 (b) and the hypothesis. So, we get from $(S1)$ and (6) that $c_3|(c_1|c_1) \leq c_3|(c_2|c_2)$.

(ii) We know $c_1 \leq c_2 \leq (c_2|c_3)|c_3$ by (9) and the hypothesis. Therefore, it follows from $(S1)$, $(S2)$ and (6) that $(c_1|c_3)|(c_1|c_3) \leq (c_2|c_3)|(c_2|c_3)$.

(iii) It is obtained from (ii) and (10) (b) that

$$c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) \leq c_2|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) \leq c_3.$$

Then we get from $(S1)$ and (6) that $c_2|(c_3|c_3) \leq c_1|(c_3|c_3)$.

(12) Because we know $(c_3|(c_3|(c_1|c_1)))|(c_3|(c_3|(c_1|c_1))) \leq c_1$ from (10) (b), we have from (11) (i) that $c_1|(c_2|c_2) \leq ((c_3|(c_3|(c_1|c_1)))|(c_3|(c_3|(c_1|c_1))))|(y|y)$. Then it is obtained from $(S1)$ and $(S3)$ that $c_1|(c_2|c_2) \leq (c_3|(c_1|c_1))|((c_3|(c_2|c_2))|(c_3|(c_2|c_2)))$.

(13) Since it is known $(c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2))) \leq c_2$ from (10) (b), it is obtained from (11); (i) that $c_2|(c_3|c_3) \leq ((c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2))))|(c_3|c_3)$. Then we get $c_2|(c_3|c_3) \leq (c_1|(c_2|c_2))|((c_1|(c_3|c_3))|(c_1|(c_3|c_3)))$ from $(S1)$ and $(S3)$. Thus, it follows from (6) and $(S1)$ that $c_1|(c_2|c_2) \leq (c_2|(c_3|c_3))|((c_1|(c_3|c_3))|(c_1|(c_3|c_3)))$.

(14) Because $c_1, c_2 \leq c_1 \vee c_2$, we obtain from (11) (ii) that $(c_1|c_3)|(c_1|c_3), (c_2|c_3)|(c_2|c_3) \leq ((c_1 \vee c_2)|c_3)|((c_1 \vee c_2)|c_3)$. Then it follows

$$((c_1|c_3)|(c_1|c_3)) \vee ((c_2|c_3)|(c_2|c_3)) \leq ((c_1 \vee c_2)|c_3)|((c_1 \vee c_2)|c_3).$$

Since $(c_1|c_3)|(c_1|c_3), (c_2|c_3)|(c_2|c_3) \leq ((c_1|c_3)|(c_1|c_3)) \vee ((c_2|c_3)|(c_2|c_3))$, we have from (6) that

$$c_1, c_2 \leq c_3|((((c_1|c_3)|(c_1|c_3)) \vee ((c_2|c_3)|(c_2|c_3)))|(((c_1|c_3)|(c_1|c_3)) \vee ((c_2|c_3)|(c_2|c_3)))).$$

Then

$$c_1 \vee c_2 \leq c_3|((((c_1|c_3)|(c_1|c_3)) \vee ((c_2|c_3)|(c_2|c_3)))|(((c_1|c_3)|(c_1|c_3)) \vee ((c_2|c_3)|(c_2|c_3)))).$$

So, it follows from (6) that $((c_1 \vee c_2)|c_3)|((c_1 \vee c_2)|c_3) \leq ((c_1|c_3)|(c_1|c_3)) \vee ((c_2|c_3)|(c_2|c_3))$.

(15) We have $c_1, c_2 \leq (c_1|(c_2|c_2))|(c_2|c_2)$ and $c_1, c_2 \leq (c_2|(c_1|c_1))|(c_1|c_1)$. Then $c_1, c_2 \leq ((c_1|(c_2|c_2))|(c_2|c_2)) \wedge ((c_2|(c_1|c_1))|(c_1|c_1))$, and so

$$c_1 \vee c_2 \leq ((c_1|(c_2|c_2))|(c_2|c_2)) \wedge ((c_2|(c_1|c_1))|(c_1|c_1)).$$

Also, we obtain

$$\begin{aligned}
&((c_1|(c_2|c_2))|(c_2|c_2)) \wedge ((c_2|(c_1|c_1))|(c_1|c_1)) \\
=&((((c_1|(c_2|c_2))|(c_2|c_2)) \wedge ((c_2|(c_1|c_1))|(c_1|c_1))) \\
&|((c_1|(c_2|c_2)) \vee (c_2|(c_1|c_1))))|((((c_1|(c_2|c_2))|(c_2|c_2)) \\
&\wedge ((c_2|(c_1|c_1))|(c_1|c_1)))|((c_1|(c_2|c_2)) \vee (c_2|(c_1|c_1)))) \quad ((5) \text{ and } (sBL-4)) \\
=&(((c_1|(c_2|c_2))|(((c_1|(c_2|c_2))||(c_2|c_2)) \wedge ((c_2|(c_1|c_1)) \\
&|(c_1|c_1))))|((c_1|(c_2|c_2))|(((c_1|(c_2|c_2))||(c_2|c_2)) \wedge ((c_2|(c_1|c_1))|(c_1|c_1))))) \\
&|(((c_2|(c_1|c_1))|(((c_1|(c_2|c_2))|(c_2|c_2)) \wedge ((c_2|(c_1|c_1))|(c_1|c_1))))|((c_2|(c_1|c_1)) \\
&|(((c_1|(c_2|c_2))||(c_2|c_2)) \wedge ((c_2|(c_1|c_1))|(c_1|c_1))))) \qquad ((S1) \text{ and } (14)) \\
\leq&(((c_1|(c_2|c_2))|((c_1|(c_2|c_2))||(c_2|c_2)))|((c_1|(c_2|c_2)) \\
&|((c_1|(c_2|c_2))||(c_2|c_2)))) \vee (((c_2|(c_1|c_1))|((c_2|(c_1|c_1))|(c_1|c_1)))| \\
&((c_2|(c_1|c_1))|((c_2|(c_1|c_1))||(c_1|c_1)))) \qquad ((S1) \text{ and } (11)\ (ii)) \\
=&((c_1|(c_2|c_2)) \wedge c_2) \vee ((c_2|(c_1|c_1)) \wedge c_1) \quad (sBL-3) \\
=&c_2 \vee c_1 \quad (8) \\
=&c_1 \vee c_2.
\end{aligned}$$

□

Lemma 2.1. *Let C be a Sheffer stroke BL-algebra. Then* $(c_1|(c_2|c_2))|(c_2|c_2) = (c_2|(c_1|c_1))|(c_1|c_1)$ *for all* $c_1, c_2 \in C$.

Proof. Let C be a Sheffer stroke BL-algebra. Then it is obtained from Proposition 2.1 (13), $(S1)$ and $(S2)$ that

$$\begin{aligned}(c_1|(c_2|c_2))|(c_2|c_2) &\leq (c_2|(c_1|c_1))|(((c_1|(c_2|c_2))|(c_1|c_1))|((c_1|(c_2|c_2))|(c_1|c_1))) \\ &= (c_2|(c_1|c_1))|(c_1|c_1),\end{aligned}$$

and similarly, $(c_2|(c_1|c_1))|(c_1|c_1) \leq (c_1|(c_2|c_2))|(c_2|c_2)$. Therefore, $(c_1|(c_2|c_2))|(c_2|c_2) = (c_2|(c_1|c_1))|(c_1|c_1)$ for all $c_1, c_2 \in C$. □

Corollary 2.1. *Let C be a Sheffer stroke BL-algebra. Then* $c_1 \vee c_2 = (c_1|(c_2|c_2))|(c_2|c_2)$ *for all* $c_1, c_2 \in C$.

Lemma 2.2. *Let C be a Sheffer stroke BL-algebra. Then* $((c_1|(c_2|c_2))|(c_2|c_2))|(c_2|c_2) = c_1|(c_2|c_2)$ *for all* $c_1, c_2 \in C$.

Proof. Let C be a Sheffer stroke BL-algebra. Then it is known from Proposition 2.1 (9) that $c_1|(c_2|c_2) \leq ((c_1|(c_2|c_2))|(c_2|c_2))|(c_2|c_2)$. Also, it follows from Proposition 2.1 (12) and (1)–(3), respectively, that

$$\begin{aligned}((c_1|(c_2|c_2))|(c_2|c_2))|(c_2|c_2) \leq &(c_1|(((c_1|(c_2|c_2))|(c_2|c_2))|((c_1|(c_2|c_2)) \\ &|(c_2|c_2))))|((c_1|(c_2|c_2))|(c_1|(c_2|c_2))) \\ =&((c_1|(c_2|c_2))|((c_1|(c_2|c_2))|(c_1|(c_2|c_2)))) \\ &|((c_1|(c_2|c_2))|(c_1|(c_2|c_2))) \\ =&(c_1|(c_2|c_2)).\end{aligned}$$

Thus, $((c_1|(c_2|c_2))|(c_2|c_2))|(c_2|c_2) = c_1|(c_2|c_2)$ for all $c_1, c_2 \in C$. □

Lemma 2.3. *Let C be a Sheffer stroke BL-algebra. Then* $c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) = (c_1|(c_2|c_2))|((c_1|(c_3|c_3))|(c_1|(c_3|c_3)))$ *for all* $c_1, c_2, c_3 \in C$.

Proof. Let C be a Sheffer stroke BL-algebra. Since $c_2 \leq c_1|(c_2|c_2)$ from Proposition 2.1 (8), it is obtained from Proposition 2.1 (11) (iii) and (1), respectively, that

$$\begin{aligned}(c_1|(c_2|c_2))|((c_1|(c_3|c_3))|(c_1|(c_3|c_3))) \leq &c_2|((c_1|(c_3|c_3))|(c_1|(c_3|c_3))) \\ =&c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))).\end{aligned}$$

Besides, it follows from Proposition 2.1 (1), (12), $(S3)$ and $(S2)$, respectively, that

$$\begin{aligned}c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) =&c_2|((c_1|(c_3|c_3))|(c_1|(c_3|c_3))) \\ \leq&(c_1|(c_2|c_2))|((c_1|((c_1|(c_3|c_3))|(c_1|(c_3|c_3)))) \\ &|(c_1|((c_1|(c_3|c_3))|(c_1|(c_3|c_3))))) \\ =&(c_1|(c_2|c_2))|((((c_1|c_1)|(c_1|c_1))|(c_3|c_3)) \\ &|(((c_1|c_1)|(c_1|c_1))|(c_3|c_3))) \\ =&(c_1|(c_2|c_2))|((c_1|(c_3|c_3))|(c_1|(c_3|c_3))).\end{aligned}$$

Therefore, $c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) = (c_1|(c_2|c_2))|((c_1|(c_3|c_3))|(c_1|(c_3|c_3)))$. □

3. Various Filters of Sheffer Stroke BL-Algebras

In this section, we give some types of filters on a Sheffer stroke BL-algebra. Unless orherwise specified, C represents a Sheffer stroke BL-algebra.

Definition 3.1. A filter of C is a nonempty subset $P \subseteq C$ satisfying
$(SF-1)$ if $c_1, c_2 \in P$, then $(c_1|c_2)|(c_1|c_2) \in P$;
$(SF-2)$ if $c_1 \in P$ and $c_1 \leq c_2$, then $c_2 \in P$.

Example 3.1. For the Sheffer stroke BL-algebra in Example 2.2, C, $\{1\}$, $\{a, d, e, 1\}$ and $\{c, e, f, 1\}$ are filters of C.

Proposition 3.1. *Let P be a nonempty subset of C. Then P is a filter of C if and only if the following hold:*
$(SF-3)$ $1 \in P$;
$(SF-4)$ $c_1 \in P$ *and* $c_1|(c_2|c_2) \in P$ *imply* $c_2 \in P$.

Lemma 3.1. *Let P be a filter of C. Then* $c_3|(((c_2|(c_1|c_1))|(c_1|c_1))|((c_2|(c_1|c_1))|(c_1|c_1))) \in P$ *and* $c_3 \in P$ *imply* $(c_1|(c_2|c_2))|(c_2|c_2) \in P$ *for any* $c_1, c_2, c_3 \in C$.

Proof. Let P be a filter of C. Since $c_3|(((c_1|(c_2|c_2))|(c_2|c_2))|((c_1|(c_2|c_2))|(c_2|c_2))) = c_3|(((c_2|(c_1|c_1))|(c_1|c_1))|((c_2|(c_1|c_1))|(c_1|c_1))) \in P$, from Lemma 2.1 and $c_3 \in P$, it follows from $(SF-4)$ that $(c_1|(c_2|c_2))|(c_2|c_2) \in P$. □

Lemma 3.2. *Let P be a filter of C. Then*
(a) $c_3|((c_2|(c_1|c_1))|(c_2|(c_1|c_1))) \in P$ *and* $c_3 \in P$ *imply* $((c_1|(c_2|c_2))|(c_2|c_2))|(c_1|c_1) \in P$;
(b) $c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) \in P$ *and* $c_1|(c_2|c_2) \in P$ *imply* $c_1|(c_3|c_3) \in P$;
(c) $c_1|(((c_2|(c_3|c_3))|(c_2|c_2))|((c_2|(c_3|c_3))|(c_2|c_2))) \in P$ *and* $c_1 \in P$ *imply* $c_2 \in P$,
for any $c_1, c_2, c_3 \in C$.

Proof. Let P be a filter of C.
(a) Because $c_3|((c_2|(c_1|c_1))|(c_2|(c_1|c_1))) \in P$ and $c_3 \in P$, we get from Lemma 2.1, Lemma 2.2 and $(SF-4)$ that

$$((c_1|(c_2|c_2))|(c_2|c_2))|(c_1|c_1) = ((c_2|(c_1|c_1))|(c_1|c_1))|(c_1|c_1) = c_2|(c_1|c_1) \in P.$$

(b) Since $(c_1|(c_2|c_2))|((c_1|(c_3|c_3))|(c_1|(c_3|c_3))) = c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) \in P$ from Lemma 2.3 and $c_1|(c_2|c_2) \in P$, it follows from $(SF-4)$ that $c_1|(c_3|c_3) \in P$.

(c) Because

$$\begin{aligned} &c_1|(((c_2|(c_3|c_3))|(c_2|c_2))|((c_2|(c_3|c_3))|(c_2|c_2))) \\ =&c_1|(((c_2|c_2)|(c_2|(c_3|c_3)))|((c_2|c_2)|(c_2|(c_3|c_3)))) \\ =&c_1|(c_2|c_2) \in P, \end{aligned}$$

from $(S1)$-$(S2)$ and $c_1 \in P$, we have from $(SF-4)$ that $c_2 \in P$. □

Lemma 3.3. *Let P be a filter of C. Then $c \vee (c|c) \in P$ for any $c \in C$.*

Proof. Let P be a filter of C, and c be any element of C. Since

$$\begin{aligned} c \vee (c|c) &= (c|((c|c)|(c|c)))|((c|c)|(c|c)) \qquad \text{(Corollary 2.1)} \\ &= (c|(c|c)) \qquad (S1)\text{-}(S2) \\ &= 1 \qquad \text{(Proposition 2.1 (2))} \end{aligned}$$

and $1 \in P$, it is obtained $c \vee (c|c) \in P$. □

Definition 3.2. Let P be a filter of C. Then P is called an ultra filter of C if it satisfies $c \in P$ or $c|c \in P$ for all $c \in C$.

Example 3.2. Consider the Sheffer stroke BL-algebra in Example 2.2. Then the filter $\{a, d, e, 1\}$ of C is ultra while the filter $\{1\}$ of C is not an ultra filter of C.

Lemma 3.4. *A filter P of C is an ultra filter of C if and only if $c_1 \notin P$ and $c_2 \notin P$ imply $c_1|(c_2|c_2) \in P$ for all $c_1, c_2 \in C$.*

Proof. (⇒) Let P be an ultra filter of C. Assume that $c_1 \notin P$ and $c_2 \notin P$. Because P is an ultra filter of C, $c_1|c_1 \in P$ and $c_2|c_2 \in P$. Then $c_1|c_1 \leq (c_2|c_2)|((c_1|c_1)|(c_1|c_1)) = c_1|(c_2|c_2)$ from Proposition 2.1 (8) and $(S1)$-$(S2)$. So, $c_1|(c_2|c_2) \in P$.

(⇐) Let $c_1 \notin P$ and $c_2 \notin P$. Then $c_1|(c_2|c_2) \in P$ for $c_1, c_2 \in C$. Suppose that $c|c \notin P$ and $c \notin P$ for any $c \in C$. Then $(c|c)|(c|c) = c \in P$ by the hypothesis and $(S2)$, which is a contradiction. Hence, $c|c \in P$ and $c \in P$ for any $c \in C$, i.e., P is an ultra filter of C. □

Lemma 3.5. *A filter P of C is an ultra filter of C if and only if $c_1 \vee c_2 \in P$ implies $c_1 \in P$ or $c_2 \in P$ for all $c_1, c_2 \in C$.*

Proof. (⇒) Let P be an ultra filter of C and $c_1 \vee c_2 \in P$. Suppose that $c_1 \notin P$ or $c_2 \notin P$. Then we have $c_1|(c_2|c_2) \in P$ from Lemma 3.4. Since $(c_1|(c_2|c_2))|(c_2|c_2) \in P$, from Corollary 2.1 and $c_1|(c_2|c_2) \in P$, we get $c_2 \in P$ which is a contrradiction. Thus, $c_1 \in P$ or $c_2 \in P$.

(⇐) Let c_1 and c_2 be any elements in C such that $c_1 \vee c_2 \in P$ implies $c_1 \in P$ or $c_2 \in P$. Because $c \vee (c|c) \in P$ for all $c \in C$ from Lemma 3.3, it follows $c \in P$ or $c|c \in P$, i.e., P is an ultra filter of C. □

4. Some Fuzzy Filters of Sheffer Stroke BL-Algebras

In this section, we introduce some fuzzy filters in Sheffer stroke BL-algebras. Unless orherwise specified, C represents a Sheffer stroke BL-algebra.

Definition 4.1. A fuzzy filter of C is a fuzzy subset f of C such that for all $c_1, c_2 \in C$

(1) $f(c_1) \leq f(1)$;
(2) $f(c_1) \wedge f(c_1|(c_2|c_2)) \leq f(c_2)$.

Example 4.1. Consider the Sheffer stroke BL-algebra C in Example 2.1. Let $f(0) = f(u) = f(v) = 0,5$ and $f(1) = 1$. Then f is a fuzzy filter of C.

Proposition 4.1. *Let f be a fuzzy subset of C. f is a fuzzy filter of C if and only if for all $c_1, c_2, c_3 \in C$ $c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) = 1$ implies $f(c_1) \wedge f(c_2) \leq f(c_3)$.*

Proof. ($\Rightarrow$) Let f be a fuzzy filter of C and $c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) = 1$. Since

$$\begin{aligned} f(c_1) &= f(c_1) \wedge f(1) \qquad \text{(Definition 4.1 (1))} \\ &= f(c_1) \wedge f(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))) \\ &\leq f(c_2|(c_3|c_3)) \qquad \text{(Definition 4.1 (2))}, \end{aligned}$$

it follows from Definition 4.1 (2) that

$$f(c_1) \wedge f(c_2) \leq f(c_2|(c_3|c_3)) \wedge f(c_2) = f(c_2) \wedge f(c_2|(c_3|c_3)) \leq f(c_3).$$

($\Leftarrow$) Let $c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3))) = 1$ imply $f(c_1) \wedge f(c_2) \leq f(c_3)$. By substituting $[c_2 := c_1]$ and $[c_3 := 1]$, it is obtained from Proposition 2.1 (4) that $c_1|((c_1|(1|1))|(c_1|(1|1))) = 1$ implies $f(c_1) = f(c_1) \wedge f(c_1) \leq f(1)$. Besides, substituting $[c_2 := c_1|(c_2|c_2)]$ and $[c_3 := c_2]$, simultaneously, it is concluded from $(S1)$, $(S3)$ and Proposition 2.1 (2) that $c_1|(((c_1|(c_2|c_2))|(c_2|c_2))|((c_1|(c_2|c_2))|(c_2|c_2))) = 1$, which implies $f(c_1) \wedge f(c_1|(c_2|c_2)) \leq f(c_2)$. Thus, f is a fuzzy filter of C. □

Corollary 4.1. *Let f be a fuzzy subset of C. f is a fuzzy filter of C if and only if for all $c_1, c_2, c_3 \in C$ $(c_1|c_2)|(c_1|c_2) \leq c_3$ implies $f(c_1) \wedge f(c_2) \leq f(c_3)$.*

Proposition 4.2. *Let f be a fuzzy subset of C. f is a fuzzy filter of C if and only if*

(1) *f is order-preserving;*

(2) *$f(c_1) \wedge f(c_2) \leq f((c_1|c_2)|(c_1|c_2))$ for any $c_1, c_2 \in C$.*

Proof. ($\Rightarrow$) Let f be a fuzzy filter of C.

(1) Assume that $c_1 \leq c_2$, i.e., $c_1|(c_2|c_2) = 1$ from Proposition 2.1 (7). Then

$$\begin{aligned} f(c_1) &= f(c_1) \wedge f(1) \qquad \text{(Definition 4.1 (1))} \\ &= f(c_1) \wedge f(c_1|(c_2|c_2)) \\ &\leq f(c_2) \qquad \text{(Definition 4.1 (2))}. \end{aligned}$$

(2) Since

$$\begin{aligned} &c_1|((c_2|(((c_1|c_2)|(c_1|c_2))|((c_1|c_2)|(c_1|c_2))))|(c_2|(((c_1|c_2)|(c_1|c_2))|((c_1|c_2)|(c_1|c_2))))) \\ =&c_1|((c_2|(c_1|c_2))|(c_2|(c_1|c_2))) \qquad (S2) \\ =&((c_1|c_2)|(c_1|c_2))|(c_1|c_2) \qquad (S3) \\ =&1, \qquad ((S1) \text{ and Proposition 2.1 (2)}) \end{aligned}$$

it follows from Proposition 4.1 that $f(c_1) \wedge f(c_2) \leq f((c_1|c_2)|(c_1|c_2))$.

($\Leftarrow$) Let f be a fuzzy subset of C satisfying (1) and (2) for all $c_1, c_2, c_3 \in C$. By (1) and the fact that $c_1 \leq 1$ for all $c_1 \in C$, $f(c_1) \leq f(1)$. It is known from Proposition 2.1 (9) that $c_1 \leq (c_1|(c_2|c_2))|(c_2|c_2)$, and so $(c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2))) \leq c_2$ by Proposition 2.1 (6). Then it is obtained from $(1)-(2)$ that $f(c_1) \wedge f(c_1|(c_2|c_2)) \leq f((c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2)))) \leq f(c_2)$. Therefore, f is a fuzzy filter of C. □

Corollary 4.2. *Let f be an order-preserving fuzzy subset of C. f is a fuzzy filter of C if and only if $f((c_1|c_2)|(c_1|c_2)) = f(c_1) \wedge f(c_2)$ for any $c_1, c_2 \in C$.*

Proof. ($\Rightarrow$) Let f be a fuzzy filter of C. By Proposition 4.2 (2), it is sufficient to show that $f((c_1|c_2)|(c_1|c_2)) \leq f(c_1) \wedge f(c_2)$ for any $c_1, c_2 \in C$. Since $c_1 \leq 1$ for all $c_1 \in C$, it follows from $(S1)$, Proposition 2.1 (2) and (6) that $(c_1|c_2)|(c_1|c_2) \leq c_1, c_2$ for all $c_1, c_2 \in C$. Because f is order-preserving, $f((c_1|c_2)|(c_1|c_2)) \leq f(c_1), f(c_2)$, so $f((c_1|c_2)|(c_1|c_2)) \leq f(c_1) \wedge f(c_2)$.

($\Leftarrow$) It is clear by Proposition 4.2. □

Corollary 4.3. *Let f be a fuzzy filter of C. Then $f(c_1 \wedge c_2) = f(c_1) \wedge f(c_2)$ for any $c_1, c_2 \in C$.*

Proof. Let f be a fuzzy filter of C. Since $c_1 \wedge c_2 \leq c_1$, $c_1 \wedge c_2 \leq c_2$ and f is an order-preserving, it is obtained $f(c_1 \wedge c_2) \leq f(c_1)$ and $f(c_1 \wedge c_2) \leq f(c_2)$. Then $f(c_1 \wedge c_2) \leq f(c_1) \wedge f(c_2)$. Because we know $c_2 \leq c_1|(c_2|c_2)$ from Proposition 2.1 (8), it follows from Proposition 2.1 (11) (ii), $(S1)$, and $(sBL-3)$ that $(c_1|c_2)|(c_1|c_2) \leq (c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2))) = c_1 \wedge c_2$. Thus, $f(c_1) \wedge f(c_2) \leq f(c_1 \wedge c_2)$ from Corollary 4.1. □

Theorem 4.1. *Let f be a fuzzy filter of C.*

(a) If $f(c_1|(c_2|c_2)) = f(1)$, then $f(c_1) \leq f(c_2)$.

(b) $f(c_3|(((c_2|(c_1|c_1))|(c_1|c_1))|((c_2|(c_1|c_1))|(c_1|c_1)))) \wedge f(c_3) \leq f((c_1|(c_2|c_2))|(c_2|c_2))$.

(c) $f(c_3|((c_2|(c_1|c_1))|(c_2|(c_1|c_1)))) \wedge f(c_3) \leq f(((c_1|(c_2|c_2))|(c_2|c_2))|(c_1|c_1))$.

(d) $f(c_1|((c_2|(c_3|c_3))|(c_2|(c_3|c_3)))) \wedge f(c_1|(c_2|c_2)) \leq f(c_1|(c_3|c_3))$,

for any $c_1, c_2, c_3 \in C$.

Proof. (a) Since

$$\begin{aligned} f(c_1) &= f(c_1) \wedge f(1) \qquad \text{(Definition 4.1)} \\ &= f(c_1) \wedge f(c_1|(c_2|c_2)) \\ &= f((c_1|(c_1|(c_2|c_2)))|(c_1|(c_1|(c_2|c_2)))) \qquad \text{(Corollary 4.2)} \\ &= f(c_1 \wedge c_2) \qquad (sBL-3) \\ &= f(c_1) \wedge f(c_2), \qquad \text{(Corollary 4.3)}, \end{aligned}$$

we get $f(c_1) \leq f(c_2)$.

(b) It is proved from Definition 4.1 (2) and Lemma 2.1.

(c) We have from Definition 4.1 (2), Lemma 2.2 and Lemma 2.1 that

$$\begin{aligned} f(c_3|((c_2|(c_1|c_1))|(c_2|(c_1|c_1)))) \wedge f(c_3) &\leq f(c_2|(c_1|c_1)) \\ &= f(((c_2|(c_1|c_1))|(c_1|c_1))|(c_1|c_1)) \\ &= f(((c_1|(c_2|c_2))|(c_2|c_2))|(c_1|c_1)). \end{aligned}$$

(d) It is proved from Lemma 2.3 and Definition 4.1 (2). □

Definition 4.2. A fuzzy subset f of C is called a fuzzy ultra filter of C if it is a fuzzy filter of C that satisfies the following conditions $f(c_1) = f(1)$ or $f(c_1|c_1) = f(1)$, for all $c_1 \in C$.

Example 4.2. Consider the Sheffer stroke BL-algebra C in Example 2.2. Let the fuzzy filter f of C be defined by $f(a) = f(d) = f(e) = f(1) = 1$ and $f(b) = f(c) = f(f) = f(0) = 0$. Since $f(a) = f(d) = f(e) = f(1) = 1$ and $1 = f(a) = f(d) = f(e) = f(1) = f(f|f) = f(c|c) = f(b|b) = f(0|0)$, f is a fuzzy ultra filter of C.

Theorem 4.2. *A fuzzy subset f of C is a fuzzy ultra filter of C if and only if $f(c_1) \neq f(1)$ and $f(c_2) \neq f(1)$ imply $f(c_1|(c_2|c_2)) = f(1)$ and $f(c_2|(c_1|c_1)) = f(1)$ for all $c_1, c_2 \in C$.*

Proof. Let $f(c_1) \neq f(1)$ and $f(c_2) \neq f(1)$ imply $f(c_1|(c_2|c_2)) = f(1)$ and $f(c_2|(c_1|c_1))$ $= f(1)$. Supoose that $f(c_1) \neq f(1)$ and $f(1|1) \neq f(1)$ for any $c_1 \in C$. Then we have from Proposition 2.1 (4)-(5) and $(S1)$-$(S2)$ that $f(c_1|c_1) = f(c_1|1) = f(c_1|((1|1)|(1|1))) = f(1)$ and $f(1) = f((c_1|c_1)|(1|1)) = f((1|1)|(c_1|c_1)) = f(1)$. Similarly, $f(c_1) = f(1)$ whenever $f(c_1|c_1) \neq f(1)$ and $f(1|1) \neq f(1)$. Thus, f is a fuzzy ultra filter of C.

Conversely, let f be a fuzzy ultra filter of C. Assume that c_1 and c_2 are any elements in C such that $f(c_1) \neq f(1)$ and $f(c_2) \neq f(1)$. So, $f(c_1|c_1) = f(1)$ and $f(c_2|c_2) = f(1)$. Because

$$(c_1|c_1)|((c_1|(c_2|c_2))|(c_1|(c_2|c_2))) = (c_2|c_2)|((c_1|(c_1|c_1))|(c_1|(c_1|c_1))) = 1,$$

from $(S1)$, $(S3)$, Proposition 2.1 (2) and (4), it is obtained

$$\begin{aligned} f(1) &= f(1) \wedge f(1) \\ &= f(c_1|c_1) \wedge f((c_1|c_1)|((c_1|(c_2|c_2))|(c_1|(c_2|c_2)))) \\ &\leq f(c_1|(c_2|c_2)) \qquad \text{(Definition 4.1 (2))}, \end{aligned}$$

which gives $f(c_1|(c_2|c_2)) = f(1)$. Similarly, $f(c_2|(c_1|c_1)) = f(1)$. $\square$

Theorem 4.3. *A fuzzy subset f of C is a fuzzy ultra filter of C if and only if $f(c_1 \vee c_2) \leq f(c_1) \vee f(c_2)$ for all $c_1, c_2 \in C$.*

Proof. Let f be a fuzzy ultra filter of C. When $f(c_1) = f(1)$ or $f(c_2) = f(1)$, the proof is completed from Definition 4.1 (1). So, let $f(c_1) \neq f(1)$ or $f(c_2) \neq f(1)$. Then $f(c_1|(c_2|c_2)) = f(1)$ and $f(c_2|(c_1|c_1)) = f(1)$ by Theorem 4.2. Since

$$\begin{aligned} f(c_1 \vee c_2) &= f(1) \wedge f(c_1 \vee c_2) \qquad \text{(Definition 4.1 (1))} \\ &= f(c_1|(c_2|c_2)) \wedge f((c_1|(c_2|c_2))|(c_2|c_2)) \qquad \text{(Corollary 2.1)} \\ &\leq f(c_2) \qquad \text{(Definition 4.1 (2))} \end{aligned}$$

and

$$\begin{aligned} f(c_1 \vee c_2) &= f(c_2 \vee c_1) \\ &= f(1) \wedge f(c_2 \vee c_1) \qquad \text{(Definition 4.1 (1))} \end{aligned}$$

$$= f(c_2|(c_1|c_1)) \wedge f((c_2|(c_1|c_1))|(c_1|c_1)) \qquad \text{(Corollary 2.1)}$$
$$\leq f(c_1) \qquad \text{(Definition 4.1 (2))},$$

$f(c_1 \vee c_2) \leq f(c_1), f(c_2)$ and so, $f(c_1 \vee c_2) \leq f(c_1) \vee f(c_2)$.

Conversely, let c_1 and c_2 be any elements in C such that $f(c_1 \vee c_2) \leq f(c_1) \vee f(c_2)$. Then

$$\begin{aligned} f(1) &= f(c|(c|c)) \qquad \text{(Proposition 2.1 (2))} \\ &= f((c|((c|c)|(c|c)))|((c|c)|(c|c))) \qquad (S2) \\ &= f(c \vee (c|c)) \qquad \text{(Corollary 2.1)} \\ &\leq f(c) \vee f(c|c), \end{aligned}$$

i.e., $f(c) \vee f(c|c) = f(1)$. Thus, $f(c) = f(1)$ or $f(c|c) = f(1)$, which implies that f is a fuzzy ultra filter of C. □

Proposition 4.3. *f is a fuzzy filter of C if and only if $f_p = \{c_1 \in C : p \leq f(c_1)\} \neq \emptyset$ is a filter of C for any $p \in (0, 1]$.*

Proof. ($\Rightarrow$) Let f be a fuzzy filter of C.

• Since $f_p \neq \emptyset$, there exists some $c \in C$ such that $p \leq f(c)$. Then we obtain from Definition 4.1 (1) that $p \leq f(c) \leq f(1)$, i.e., $1 \in f_p$.

• Let $c_1, c_1|(c_2|c_2) \in f_p$, i.e., $p \leq f(c_1), f(c_1|(c_2|c_2))$. It is concluded from Definition 4.1 (2) that $p \leq f(c_1) \wedge f(c_1|(c_2|c_2)) \leq f(c_2)$, that is, $c_2 \in f_p$. Therefore, f_p is a filter of C.

($\Leftarrow$) Let $f_p \neq \emptyset$ is a filter of C.

• Let $c \in C$ such that $f(c) > f(1)$. If $p = \frac{f(c)+f(1)}{2}$, then $f(1) < p < f(c)$. So, $1 \notin f_p$ which contradicts with $(SF-3)$. Hence, $f(c) \leq f(1)$.

• Suppose that $c_1, c_2 \in C$ such that $f(c_2) < f(c_1) \wedge f(c_1|(c_2|c_2))$. If $f(c_1) = \gamma$, $f(c_2) = \theta$ and $f(c_1|(c_2|c_2)) = \lambda$, then $\theta < \min(\gamma, \lambda)$. Consider $\lambda_1 = \frac{1}{2}(\theta + \min(\gamma, \lambda))$. Then $\theta < \lambda_1 < \gamma$ and $\theta < \lambda_1 < \lambda$. For $p = \lambda_1 \in (0, 1]$, $c_1 \in f_p$ and $c_1|(c_2|c_2) \in f_p$ but $c_2 \notin f_p$ which contradicts with $(SF-4)$. Thus, $f(c_1) \wedge f(c_1|(c_2|c_2)) \leq f(c_2)$. □

Theorem 4.4. *Let f be a fuzzy filter of C. Then f is a fuzzy ultra filter of C if and only if f_p is either empty or an ultra filter of C for each $p \in [0, 1]$.*

Proof. Assume that f is a fuzzy ultra filter of C, and $f_p \neq \emptyset$. Let $c_1 \vee c_2 \in f_p$, i.e., $p \leq f(c_1 \vee c_2)$. Then $p \leq f(c_1 \vee c_2) \leq f(c_1) \vee f(c_2)$ from Theorem 4.3. So, $p \leq f(c_1)$ or $p \leq f(c_2)$, i.e., $c_1 \in f_p$ or $c_2 \in f_p$. Hence, f_p is an ultra filter of C.

Conversely, suppose that f_p is an ultra filter of C. Let $p = f(c_1 \vee c_2)$, i. e., $c_1 \vee c_2 \in f_p$. Then $c_1 \in f_p$ or $c_2 \in f_p$ from Lemma 3.5. Thus, $f(c_1 \vee c_2) = p \leq f(c_1)$ or $f(c_1 \vee c_2) \leq f(c_2)$, and so, $f(c_1 \vee c_2) \leq f(c_1) \vee f(c_2)$. Therefore, f is a fuzzy ultra filter of C. □

Corollary 4.4. *Let f be a fuzzy filter of C. Then f is a fuzzy ultra filter of C if and only if $f_{f(1)}$ is an ultra filter of C.*

Corollary 4.5. *Let P be a nonempty subset of C. Then P is an ultra filter of C if and only if χ_P is a fuzzy ultra filter of C in which χ_P is the characterictic function of P.*

5. Conclusion

In the present work, we have studied on (fuzzy) filters of Sheffer stroke BL-algebras, and the relationships between them. After giving basic definitions and notions about Sheffer stroke BL-algebra, we introduce some types of (fuzzy) filters of a Sheffer stroke BL-algebra, and present their some properties. Then we show that f is a fuzzy filter of a Sheffer stroke BL-algebra if and only if f_p is empty or is its filter for any $p \in (0, 1]$, and it holds in the case of (fuzzy) ultra filter. Indeed, it is concluded that above property holds for $p = f(1)$ and for the characteristic function of a nonempty subset of a Sheffer stroke BL-algebra. In a similar way, it can be examined relationships between them by defining some kinds of (fuzzy) ideals of Sheffer stroke BL-algebras.

Acknowledgements. We like to thank the anonymous referees for their helpful comments and suggestions for the improvement of this paper.

This study is partially funded by Ege University Scientific Research Projects Directorate with the project number 20772.

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