Kragujevac Journal of Mathematics
Volume 46(1) (2022), Pages 115–124.

# DIFFERENTIAL SUBORDINATION RESULTS FOR HOLOMORPHIC FUNCTIONS RELATED TO GENERALIZED DIFFERENTIAL OPERATOR

ABBAS KAREEM WANAS[1]

Abstract. In the present investigation, we use the principle of subordination to introduce a new family for holomorphic functions defined by generalized differential operator. Also we establish some interesting geometric properties for functions belonging to this family.

## 1. Introduction and Preliminaries

Let $\mathcal{A}_m$ stands for the family of functions $f$ of the form:

$$f(z) = z + \sum_{n=m+1}^{\infty} a_n z^n \quad (m \in \mathbb{N} = \{1, 2, \dots\}, z \in U), \tag{1.1}$$

which are holomorphic in the open unit disk $U = \{z \in C : |z| < 1\}$.

For two functions $f$ and $g$ holomorphic in $U$, we say that the function $f$ is subordinate to $g$, written $f \prec g$ or $f(z) \prec g(z)(z \in U)$, if there exists a Schwarz function $w$ holomorphic in $U$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. In particular, if the function $g$ is univalent in $U$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$ (see [6]).

If $f \in \mathcal{A}_m$ is given by (1.1) and $g \in \mathcal{A}_m$ given by

$$g(z) = z + \sum_{n=m+1}^{\infty} b_n z^n \quad (m \in \mathbb{N} = \{1, 2, \dots\}, z \in U),$$

*Key words and phrases.* Holomorphic functions, Differential subordination, Convex univalent, Hadamard product, Generalized differential operator.

2010 *Mathematics Subject Classification.* Primary: 30C45. Secondary: 30A20.

*Received*: March 29, 2019.

*Accepted*: September 06, 2019.

then the Hadamard product (or convolution) $f * g$ of $f$ and $g$ is defined by

$$(f * g)(z) = z + \sum_{n=m+1}^{\infty} a_n b_n z^n = (g * f)(z).$$

A function $f \in \mathcal{A}_m$ is said to be starlike of order $\rho$ in $U$ if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \rho \quad (0 \leq \rho < 1, z \in U).$$

Indicate the class of all starlike functions of order $\rho$ in $U$ by $S^*(\rho)$.

A function $f \in \mathcal{A}_m$ is said to be prestarlike of order $\rho$ in $U$ if

$$\frac{z}{(1-z)^{2(1-\rho)}} * f(z) \in S^*(\rho) \quad (\rho < 1).$$

Indicate the class of all prestarlike functions of order $\rho$ in $U$ by $\operatorname{Re}(\rho)$.

Clearly a function $f \in \mathcal{A}_m$ is in the class $\operatorname{Re}(0)$ if and only if $f$ is convex univalent in $U$ and $\operatorname{Re}(\frac{1}{2}) = S^*(\frac{1}{2})$.

For $\sigma \in N_0 = N \cup \{0\}$, $\alpha, \delta \geq 0$, $\tau, \lambda, \beta > 0$ and $\alpha \neq \lambda$, we consider the generalized differential operator $A^{\sigma}_{\tau,\lambda,\delta}(\alpha, \beta) : \mathcal{A}_m \to \mathcal{A}_m$, introduced by Amourah and Darus [2], where

$$A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f(z) = z + \sum_{n=m+1}^{\infty}\left[1 + \frac{(n-1)\left((\lambda-\alpha)\beta + n\delta\right)}{\tau+\lambda}\right]^{\sigma} a_n z^n. \tag{1.2}$$

It is readily verified from (1.2) that

$$\begin{aligned} z\left(A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)\right)' =& \frac{\tau+\lambda}{(\lambda-\alpha)\beta+n\delta} A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f(z) \\ &+ \left(1 - \frac{\tau+\lambda}{(\lambda-\alpha)\beta+n\delta}\right) A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f(z). \end{aligned} \tag{1.3}$$

Here, we would point out some of the special cases of the operator defined by (1.2) can be found in [1, 4, 5, 11].

Let $H$ be the class of functions $h$ with $h(0) = 1$, which are holomorphic and convex univalent in $U$.

**Definition 1.1.** A function $f \in \mathcal{A}_m$ is said to be in the class $\mathcal{M}(\eta, \sigma, \tau, \lambda, \delta, \alpha, \beta, m; h)$ if it satisfies the subordination condition:

(1.4)

$$\frac{1}{z}\left[\left(1 - \frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right) A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f(z) + \frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta} A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)\right] \prec h(z),$$

where $\eta \in C$, $\sigma \in N_0 = N \cup \{0\}$, $\alpha, \delta \geq 0$, $\tau, \lambda, \beta > 0$, $\alpha \neq \lambda$ and $h \in H$.

Now, we need the following lemmas that will be used to prove our main results.

**Lemma 1.1** ([8])**.** *Let $g$ be holomorphic in $U$ and let $h$ be holomorphic and convex univalent in $U$ with $h(0) = g(0)$. If*

$$g(z) + \frac{1}{\mu} z g'(z) \prec h(z), \tag{1.5}$$

*where* $\mathrm{Re}(\mu) \geq 0$ *and* $\mu \neq 0$, *then*

$$g(z) \prec \breve{h}(z) = \mu z^{-\mu} \int_0^z t^{\mu-1} h(t) dt \prec h(z)$$

*and $\breve{h}$ is the best dominant of* (1.5)*.*

**Lemma 1.2** ([10])**.** *Let $\rho < 1$, $f \in S^*(\rho)$ and $g \in \mathrm{Re}(\rho)$. Then, for any holomorphic function $F$ in $U$*

$$\frac{g * (fF)}{g * f}(U) \subset \bar{co}\left(F(U)\right),$$

*where $\bar{co}\left(F(U)\right)$ denotes the closed convex hull of $F(U)$.*

Such type of study was carried out by various authors for another classes, like, Liu [7], Prajapat and Raina [9], Atshan and Wanas [3], Wanas [12] and Wanas and Majeed [13].

## 2. Main Results

**Theorem 2.1.** *Let $0 \leq \eta < \varepsilon$. Then*

$$\mathcal{M}(\varepsilon, \sigma, \tau, \lambda, \delta, \alpha, \beta, m; h) \subset \mathcal{M}(\eta, \sigma, \tau, \lambda, \delta, \alpha, \beta, m; h).$$

*Proof.* Let $0 \leq \eta < \varepsilon$ and $f \in \mathcal{M}(\varepsilon, \sigma, \tau, \lambda, \delta, \alpha, \beta, m; h)$. Assume that

$$g(z) = \frac{A^{\sigma}_{\tau,\lambda,\delta}(\alpha, \beta) f(z)}{z} = 1 + \sum_{n=m+1}^{\infty} \left[ 1 + \frac{(n-1)\left((\lambda - \alpha)\beta + n\delta\right)}{\tau + \lambda} \right]^{\sigma} a_n z^{n-1}. \tag{2.1}$$

It is obvious that the function $g$ is holomorphic in $U$ and $g(0) = 1$. Since $f \in \mathcal{M}(\varepsilon, \sigma, \tau, \lambda, \delta, \alpha, \beta, m; h)$, then we deduce that

$$\frac{1}{z} \left[ \left( 1 - \frac{\varepsilon(\tau + \lambda)}{(\lambda - \alpha)\beta + n\delta} \right) A^{\sigma}_{\tau,\lambda,\delta}(\alpha, \beta) f(z) + \frac{\varepsilon(\tau + \lambda)}{(\lambda - \alpha)\beta + n\delta} A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha, \beta) f(z) \right] \prec h(z). \tag{2.2}$$

Differentiating both sides of (2.1) with respect to $z$ and using (1.3) and (2.2), we find that

$$\begin{aligned}
&\frac{1}{z} \left[ \left( 1 - \frac{\varepsilon(\tau + \lambda)}{(\lambda - \alpha)\beta + n\delta} \right) A^{\sigma}_{\tau,\lambda,\delta}(\alpha, \beta) f(z) + \frac{\varepsilon(\tau + \lambda)}{(\lambda - \alpha)\beta + n\delta} A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha, \beta) f(z) \right] \\
=&\frac{1}{z} \left[ (1 - \varepsilon) A^{\sigma}_{\tau,\lambda,\delta}(\alpha, \beta) f(z) + \varepsilon \left( 1 - \frac{\tau + \lambda}{(\lambda - \alpha)\beta + n\delta} \right) A^{\sigma}_{\tau,\lambda,\delta}(\alpha, \beta) f(z) \right. \\
&\left. + \frac{\varepsilon(\tau + \lambda)}{(\lambda - \alpha)\beta + n\delta} A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha, \beta) f(z) \right]
\end{aligned}$$

$$
\begin{aligned}
=&(1-\varepsilon)\frac{A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)}{z}+\varepsilon\left(A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)\right)'\\
=&\frac{A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)}{z}+\varepsilon z\left(\frac{A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)}{z}\right)'\\
=&g(z)+\varepsilon z g'(z)\prec h(z).
\end{aligned}
$$

An application of Lemma 1.1 with $\mu=\frac{1}{\varepsilon}$, yields

$$g(z)\prec h(z). \tag{2.3}$$

Evidently, $0\leq\frac{\eta}{\varepsilon}<1$ and that $h$ is convex univalent in $U$, it follows from (2.1), (2.2) and (2.3) that

$$
\begin{aligned}
&\frac{1}{z}\left[\left(1-\frac{\eta\left(\tau+\lambda\right)}{\left(\lambda-\alpha\right)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)+\frac{\eta\left(\tau+\lambda\right)}{\left(\lambda-\alpha\right)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)\right]\\
=&\frac{\eta}{\varepsilon z}\left[\left(1-\frac{\varepsilon\left(\tau+\lambda\right)}{\left(\lambda-\alpha\right)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)+\frac{\varepsilon\left(\tau+\lambda\right)}{\left(\lambda-\alpha\right)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)\right]\\
&+\left(1-\frac{\eta}{\varepsilon}\right)g(z)\prec h(z).
\end{aligned}
$$

Hence, $f\in\mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;h)$ and the proof of Theorem 2.1 is completed. $\square$

**Theorem 2.2.** *Let* $\operatorname{Re}\left\{\frac{\tau+\lambda}{(\lambda-\alpha)\beta+n\delta}\right\}\geq 0$ *and* $\frac{\tau+\lambda}{(\lambda-\alpha)\beta+n\delta}\neq 0$. *Then*

$$\mathcal{M}(\eta,\sigma+1,\tau,\lambda,\delta,\alpha,\beta,m;h)\subset\mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;h).$$

*Proof.* Let $f\in\mathcal{M}(\eta,\sigma+1,\tau,\lambda,\delta,\alpha,\beta,m;h)$ and suppose that

$$g(z)=\frac{1}{z}\left[\left(1-\frac{\eta\left(\tau+\lambda\right)}{\left(\lambda-\alpha\right)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)+\frac{\eta\left(\tau+\lambda\right)}{\left(\lambda-\alpha\right)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)\right]. \tag{2.4}$$

By taking the derivatives in the both sides of (2.4) with respect to $z$ and using (1.3), we conclude that

$$
\begin{aligned}
&g(z)+zg'(z)\\
=&\frac{1}{z}\left[\left(1-\frac{\eta\left(\tau+\lambda\right)}{\left(\lambda-\alpha\right)\beta+n\delta}\right)\left(1-\frac{\tau+\lambda}{\left(\lambda-\alpha\right)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)\right.\\
&+\left(1+\eta\left(1-\frac{2\left(\tau+\lambda\right)}{\left(\lambda-\alpha\right)\beta+n\delta}\right)\right)\frac{\tau+\lambda}{\left(\lambda-\alpha\right)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)\\
&\left.+\eta\left(\frac{\tau+\lambda}{\left(\lambda-\alpha\right)\beta+n\delta}\right)^{2}A^{\sigma+2}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)\right].
\end{aligned} \tag{2.5}
$$

In the light of (2.4) and (2.5), we deduce that

$$\frac{\tau+\lambda}{\left(\lambda-\alpha\right)\beta+n\delta}g(z)+zg'(z)$$

$$
\begin{aligned}
=&\frac{1}{z}\left[\left(1-\frac{\eta\left(\tau+\lambda\right)}{\left(\lambda-\alpha\right)\beta+n\delta}\right)\left(\frac{\tau+\lambda}{\left(\lambda-\alpha\right)\beta+n\delta}\right)A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)\right.\\
&\left.+\eta\left(\frac{\tau+\lambda}{\left(\lambda-\alpha\right)\beta+n\delta}\right)^{2}A^{\sigma+2}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)\right],
\end{aligned}
$$

that is

$$
\begin{aligned}
g(z)+\frac{\left(\lambda-\alpha\right)\beta+n\delta}{\tau+\lambda}zg'(z)=&\frac{1}{z}\left[\left(1-\frac{\eta\left(\tau+\lambda\right)}{\left(\lambda-\alpha\right)\beta+n\delta}\right)A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)\right.\\
&\left.+\frac{\eta\left(\tau+\lambda\right)}{\left(\lambda-\alpha\right)\beta+n\delta}A^{\sigma+2}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)\right].
\end{aligned}
\tag{2.6}
$$

Since $f\in\mathcal{M}(\eta,\sigma+1,\tau,\lambda,\delta,\alpha,\beta,m;h)$, then it follows from (2.6) that

$$
g(z)+\frac{\left(\lambda-\alpha\right)\beta+n\delta}{\tau+\lambda}zg'(z)\prec h(z),
$$

where

$$
\operatorname{Re}\left\{\frac{\tau+\lambda}{\left(\lambda-\alpha\right)\beta+n\delta}\right\}\geq 0,\qquad \frac{\tau+\lambda}{\left(\lambda-\alpha\right)\beta+n\delta}\neq 0.
$$

An application of Lemma 1.1, with $\mu=\frac{\tau+\lambda}{(\lambda-\alpha)\beta+n\delta}$, yields $g(z)\prec h(z)$. In view of (2.4), we have

$$
\frac{1}{z}\left[\left(1-\frac{\eta\left(\tau+\lambda\right)}{\left(\lambda-\alpha\right)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)+\frac{\eta\left(\tau+\lambda\right)}{\left(\lambda-\alpha\right)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)\right]\prec h(z).
$$

This shows that $f\in\mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;h)$ and the proof of Theorem 2.2 is completed. $\square$

**Theorem 2.3.** *Let $\eta>0$, $\gamma>0$ and $f\in\mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;\gamma h+1-\gamma)$. If $\gamma\leq\gamma_0$, where*

$$
\gamma_0=\frac{1}{2}\left(1-\frac{1}{\eta}\int_0^1\frac{u^{\frac{1}{\eta}-1}}{1+u}du\right)^{-1},
\tag{2.7}
$$

*then $f\in\mathcal{M}(0,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;h)$. The bound $\gamma_0$ is the sharp when $h(z)=\frac{1}{1-z}$.*

*Proof.* Assume that

$$
g(z)=\frac{A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)}{z}.
\tag{2.8}
$$

Let $f\in\mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;\gamma h+1-\gamma)$ with $\eta>0$ and $\gamma>0$. Then we obtain

$$
\begin{aligned}
&g(z)+\eta zg'(z)\\
=&\frac{1}{z}\left[\left(1-\frac{\eta\left(\tau+\lambda\right)}{\left(\lambda-\alpha\right)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)+\frac{\eta\left(\tau+\lambda\right)}{\left(\lambda-\alpha\right)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f\left(z\right)\right]\\
\prec&\gamma h(z)+1-\gamma.
\end{aligned}
$$

Making use of Lemma 1.1, we observe that

$$g(z) \prec \frac{\gamma}{\eta} z^{-\frac{1}{\eta}} \int_0^z t^{\frac{1}{\eta}-1} h(t)dt + 1 - \gamma = (h * \phi)(z), \tag{2.9}$$

where

$$\phi(z) = \frac{\gamma}{\eta} z^{-\frac{1}{\eta}} \int_0^z \frac{t^{\frac{1}{\eta}-1}}{1-t} dt + 1 - \gamma. \tag{2.10}$$

If $0 < \gamma \le \gamma_0$, where $\gamma_0 > 1$ is given by (2.7), then we find from (2.10) that

$$\operatorname{Re}(\phi(z)) = \frac{\gamma}{\eta} \int_0^1 u^{\frac{1}{\eta}-1} \operatorname{Re}\left(\frac{1}{1-uz}\right) du + 1 - \gamma > \frac{\gamma}{\eta} \int_0^1 \frac{u^{\frac{1}{\eta}-1}}{1+u} du + 1 - \gamma \ge \frac{1}{2}. \tag{2.11}$$

By using (2.8) and (2.9), we have

$$\frac{A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta) f(z)}{z} \prec (h * \phi)(z). \tag{2.12}$$

In the light of (2.11), we note that the function $\phi(z)$ has the Herglotz representation

$$\phi(z) = \int_{|x|=1} \frac{d\mu(x)}{1-xz} \quad (z \in U), \tag{2.13}$$

where $\mu(x)$ is a probability measure defined on the unit circle $|x| = 1$ and

$$\int_{|x|=1} d\mu(x) = 1.$$

Since $h$ is convex univalent in $U$, then we deduce from (2.12) and (2.13) that

$$\frac{A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta) f(z)}{z} \prec (h * \phi)(z) = \int_{|x|=1} \phi(xz)\, d\mu(x) \prec h(z).$$

This shows that $f \in \mathcal{M}(0, \sigma, \tau, \lambda, \delta, \alpha, \beta, m; h)$. For $h(z) = \frac{1}{1-z}$ and $f \in \mathcal{A}_m$ defined by

$$\frac{A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta) f(z)}{z} = \frac{\gamma}{\eta} z^{-\frac{1}{\eta}} \int_0^z \frac{t^{\frac{1}{\eta}-1}}{1-t} dt + 1 - \gamma,$$

we obtain

$$\frac{1}{z}\left[\left(1 - \frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta + n\delta}\right) A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta) f(z) + \frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta + n\delta} A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta) f(z)\right]$$
$$= \gamma h(z) + 1 - \gamma.$$

Thus, $f \in \mathcal{M}(\eta, \sigma, \tau, \lambda, \delta, \alpha, \beta, m; \gamma h + 1 - \gamma)$. Also, for $\gamma > \gamma_0$ , we have

$$Re\left\{\frac{A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta) f(z)}{z}\right\} \longrightarrow \frac{\gamma}{\eta} \int_0^1 \frac{u^{\frac{1}{\eta}-1}}{1+u} du + 1 - \gamma < \frac{1}{2} \quad (z \to -1),$$

which implies that $f \notin \mathcal{M}(0, \sigma, \tau, \lambda, \delta, \alpha, \beta, m; h)$. Thus, the bound $\gamma_0$ cannot be increased when $h(z) = \frac{1}{1-z}$. This completes the proof of the theorem. $\square$

**Theorem 2.4.** *Let $f \in \mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;h)$ be defined as in* (1.1). *Then the function $I$ defined by*

$$I(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \quad (\mathrm{Re}(c)>-1),$$

*is also in the class $\mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;h)$.*

*Proof.* Let $f \in \mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;h)$ be defined as in (1.1). Then, we find that

(2.14)

$$\frac{1}{z}\left[\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)+\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)\right]\prec h(z).$$

We can easily see that

$$I(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt=z+\sum_{n=m+1}^{\infty}\frac{c+1}{c+n}a_nz^n. \tag{2.15}$$

We have from (2.15) that $I\in\mathcal{A}_m$ and

$$f(z)=\frac{cI(z)+zI'(z)}{c+1}. \tag{2.16}$$

Define the function $J$ by

(2.17)

$$J(z)=\frac{1}{z}\left[\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)I(z)+\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)I(z)\right].$$

Differentiating both sides of (2.17) with respect to $z$ and using (2.14) and (2.16), we obtain

$$\begin{aligned}
&\frac{1}{z}\left[\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)+\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)\right]\\
=&\frac{1}{z}\left[\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)\left(\frac{cI(z)+zI'(z)}{c+1}\right)\right.\\
&\left.+\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)\left(\frac{cI(z)+zI'(z)}{c+1}\right)\right]\\
=&\frac{c}{z(c+1)}\left[\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)I(z)\right.\\
&\left.+\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)I(z)\right]\\
&+\frac{1}{z(c+1)}\left[\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)\left(zI'(z)\right)\right.\\
&\left.+\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)\left(zI'(z)\right)\right]\\
=&\frac{c}{c+1}J(z)+\frac{1}{c+1}\left(zJ'(z)+J(z)\right)=J(z)+\frac{1}{c+1}zJ'(z)\prec h(z).
\end{aligned}$$

An application of Lemma 1.1 with $\mu = c+1$, yields $J(z) \prec h(z)$. By using (2.17), we get

$$\frac{1}{z}\left[\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)I(z)+\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)I(z)\right] \prec h(z),$$

which implies that $I \in \mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;h)$. □

**Theorem 2.5.** *Let $f \in \mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;h)$, $g \in \mathcal{A}_m$ and*

$$\operatorname{Re}\left\{\frac{g(z)}{z}\right\} > \frac{1}{2}. \tag{2.18}$$

*Then $f * g \in \mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;h)$.*

*Proof.* Let $f \in \mathcal{M}(\eta,\sigma,\tau,\lambda,\delta,\alpha,\beta,m;h)$ and $g \in \mathcal{A}_m$. Then, we have

$$\begin{aligned}
&\frac{1}{z}\left[\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)(f*g)(z)\right.\\
&\quad\left.+\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)(f*g)(z)\right]\\
&=\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)\left(\frac{g(z)}{z}\right)*\left(\frac{A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)}{z}\right)\\
&\quad+\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\left(\frac{g(z)}{z}\right)*\left(\frac{A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)}{z}\right)=\left(\frac{g(z)}{z}\right)*\varphi(z),
\end{aligned} \tag{2.19}$$

where

$$\begin{aligned}
\varphi(z)=\frac{1}{z}&\left[\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)+\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)f(z)\right]\\
&\prec h(z).
\end{aligned} \tag{2.20}$$

In view of (2.18), we note that the function $\frac{g(z)}{z}$ has the Herglotz representation

$$\frac{g(z)}{z}=\int_{|x|=1}\frac{d\mu(x)}{1-xz} \quad (z \in U), \tag{2.21}$$

where $\mu(x)$ is a probability measure defined on the unit circle $|x|=1$ and

$$\int_{|x|=1} d\mu(x) = 1.$$

Since $h$ is convex univalent in $U$, then we find from (2.19), (2.20) and (2.21) that

$$\begin{aligned}
&\frac{1}{z}\left[\left(1-\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right)A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)(f*g)(z)+\frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta+n\delta}\right.\\
&\quad\left.\times A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)(f*g)(z)\right]
\end{aligned}$$

$$= \int_{|x|=1} \varphi(xz)\, d\mu(x) \prec h(z).$$

This shows that $f * g \in \mathcal{M}(\eta, \sigma, \tau, \lambda, \delta, \alpha, \beta, m; h)$. $\square$

**Theorem 2.6.** *Let $f \in \mathcal{M}(\eta, \sigma, \tau, \lambda, \delta, \alpha, \beta, m; h)$ and $g \in \mathcal{A}_m$ be prestarlike of order $\alpha$, $(\alpha < 1)$. Then $f * g \in \mathcal{M}(\eta, \sigma, \tau, \lambda, \delta, \alpha, \beta, m; h)$.*

*Proof.* For $f \in \mathcal{M}(\eta, \sigma, \tau, \lambda, \delta, \alpha, \beta, m; h)$ and $g \in \mathcal{A}_m$, from (2.19) (used in the proof of Theorem 2.5), we can write

$$\frac{1}{z}\left[\left(1 - \frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta + n\delta}\right) A^{\sigma}_{\tau,\lambda,\delta}(\alpha,\beta)(f*g)(z) + \frac{\eta(\tau+\lambda)}{(\lambda-\alpha)\beta + n\delta}\right. \tag{2.22}$$

$$\left. \times A^{\sigma+1}_{\tau,\lambda,\delta}(\alpha,\beta)(f*g)(z)\right] \tag{2.23}$$

$$= \frac{g(z) * (z\varphi(z))}{g(z) * z} \quad (z \in U),$$

where $\varphi(z)$ is defined as in (2.20). Since $h$ is convex univalent in $U$, $\psi(z) \prec h(z)$, $g(z) \in \mathrm{Re}(\alpha)$ and $z \in S^*(\alpha)$, $\alpha < 1$, it follows from (2.22) and Lemma 1.2, we obtain the result. $\square$

## References


[1] F. M. Al-Oboudi, *On univalent functions defined by a generalized Sălăgean operator*, Int. J. Math. Math. Sci. **27** (2004), 1429–1436.

[2] A. Amourah and M. Darus, *Some properties of a new class of univalent functions involving a new generalized differential operator with negative coefficients*, Indian Journal of Science and Technology **9**(36) (2016), 1–7.

[3] W. G. Atshan and A. K. Wanas, *Differential subordinations of multivalent analytic functions associated with Ruscheweyh derivative*, Analele Universitătii din Oradea - Fascicola Matematică **XX**(1) (2013), 27–33.

[4] N. E. Cho and T. H. Kim, *Multiplier transformations and strongly close-to-convex functions*, Bull. Korean Math. Soc. **40**(3) (2003), 399–410.

[5] M. Darus and R. W. Ibrahim, *On subclasses for generalized operators of complex order*, Far East Journal of Mathematical Sciences **33**(3) (2009), 299–308.

[6] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.

[7] J. L. Liu, *On a class of multivalent analytic functions associated with an integral operator*, Bulletin of the Institute of Mathematics **5**(1) (2010), 95–110.

[8] S. S. Miller and P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J. **28** (1981), 157–171.

[9] J. K. Prajapat and R. K. Raina, *Some applications of differential subordination for a general class of multivalently analytic functions involving a convolution structure*, Math. J. Okayama Univ. **52** (2010), 147–158.

[10] S. Ruscheweyh, *Convolutions in Geometric Function Theory*, Les Presses de I'Université de Montréal, Montréal, 1982.

[11] G. St. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Mathematics **1013**, Springer Verlag, Berlin, 1983, 362–372.

[12] A. K. Wanas, *Differential subordination results defined by new class for higher-order derivatives of multivalent analytic functions*, International Journal of Advanced Research in Science, Engineering and Technology **4**(8) (2017), 4363–4368.
[13] A. K. Wanas and A. H. Majeed, *Differential subordinations for higher-order derivatives of multivalent analytic functions associated with Dziok-Srivastava operator*, Analele Universitătii din Oradea - Fascicola Matematică **XXV**(1) (2018), 33–42.

[1]Department of Mathematics,
College of Science, University of Al-Qadisiyah,
Iraq
*Email address*: abbas.kareem.w@qu.edu.iq