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ANALYZING THE SEMILOCAL CONVERGENCE OF A FOUR-STEP SCHEME WITH NOVEL MAJORANT AND AVERAGE LIPSCHITZ CONDITIONS

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Abstract. The primary goal of this study is to examine the semilocal convergence (s.c.) of the classical four-step nonlinear scheme (fsns), which is used to identify nonlinear operators in Banach spaces (b.s.). The first derivative of the operator is assumed to satisfy a generalized Lipschitz condition (g.l.c.), which leads to a novel s.c. analysis of the fsns. Majorizing functions (m.f.) and average Lipschitz conditions are used in the study because they have been shown to be effective in studying the convergence of Newton-type approaches for nonlinear operator equations. The study expands on prior research on the topic by examining higher-order iterative algorithms and increasing their range of applications. The fsns's s.c. analysis is then presented under the average Lipschitz condition. The scheme's applicability to nonlinear integral equations is also demonstrated.

1. Introduction

Let $\Im$ be an operator that maps from a nonempty open convex subset $\mho$ of a b.s. $\bar{\bar{U}}$ to another b.s. $\bar{\bar{V}}$. Our aim is to approximate a locally unique solution Υ^* of a nonlinear equation given by:

$$\Im(x) = 0. \tag{1.1}$$

It is important to highlight that within the field of computer science, the realm of numerical analysis is closely related to various adaptations of Newton's method (NM),

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given by

$$x_{n+1} = x_n - [\Im'(x_n)]^{-1}\,\Im(x_n), \quad n \geq 0. \tag{1.2}$$

Although NM is known for its relatively slow convergence, it remains a popular and widely used iterative method. For a more comprehensive understanding of this method, interested readers can refer to Ortega's survey on NM [12].

Numerous studies have focused on analyzing the local convergence (l.c.) of iterative methods (IM) under Lipschitz, Hölder, and w-continuity conditions. However, these conditions may not be satisfied by certain nonlinear problems, which limits the applicability of these methods. In order to overcome this limitation, Wang [15] introduced the concept of g.l.c. to analyze the l.c. of NM. However, it was realized by Saxena et al. [13] that the earlier definition of g.l.c. cannot be directly applied to multi-step Newton-type methods. Therefore, they proposed a modified definition of generalized Lipschitz conditions to address this issue.

To examine the l.c. of the fsns method, we will employ the four-step Newton-like sequences fsns proposed by Regmi [10], subject to the χ-average condition. The fsns method is defined as follows:

$$\begin{aligned}
y_n &= x_n - [\Im'(x_n)]^{-1}\Im(x_n),\\
z_n &= y_n - [\Im'(x_n)]^{-1}\Im(y_n),\\
q_n &= z_n - [\Im'(x_n)]^{-1}\Im(z_n),\\
x_{n+1} &= q_n - [\Im'(x_n)]^{-1}\Im(q_n), \quad n \geq 0.
\end{aligned} \tag{1.3}$$

Extensive investigations have been conducted by Saxena et al. [14] to analyze the l.c. of the aforementioned iterative operator under a generalized Lipschitz condition. The m.f. technique has proven to be a valuable analytical tool in studying the convergence of various Newton-type methods, including those employed for solving nonlinear operator equations [4, 5, 16].

Previous investigations have extensively examined the s.c. of iterative methods under various assumptions. Argyros [1] advanced the analysis by considering a generalized notion of the second Fréchet derivative of the operator $\Im$, i.e., the mapping $\Im : D \subset \mathbb{R}^n \to \mathbb{R}^m$ is Fréchet-differentiable at $x \in \operatorname{int}(D)$ if there is an $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{h\to 0} \frac{||\Im(x+h) - \Im x - Ah||}{||h||} = 0.$$

The linear operator A is again denoted by $\Im'(x)$, and is called the first $\Im$-derivative of $\Im$ at x. $\Im$ is second order Fréchet-differentiable at x, $\Im''(x)$ if the first derivative $\Im'(x)$ is itself Fréchet-differentiable as a map from $\mathbb{R}^n$ to the space of bounded linear operators $L(\mathbb{R}^n, \mathbb{R}^m)$ [12]. Ruiz and Argyros [11] relaxed the conditions and introduced a novel convergence analysis framework based on Lipschitz and Lipschitz-like criteria for the first Fŕechet derivative of $\Im$. George [2] focused on the l.c. of multiple Newton-like methods, while Ling [9] proposed a two-step NM with a generalized Lipschitz condition to ensure convergence. Ling's approach achieved Q-cubic convergence and provided a fresh perspective on s.c. analysis.

Recognizing the potential for exploring higher-order iterative methods and expanding their applicability, we embark on an in-depth investigation of s.c. Using our analysis and findings, we aim to improve our understanding of numerical methods for solving nonlinear equations and systems. Through our analysis and findings, our aim is to improve the understanding of numerical methods for solving nonlinear equations and systems. The improved convergence criteria and error estimates provided by our approach have the potential to enhance the efficiency and reliability of numerical algorithms in practice.

The paper is organized as follows. Section 2 discusses the generalized/average Lipschitz condition. Section 3 provides a review of preliminary notions and properties related to m.f.'s and majorizing sequences. The s.c. analysis of the fsns under the χ-average Lipschitz condition is presented in Section 4. In Section 5, we demonstrate the application of the proposed approach to nonlinear integral equations. In conclusion, Section 6 provides final remarks to wrap up the paper.

2. Preliminary Results and Notations

In order to make this research self-contained, we provide the necessary concepts and notations extracted from references [9, 16].

This section concludes by introducing the concepts of generalized/average Lipschitz condition.

Criteria 1. Let $x_0 \in \mathbb{D}$ and $[\Im'(x_0)]^{-1}$ be nonsingular, and let $\epsilon > 0$, such that $O(x_0, \epsilon) \subseteq \mathbb{D}$. We say that the χ-average Lipschitz condition is satisfied by $\Im'$ on $O(x_0, \epsilon)$ if, for any $x, y \in O(x_0, \epsilon)$ with $\|x - x_0\| + \|y - x\| < \epsilon$

$$\left\|[\Im'(x_0)]^{-1}(\Im'(y) - \Im'(x))\right\| \leq \int_{\|x-x_0\|}^{\|x-x_0\|+\|y-x\|} \chi(h)du. \tag{2.1}$$

In the work of Wang [16], the generalized Lipschitz condition also referred to as the center Lipschitz condition in the inscribed sphere with χ-average, was introduced.

Clearly, the conventional Lipschitz condition with the Lipschitz constant $\chi(\epsilon)$ can be inferred from the χ-average Lipschitz condition on $O(x_0, \epsilon)$, as defined earlier. By utilizing the χ-average Lipschitz condition, the NM can be improved, offering a more accurate convergence criterion and an estimation of the convergence radius.

3. Application of fsns (1.3) to the Majorizing Function

Consider the notation $O(x, \wp)$ representing an open ball with radius $\wp$ centered at x, and $\overline{O(x, \wp)}$ denoting its closure set. The identity operator is denoted by I. Let us consider $\chi(\cdot)$ as a positive non-decreasing (n.d.) function defined on the interval $[0, \varrho)$, where $\varrho > 0$ satisfies the following relation:

$$\frac{\int_0^{\varrho} \chi(h)(\varrho - h)du}{\varrho} = 1. \tag{3.1}$$

The m.f. $\hbar : [0, \varrho] \to \mathbb{R}$ is defined as

$$\hbar(a) = \flat - a + \int_0^a \chi(h)(a-h)du, \quad a \in [0, \varrho]. \tag{3.2}$$

In the early 2000s, Wang [16] made a significant contribution through an analysis of the convergence properties of Newton's method (1.2), we can gain insights into the behavior of the method. It is important to mention that the m.f. employed in this study shares similarities with the one utilized by Ferreira [4]. The motivation behind adopting the aforementioned m.f. in this study is its potential to provide more accurate convergence criteria and error estimates for the three-step classical approach. With this in mind, we can clearly observe that:

$$\hbar'(a) = -1 + \int_0^a \chi(h)du, \quad a \in [0, \varrho), \tag{3.3}$$

and

$$\hbar''(a) = \chi(a) > 0, \quad \text{for a.e. } a \in [0, \varrho). \tag{3.4}$$

Consequently, it can be inferred that

$$\int_j^a \chi(h)du = \hbar'(a) - \hbar'(j), \quad \text{for any } j, a \in [0, \varrho) \text{ with } j < a.$$

The relationship between the m.f. and the χ function discussed here will be frequently employed in the convergence analysis of the fsns provided in equation (1.3). Let's assume that $\wp_0$ satisfies

$$\int_0^{\wp_0} \chi(h)du = 1. \tag{3.5}$$

Consequently, it can be observed that $\hbar(a)$ is strictly convex, $\hbar'(a)$ is convex, increasing, and satisfies $-1 \leq \hbar'(a) < 0$ for any $a \in [0, \wp_0)$.

In this section, we will commence by examining some essential intermediate results concerning the error estimates for the majorizing sequences c_ι, b_ι, and a_ι. Additionally, we will explore the connection between the m.f. $\hbar(a)$, defined in equation (3.2), and the nonlinear operator $\Im$. Subsequently, we will delve into the convergence analysis of the fsns (1.3) under the χ-average Lipschitz condition, as presented in b.s.

3.1. **Important Intermediary Findings.** The subsequent auxiliary result concerning scalar-valued functions is drawn from standard convex analysis literature (refer to Remark 4.1.2 and Theorem 4.1.1 in [7, p. 21]. These findings hold great significance and will be integrated into our analysis.

Lemma 3.1. *Assuming $G : (0, \varrho) \to \mathbb{R}$ is a continuously differentiable and convex function, in which $\varrho > 0$ and $0 \leq \mho \leq 1$. Then,*

(i) $(1-\mho)G'(\mho a) \leq \frac{G(a)-G(\mho a)}{a} \leq (1-\mho)G'(a)$, *for all* $a \in (0, \varrho)$,

(ii) $\frac{G(l)-G(\mho l)}{l} \leq \frac{G(m)-G(\mho m)}{m}$, *for all* $l, m \in (0, \varrho)$, $l < m$.

Most notably, if G exhibits strict convexity, then the aforementioned inequalities are strict.

Also, establish

$$\natural := \int_0^{\wp_0} \chi(h) u du, \tag{3.6}$$

in which $\wp_0$ is defined by equation (3.5). The preceding lemma is adapted from [16, Lemma 1.2] and provides some fundamental characteristics for the m.f. $\hbar$ given by expression (3.2).

Lemma 3.2 ([16]). *In the case where* $0 < \flat < \natural$, *the function* $\hbar$ *exhibits a decreasing behavior on the interval* $[0, \wp_0]$ *and*

$$\hbar(\flat) > 0, \quad \hbar(\wp_0) = \flat - \natural < 0, \quad \hbar(\varrho) = \flat > 0. \tag{3.7}$$

Furthermore, within each interval, $\hbar$ *possesses a unique zero denoted by* a^* *and* a^{**} *respectively. These zeros satisfy the following conditions:*

$$\flat < a^* < \frac{\wp_0}{\natural}\flat < \wp_0 < a^{**} < \varrho. \tag{3.8}$$

We initialize the position as $a_0 = 0$. The sequences b_ι, c_ι, and a_ι represent the values generated using the fsns for the m.f. $\hbar$ as described in [9]. This scheme can be expressed as follows:

$$\begin{cases} b_\iota = a_\iota - \dfrac{\hbar(a_\iota)}{\hbar'(a_\iota)}, \\ c_\iota = b_\iota - \dfrac{\hbar(b_\iota)}{\hbar'(a_\iota)}, \\ d_\iota = c_\iota - \dfrac{\hbar(c_\iota)}{\hbar'(a_\iota)}, \\ a_{\iota+1} = d_\iota - \dfrac{\hbar(d_\iota)}{\hbar'(a_\iota),} \quad \iota = 0, 1, 2, \dots. \end{cases} \tag{3.9}$$

*Note 1.*Assuming that $0 < \flat \leq \natural$, we can utilize Lemmas 3.1 and 3.2, along with standard analytical techniques (e.g., as described in [8]), to easily establish that the sequences b_ι, c_ι, and a_ι generated by (3.9) satisfy the following relationships:

$$0 \leq a_\iota < b_\iota < c_\iota < d_\iota < a_{\iota+1} < a^*, \quad \text{for all } \iota \geq 0, \tag{3.10}$$

and progressively approach the common point a^*, which represents the unique zero of $\hbar$ on the interval $[0, \wp_0]$. Here, $\wp_0$ is determined by (3.5). Moreover, we can derive the following:

$$a^* - a_{\iota+1} \leq \frac{1}{2} \cdot \frac{\hbar''(a^*)^4}{\hbar'(a^*)^4}(a^* - a_\iota)^5, \quad \iota \geq 0. \tag{3.11}$$

Specifically, if $1 + a^*\hbar''(a^*)/\hbar'(a^*) \geq 0$, then we obtain

$$d_\iota - c_\iota \geq (a^* - c_\iota) + \frac{\hbar''(a^*)}{\hbar'(a^*)}(a^* - a_\iota)(a^* - c_\iota), \quad \iota \geq 0. \tag{3.12}$$

The convergence properties of the sequences d_ι, c_ι, b_ι, and a_ι discussed earlier will be utilized in the convergence analysis of the fsns (1.3). Consider an initial guess $x_0 \in \mathbb{D}$

such that the inverse $[\Im'(x_0)]^{-1}$ exists. Let $O(x_0, \wp_0) \subset \mathbb{D}$, where $\wp_0$ satisfies equation (3.5). Let's denote

$$\flat := \left\|[\Im'(x_0)]^{-1}\Im(x_0)\right\|. \tag{3.13}$$

Recalling that equation (3.2) defines the m.f. $\hbar$, and equation (3.6) defines $\natural$, we have the unique zeros a^* and a^{**} of $\hbar$ on the intervals $[0, \wp_0]$ and $[\wp_0, \varrho]$, respectively. Here, $\wp_0$ and ϱ satisfy equations (3.5) and (3.1), respectively. It is important to note that when $0 < \flat \leq \natural$, the sequences c_ι, b_ι, and a_ι given by equation (3.9) gradually converge to a^*, where $\natural$ is defined by equation (3.6).

The subsequent lemmas, which establish explicit connections between the m.f. $\hbar$ and the nonlinear function $\Im$, play a vital role in the subsequent analysis of the fsns given by (1.3).

Lemma 3.3. *Suppose that $|x - x_0| \leq a < a^*$ and the first derivative $\Im$ in $O(\Upsilon^*, a)$ satisfies the χ-average Lipschitz condition* (2.1)*. In such a case, $\Im'(x)$ is nonsingular, we obtain the following outcome:*

$$\left\|[\Im'(x)]^{-1}\Im'(x_0)\right\| \leq -\frac{1}{\hbar'(\|x - x_0\|)} \leq -\frac{1}{\hbar'(a)}. \tag{3.14}$$

Furthermore, $\Im$ is also nonsingular in $O(x_0, a^*)$.

Proof. Let's consider $x \in \overline{O(x_0, a)}, 0 \leq a < a^*$. By applying the χ-average Lipschitz condition (2.1), we can observe the following:

$$\left\|[\Im'(x_0)]^{-1}\Im'(x) - I\right\| \leq \int_0^{\|x-x_0\|} \chi(h)du = \hbar'(\|x - x_0\|) - \hbar'(0).$$

Since $\hbar'(0) = 1$ and $\hbar$ strictly increases in the interval $(0, a^*)$, we obtain the following:

$$\left\|[\Im'(x_0)]^{-1}\Im'(x) - I\right\| \leq \hbar'(x) + 1 < 1.$$

The last inequality holds due to the fact that $-1 < \hbar'(x) < 0$ for any $x \in (0, a^*)$. Therefore, we can apply the Banach lemma to deduce that $[\Im'(x_0)]^{-1}\Im'(x)$ is nonsingular and relation (3.14) holds. Thus, the proof is now complete. ☐

Lemma 3.4. *Consider the sequences d_ι, c_ι, b_ι, and a_ι generated by the scheme* (3.9)*. Assuming that $\Im'$ in $O(x_0, a^*)$ satisfies the χ-average Lipschitz condition* (2.1)*, if $0 < \flat \leq \natural$, the sequences x_ι, y_ι, z_ι and q_ι obtained using the four-step fsns* (1.3) *with the initial guess x_0 are well defined and contained within $O(x_0, a)$. Moreover, for all $\iota = 0, 1, 2, \ldots$, we have the following:*

- (ι) $[\Im'(x_\iota)]^{-1}$ *exists with* $\|[\Im'(x_\iota)]^{-1}\Im'(x_0)\| \leq -\frac{1}{\hbar'(\|x_\iota - x_0\|)} \leq -\frac{1}{\hbar'(a)}$,
- (ii) $\|[\Im'(x_0)]^{-1}\Im(x_\iota)\| \leq \hbar(x_\iota)$,
- (iii) $\|y_\iota - x_\iota\| \leq b_\iota - a_\iota$,
- (iv) $\|z_\iota - y_\iota\| \leq (c_\iota - b_\iota)\left(\frac{\|y_\iota - x_\iota\|}{b_\iota - a_\iota}\right)^2 \leq c_\iota - b_\iota$,
- (v) $\|q_\iota - z_\iota\| \leq (d_\iota - c_\iota)\left[\frac{\|z_\iota - y_\iota\|}{c_\iota - b_\iota} \cdot \frac{\|y_\iota - x_\iota + \tau\|z_\iota - y_\iota\|}{b_\iota - a_\iota + \tau\|c_\iota - b_\iota\|}\right] \leq d_\iota - c_\iota$,
- (vi) $\|z_\iota - x_\iota\| \leq c_\iota - a_\iota$,

(vii) $\|q_\iota - x_\iota\| \le d_\iota - a_\iota$,
$(viii)$ $\|x_{\iota+1} - q_\iota\| \le (a_{\iota+1} - d_\iota)\left[\frac{\|q_\iota - z_\iota\|}{d_\iota - c_\iota} \cdot \frac{\|z_\iota - x_\iota + \tau\|q_\iota - z_\iota\|}{c_\iota - a_\iota + \tau\|d_\iota - c_\iota\|}\right] \le a_{\iota+1} - d_\iota$,
(ix) $\|x_{\iota+1} - x_\iota\| \le (a_{\iota+1} - a_\iota)$.

Proof. We will prove the validity of the statement using induction. Let's consider the base case $\iota = 0$. In this case, we can see that $(\iota) - (iii)$, (vi), and (vii) hold true. Therefore, we have $y_0 \in O(x_0, a^*)$ because $\|y_0 - x_0\| \le b_0 - a_0 = b_0 < a^*$ and $\|z_0 - y_0\| \le c_0 - b_0$. Now, let's examine (iv), (v), $(viii)$, and (ix) by considering scheme (1.3). We can express these conditions as follows:

$$\begin{aligned}\Im(y_0) &= \Im(y_0) - \Im(y_0) - \Im'(y_0)(y_0 - x_0)\\ &= \int_0^1 \Big[\Im'(x_0 + \tau(y_0 - x_0)) - \Im'(x_0)\Big](y_0 - x_0)\, d\tau.\end{aligned}$$

By utilizing the χ-average Lipschitz condition (2.1), we can infer that:

$$\begin{aligned}\|[\Im'(x_0)]^{-1}\Im(y_0)\| &\le \int_0^1 \left\|[\Im'(x_0)]^{-1}\Big(\Im'(x_0 + \tau(y_0 - x_0)) - \Im'(x_0)\Big)\right\| \cdot \|y_0 - x_0\|\, d\tau\\ &\le \int_0^1 \left(\int_0^{\tau\|y_0 - x_0\|} \chi(h)\, du\right) \|y_0 - x_0\|\, d\tau\\ &= \int_0^1 \Big[\hbar'(\tau\|y_0 - x_0\|) - \hbar'(0)\Big] \cdot \|y_0 - x_0\|\, d\tau\end{aligned}$$

Considering the strict convexity of $\hbar'$ in $[0, \wp_0)$ and the fact that $\|y_0 - x_0\| \le b_0 - a_0$ according to (iii), Lemma 3.1 implies that:

$$\begin{aligned}\hbar'(\tau\|y_0 - x_0\|) - \hbar'(0) &= \frac{\hbar'(\tau\|y_0 - x_0\|) - \hbar'(0)}{\|y_0 - x_0\|}\|y_0 - x_0\|\\ &\le \frac{\hbar'(\tau(b_0 - a_0)) - \hbar'(0)}{(b_0 - a_0)}\|y_0 - x_0\|\end{aligned}$$

By combining the aforementioned inequality and scheme (3.9), we can derive:

$$\begin{aligned}\left\|[\Im'(x_0)]^{-1}\Im(y_0)\right\| &\le \int_0^1 \Big[\hbar'(\tau b_0) - \hbar'(0)\Big]\, b_0\, d\tau \left(\frac{\|y_0 - x_0\|}{b_0 - a_0}\right)^2\\ &= \hbar(b_0)\left(\frac{\|y_0 - x_0\|}{b_0 - a_0}\right)^2 = (c_0 - b_0)\left(\frac{\|y_0 - x_0\|}{b_0 - a_0}\right)^2.\end{aligned}$$

Consequently,

$$\|z_0 - y_0\| = \left\|[\Im'(x_0)]^{-1}\Im(y_0)\right\| \le (c_0 - b_0)\left(\frac{\|y_0 - x_0\|}{b_0 - a_0}\right)^2.$$

By utilizing these two outcomes, we find that $|z_0 - x_0| \leq |z_0 - y_0 + y_0 - x_0| \leq c_0 - b_0 + b_0 - a_0 \leq c_0 - a_0$. Likewise, for the subsequent relation, we observe that

$$\begin{aligned}\Im(z_0) &= \Im(z_0) - \Im(y_0) - \Im'(x_0)(z_0 - y_0)\\ &= \int_0^1 \Big[\Im'(y_0 + \tau(z_0 - y_0)) - \Im'(x_0)\Big](z_0 - y_0)\, d\tau.\end{aligned}$$

By employing the χ-average Lipschitz condition (2.1) as a supporting factor

$$\begin{aligned}\Big\|[\Im'(x_0)]^{-1}\Im(z_0)\Big\| &\leq \int_0^1 \Big\|[\Im'(x_0)]^{-1}\Big(\Im'(y_0 + \tau(z_0 - y_0)) - \Im'(x_0)\Big)\Big\| \cdot \|z_0 - y_0\|\, d\tau\\ &\leq \Big\|[\Im'(x_0)]^{-1}\Im'(x_0)\Big\| \int_0^1 \left(\int_0^{\|y_0 - x_0\| + \tau\|z_0 - y_0\|} \chi(h)\, du\right) \|z_0 - y_0\|\, d\tau\\ &= \int_0^1 \Big[\hbar'(\|y_0 - x_0\| + \tau\|z_0 - y_0\|) - \hbar'(0)\Big] \cdot \|z_0 - y_0\|\, d\tau.\end{aligned}$$

Lemma 3.1 states that, under the assumption of $\hbar'$ being strictly convex in $[0, \wp_0]$, along with the condition $|z_0 - y_0| \leq c_0 - b_0$ from (iv), we have the following:

$$\begin{aligned}&\hbar'(\|y_0 - x_0\| + \tau\|z_0 - y_0\|) - \hbar'(0)\\ =&\frac{\hbar'(\|y_0 - x_0\| + \tau\|z_0 - y_0\|) - \hbar'(0)}{\|y_0 - x_0\| + \tau\|z_0 - y_0\|}\Big(\|y_0 - x_0\| + \tau\|z_0 - y_0\|\Big)\\ \leq&\frac{\hbar'\Big((b_0 - a_0) + \tau(c_0 - b_0)\Big) - \hbar'(0)}{(b_0 - a_0) + \tau(c_0 - b_0)}\Big(\|y_0 - x_0\| + \tau\|z_0 - y_0\|\Big).\end{aligned}$$

By combining the inequality mentioned above and scheme (3.9), we can derive the following:

$$\begin{aligned}&\Big\|[\Im'(x_0)]^{-1}\Im(z_0)\Big\|\\ \leq&\int_0^1 \frac{\hbar'\Big((b_0 - a_0) + \tau(c_0 - b_0)\Big) - \hbar'(0)}{(b_0 - a_0) + \tau(c_0 - b_0)}\|z_0 - y_0\| \Big(\|y_0 - x_0\| + \tau\|z_0 - y_0\|\Big)\, d\tau\\ =&\hbar(c_0)\left[\frac{\|z_0 - y_0\|}{c_0 - b_0} \cdot \frac{\|y_0 - x_0 + \tau\|z_0 - y_0\|\|\|}{(b_0 - a_0) + \tau(c_0 - b_0)}\right]\\ =&(d_0 - c_0)\left[\frac{\|z_0 - y_0\|}{c_0 - b_0} \cdot \frac{\|y_0 - x_0 + \tau\|z_0 - y_0\|\|\|}{(b_0 - a_0) + \tau(c_0 - b_0)}\right].\end{aligned}$$

Similarly, for the third part, we observe that

$$\begin{aligned}\Im(q_0) &= \Im(q_0) - \Im(z_0) - \Im'(x_0)(q_0 - z_0)\\ &= \int_0^1 \Big[\Im'(z_0 + \tau(q_0 - z_0)) - \Im'(x_0)\Big](q_0 - z_0)\, d\tau.\end{aligned}$$

By utilizing the χ-average Lipschitz condition (2.1),

$$\begin{aligned}
&\left\|[\Im'(x_0)]^{-1}\Im(q_0)\right\| \\
\le& \int_0^1 \left\|[\Im'(x_0)]^{-1}\Big(\Im'(z_0+\tau(q_0-z_0))-\Im'(x_0)\Big)\right\| \cdot \|q_0-z_0\|\,d\tau \\
\le& \left\|[\Im'(x_0)]^{-1}\Im'(x_0)\right\| \int_0^1 \left(\int_0^{\|z_0-x_0\|+\tau\|q_0-z_0\|} \chi(h)\,du\right) \|q_0-z_0\|\,d\tau \\
=& \int_0^1 \Big[\hbar'(\|z_0-x_0\|+\tau\|q_0-z_0\|)-\hbar'(0)\Big]\cdot\|q_0-z_0\|\,d\tau.
\end{aligned}$$

Lemma 3.1 asserts that, provided $\hbar'$ exhibits strict convexity in the interval $[0,\wp_0]$, and taking into account the inequality $|q_0-z_0|\le d_0-c_0$ from (v), we have the following

$$\begin{aligned}
&\hbar'\Big(\|z_0-x_0\|+\tau\|q_0-z_0\|\Big)-\hbar'(0) \\
=&\frac{\hbar'\Big(\|z_0-x_0\|+\tau\|q_0-z_0\|\Big)-\hbar'(0)}{\|z_0-x_0\|+\tau\|q_0-z_0\|}\Big(\|z_0-x_0\|+\tau\|q_0-z_0\|\Big) \\
\le&\frac{\hbar'\Big((c_0-a_0)+\tau(d_0-c_0)\Big)-\hbar'(0)}{(c_0-a_0)+\tau(d_0-c_0)}\Big(\|z_0-x_0\|+\tau\|q_0-z_0\|\Big).
\end{aligned}$$

Combining the aforementioned inequality and scheme (3.9), we can deduce the following

$$\begin{aligned}
&\left\|[\Im'(x_0)]^{-1}\Im(q_0)\right\| \\
\le&\int_0^1 \frac{\hbar'\Big((c_0-a_0)+\tau(d_0-c_0)\Big)-\hbar'(0)}{(c_0-a_0)+\tau(d_0-c_0)}\|q_0-z_0\|\Big(\|z_0-x_0\|+\tau\|q_0-z_0\|\Big)\,d\tau \\
=&\hbar(d_0)\left[\frac{\|q_0-z_0\|}{d_0-c_0}\cdot\frac{\|z_0-x_0\|+\tau\|q_0-z_0\|}{(c_0-a_0)+\tau(d_0-c_0)}\right] \\
=&(a_1-d_0)\cdot\left[\frac{\|q_0-z_0\|}{d_0-c_0}\cdot\frac{\|z_0-x_0\|+\tau\|q_0-z_0\|}{(c_0-a_0)+\tau(d_0-c_0)}\right]
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|x_1-q_0\| &= \left\|[\Im'(x_0)]^{-1}\Im(q_0)\right\| \\
&\le (a_1-d_0)\left[\frac{\|q_0-z_0\|}{d_0-c_0}\cdot\frac{\|z_0-x_0\|+\tau\|q_0-z_0\|}{(c_0-a_0)+\tau(d_0-c_0)}\right]
\end{aligned}$$

In conclusion, we obtain

$$\begin{aligned}
\|x_1-x_0\| &\le \|x_1-q_0\|+\|q_0-z_0\|+\|z_0-y_0\|+\|y_0-x_0\| \\
&\le (a_1-d_0)+(d_0-c_0)+(c_0-b_0)+(b_0-a_0) \\
&= a_1-a_0
\end{aligned}$$

Let's assume that $x_\iota, y_\iota, z_\iota, q_\iota \in O(x_0, a^*)$, $|x_1 - x_0| \leq a_\iota$, and (ι)-(ix) are valid for some $\iota \geq 0$. Using the inductive hypothesis (iii), we have $|y_\iota - x_0| \leq |y_\iota - x_\iota| + |x_\iota - x_0| \leq b_\iota$. Additionally, by applying the inductive hypothesis (ix) and (3.10), we obtain

$$\|x_{\iota+1} - x_0\| \leq \sum_{a=0}^{\iota} \|x_{\iota+1} - x_\iota\| \leq \sum_{a=0}^{\iota} (a_{\iota+1} - a_\iota) = a_{\iota+1} < a^*.$$

This indicates that $x_{\iota+1} \in O(x_0, a^*)$. Combining this with relation (3.14), we can conclude that (ι) holds for $\iota + 1$. Applying scheme (1.3), we can derive the following identity for (ii)

$$\begin{aligned}\Im(x_{\iota+1}) &= \Im(x_{\iota+1}) - \Im(q_\iota) - \Im'(x_\iota)(x_{\iota+1} - q_\iota) \\ &= \int_0^1 \Big[\Im'(q_\iota + \tau(x_{\iota+1} - q_\iota)) - \Im'(x_\iota)\Big](x_{\iota+1} - q_\iota)\, d\tau.\end{aligned}$$

By considering the χ-average Lipschitz condition (2.1), we attain

$$\begin{aligned}\Big\|[\Im'(x_0)]^{-1}\Im(x_{\iota+1})\Big\| &\leq \int_0^1 \Big\|[\Im'(x_0)]^{-1}[\Im'(q_\iota + \tau(x_{\iota+1} - q_\iota)) - \Im'(x_\iota)]\Big\| \cdot \|x_{\iota+1} - q_\iota\|\, d\tau \\ &\leq \int_0^1 \left(\int_{\|x_\iota - x_0\|}^{\|x_\iota - x_0\| + \|q_\iota - x_\iota + \tau(x_{\iota+1} - q_\iota)\|} \chi(h)\, du\right) \|x_{\iota+1} - q_\iota\|\, d\tau.\end{aligned}$$

By utilizing Lemma 3.1 and the note (3.10), one can infer that if $\hbar'$ is convex and increasing in $[0, \wp_0]$, then

$$\begin{aligned}&\int_{\|x_\iota - x_0\|}^{\|x_\iota - x_0\| + \|q_\iota - x_\iota + \tau(x_{\iota+1} - q_\iota)\|} \chi(h)\, du \\ =&\hbar'\Big(\|x_\iota - x_0\| + \|q_\iota - x_\iota + \tau(x_{\iota+1} - q_\iota)\|\Big) - \hbar'(\|q_\iota - x_0\|) \\ \leq&\hbar'\Big(\|x_\iota - x_0\| + \|q_\iota - x_\iota\| + \tau\|x_{\iota+1} - q_\iota\|\Big) - \hbar'(\|q_\iota - x_0\|) \\ \leq&\frac{\hbar'(d_\iota + \tau(a_{\iota+1} - d_\iota)) - \hbar'(a_\iota)}{d_\iota - a_\iota + \tau(a_{\iota+1} - d_\iota)} \Big(\|q_\iota - x_\iota\| + \tau\|x_{\iota+1} - q_\iota\|\Big) \\ \leq&\hbar'(d_\iota + \tau(a_{\iota+1} - d_\iota)) - \hbar'(a_\iota).\end{aligned}$$

As a result, we can obtain

$$\begin{aligned}\Big\|[\Im'(x_0)]^{-1}\Im(x_{\iota+1})\Big\| &\leq \int_0^1 \Big[\hbar'(d_\iota + \tau(a_{\iota+1} - d_\iota)) - \hbar'(a_\iota)\Big] \cdot \|x_{\iota+1} - q_\iota\| d\tau \\ &= \hbar(a_{\iota+1}) - \hbar(d_\iota) - \hbar'(a_\iota)(a_{\iota+1} - d_\iota)\frac{\|x_{\iota+1} - q_\iota\|}{a_{\iota+1} - d_\iota} \\ &= \hbar(a_{\iota+1}).\end{aligned} \tag{3.15}$$

This illustrates that (ii) is valid for the case $\iota+1$. By combining equations (3.14) and (3.15), we obtain that

$$\begin{aligned}\|y_{\iota+1}-x_{\iota+1}\| &= \left\|[\Im'(x_{\iota+1})]^{-1}\Im(x_{\iota+1})\right\| \\ &\leq \left\|[\Im'(x_{\iota+1})]^{-1}\Im'(x_0)\right\| \cdot \left\|[\Im'(x_0)]^{-1}\Im(x_{\iota+1})\right\| \\ &\leq -\frac{\hbar(a_{\iota+1})}{\hbar'(a_{\iota+1})} = b_{\iota+1}-a_{\iota+1}.\end{aligned} \tag{3.16}$$

This indicates by verifying that condition (iii) holds for the case $\iota+1$, we can therefore infer that $\|y_{\iota+1}-x_0\| \leq \|y_{\iota+1}-x_{\iota+1}\| + \|x_{\iota+1}-x_0\| \leq b_{\iota+1} < a^*$, implying that $y_{\iota+1} \in O(x_0, a^*)$. Additionally, with regard to (iv), it is noteworthy that

$$\begin{aligned}\Im(y_{\iota+1}) &= \Im(y_{\iota+1}) - \Im(x_{\iota+1}) - \Im'(x_{\iota+1})(y_{\iota+1}-x_{\iota+1}) \\ &= \int_0^1 \Big[\Im'(x_{\iota+1}+\tau(y_{\iota+1}-x_{\iota+1})) - \Im'(x_{\iota+1})\Big](y_{\iota+1}-x_{\iota+1})\,d\tau.\end{aligned}$$

Given the χ-average Lipschitz condition stated in (2.1), reduces the above relation to

$$\begin{aligned}&\left\|[\Im'(x_{\iota+1})]^{-1}\Im(y_{\iota+1})\right\| \\ &\leq -\frac{1}{\hbar'(a_{\iota+1})}\int_0^1 \left(\int_{\|x_{\iota+1}-x_0\|}^{\|x_{\iota+1}-x_0\|+\|\tau(y_{\iota+1}-x_{\iota+1})\|} \chi(h)\,du\right)\|y_{\iota+1}-x_{\iota+1}\|\,d\tau.\end{aligned} \tag{3.17}$$

By utilizing Lemma 3.1 and the observation in Note 1, one can derive from the convexity and increasing nature of $\hbar'$ in the interval $[0, \wp_0]$ that

$$\begin{aligned}&\int_{\|x_{\iota+1}-x_0\|}^{\|x_{\iota+1}-x_0\|+\tau\|y_{\iota+1}-x_{\iota+1}\|} \chi(h)\,du \\ =&\hbar'\Big(\|x_{\iota+1}-x_0\|+\tau\|y_{\iota+1}-x_{\iota+1}\|\Big) - \hbar'(\|x_{\iota+1}-x_0\|) \\ \leq&\frac{\hbar'(a_{\iota+1}+\tau(b_{\iota+1}-a_{\iota+1})) - \hbar'(a_{\iota+1})}{b_{\iota+1}-a_{\iota+1}}\|y_{\iota+1}-x_{\iota+1}\| \\ \leq&\hbar'(a_{\iota+1}+\tau(b_{\iota+1}-a_{\iota+1})) - \hbar'(a_{\iota+1}).\end{aligned}$$

From the aforementioned deduction, we are able to acquire

$$\begin{aligned}&\left\|[\Im'(x_{\iota+1})]^{-1}\Im(y_{\iota+1})\right\| \\ &\leq -\frac{1}{\hbar'(a_{\iota+1})}\int_0^1 \Big[\hbar'(a_{\iota+1}+\tau(b_{\iota+1}-a_{\iota+1})) - \hbar'(a_{\iota+1})\Big]\cdot\|y_{\iota+1}-x_{\iota+1}\|\,d\tau \\ &= -\frac{1}{\hbar'(a_{\iota+1})}\Big[\hbar(b_{\iota+1}) - \hbar(a_{\iota+1}) - \hbar'(a_{\iota+1})(b_{\iota+1}-a_{\iota+1})\Big]\frac{\|y_{\iota+1}-x_{\iota+1}\|^2}{(b_{\iota+1}-a_{\iota+1})^2} \\ &= -\frac{\hbar(b_{\iota+1})}{\hbar'(a_{\iota+1})}\cdot\frac{\|y_{\iota+1}-x_{\iota+1}\|^2}{(b_{\iota+1}-a_{\iota+1})^2}.\end{aligned} \tag{3.18}$$

which is resulting in

$$\|z_{\iota+1}-y_{\iota+1}\| = \left\|[\Im'(x_{\iota+1})]^{-1}\Im(y_{\iota+1})\right\| \leq \frac{-\hbar(b_{\iota+1})}{\hbar'(a_{\iota+1})} = c_{\iota+1}-b_{\iota+1}. \tag{3.19}$$

This establishes the validity of (iv) for the case $\iota+1$. Based on these two outcomes, we can deduce that for (vi) we obtain: $\|z_{\iota+1}-x_{\iota+1}\| \le \|z_{\iota+1}-y_{\iota+1}+y_{\iota+1}-x_{\iota+1}\| \le c_{\iota+1}-b_{\iota+1}+b_{\iota+1}-a_{\iota+1} \le c_{\iota+1}-a_{\iota+1}$.

Likewise, for the subsequent relationship, it becomes apparent from the scheme (1.3) that

$$q_{\iota+1}-z_{\iota+1} = -[\Im'(x_{\iota+1})]^{-1}\int_0^1 \Big[\Im'(y_{\iota+1}+\tau(z_{\iota+1}-y_{\iota+1}))-\Im'(x_{\iota+1})\Big](z_{\iota+1}-y_{\iota+1})\,d\tau.$$

By utilizing the χ-average Lipschitz condition (2.1)

$$\begin{aligned}
&\|q_{\iota+1}-z_{\iota+1}\| \\
\le& \Big\|[\Im'(x_{\iota+1})]^{-1}\Im'(x_0)\Big\| \\
&\times \int_0^1 \Big\|[\Im'(x_0)]^{-1}[\Im'(y_{\iota+1}+\tau(z_{\iota+1}-y_{\iota+1}))-\Im'(x_{\iota+1})]\Big\|\cdot\|z_{\iota+1}-y_{\iota+1}\|\,d\tau \\
\le& \Big\|[\Im'(x_{\iota+1})]^{-1}\Im'(x_0)\Big\| \\
&\times \int_0^1 \left(\int_{\|x_{\iota+1}-x_0\|}^{\|x_{\iota+1}-x_0\|+\|y_{\iota+1}-x_{\iota+1}\|+\tau\|z_{\iota+1}-y_{\iota+1}\|} \chi(h)\,du\right)\|z_{\iota+1}-y_{\iota+1}\|\,d\tau \\
=& -\frac{1}{\hbar'(a_{\iota+1})}\int_0^1 \Big[\hbar'(\|x_{\iota+1}-x_0\|+\|y_{\iota+1}-x_{\iota+1}\|+\tau\|z_{\iota+1}-y_{\iota+1}\|)-\hbar'(\|x_{\iota+1}-x_0\|)\Big] \\
&\times \|z_{\iota+1}-y_{\iota+1}\|\,d\tau.
\end{aligned}$$

Since $\hbar'$ is strictly convex in $[0,\wp_0]$ and considering the fact that $\|z_{\iota+1}-x_{\iota+1}\| \le c_{\iota+1}-a_{\iota+1}$, we can derive from Lemma 3.1 that

$$\begin{aligned}
\|q_{\iota+1}-z_{\iota+1}\| \le& -\frac{1}{\hbar'(a_{\iota+1})}\left(\int_0^1 \frac{\hbar'(b_{\iota+1}+\tau(c_{\iota+1}-b_{\iota+1}))-\hbar'(a_{\iota+1})}{(b_{\iota+1}-a_{\iota+1})+\tau(c_{\iota+1}-b_{\iota+1})}d\tau\right. \\
&\times\|z_{\iota+1}-y_{\iota+1}\|\left(\|y_{\iota+1}-x_{\iota+1}\|+\tau\|z_{\iota+1}-y_{\iota+1}\|\right)\Big) \\
\le& \frac{-\hbar(c_{\iota+1})}{\hbar(b_{\iota+1})}\left[\frac{\|z_{\iota+1}-y_{\iota+1}\|}{c_{\iota+1}-b_{\iota+1}}\cdot\frac{\|y_{\iota+1}-x_{\iota+1}\|+\tau\|z_{\iota+1}-y_{\iota+1}\|}{b_{\iota+1}-a_{\iota+1}+\tau(c_{\iota+1}-b_{\iota+1})}\right] \\
\le& (d_{\iota+1}-c_{\iota+1})\left[\frac{\|z_{\iota+1}-y_{\iota+1}\|}{c_{\iota+1}-b_{\iota+1}}\cdot\frac{\|y_{\iota+1}-x_{\iota+1}\|+\tau\|z_{\iota+1}-y_{\iota+1}\|}{b_{\iota+1}-a_{\iota+1}+\tau(c_{\iota+1}-b_{\iota+1})}\right] \qquad (3.20)\\
\le& d_{\iota+1}-c_{\iota+1}.
\end{aligned}$$

This establishes the validity of (v) for the case $\iota+1$. Likewise, by examining the scheme (1.3), we observe that for the next relation

$$x_{\iota+2}-q_{\iota+1} = -[\Im'(x_{\iota+1})]^{-1}\int_0^1 \Big[\Im'(z_{\iota+1}+\tau(q_{\iota+1}-z_{\iota+1}))-\Im'(x_{\iota+1})\Big](q_{\iota+1}-z_{\iota+1})d\tau.$$

By utilizing the condition of χ-average Lipschitz (2.1), we have

$$\begin{aligned}
&\|x_{\iota+2}-q_{\iota+1}\| \\
\leq &\left\|[\Im'(x_{\iota+1})]^{-1}\Im'(x_0)\right\| \\
&\times \int_0^1 \left\|[\Im'(x_0)]^{-1}\Big[\Im'(z_{\iota+1}+\tau(q_{\iota+1}-z_{\iota+1}))-\Im'(x_{\iota+1})\Big](q_{\iota+1}-z_{\iota+1})\right\| d\tau \\
\leq &\left\|[\Im'(x_{\iota+1})]^{-1}\Im'(x_0)\right\| \\
&\times \int_0^1 \left(\int_{\|x_{\iota+1}-x_0\|}^{\|x_{\iota+1}-x_0\|+\|z_{\iota+1}-x_{\iota+1}\|+\tau\|q_{\iota+1}-z_{\iota+1}\|} \chi(h)\,du\right)\|q_{\iota+1}-z_{\iota+1}\|d\tau \\
= &\frac{-1}{\hbar'(a_{\iota+1})}\int_0^1 \Big[\hbar'(\|x_{\iota+1}-x_0\|+\|z_{\iota+1}-x_{\iota+1}\|+\tau\|q_{\iota+1}-z_{\iota+1}\|)-\hbar'(\|x_{\iota+1}-x_0\|)\Big] \\
&\times \|q_{\iota+1}-z_{\iota+1}\|d\tau.
\end{aligned}$$

Assuming that $\hbar'$ is strictly convex in $[0,\wp_0]$, one can deduce from Lemma 3.1

$$\begin{aligned}
\|x_{\iota+2}-q_{\iota+1}\| &\leq -\frac{1}{\hbar'(a_{\iota+1})}\left(\int_0^1 \frac{\hbar'(c_{\iota+1}+\tau(d_{\iota+1}-c_{\iota+1}))-\hbar'(a_{\iota+1})}{(c_{\iota+1}-a_{\iota+1})+\tau(d_{\iota+1}-c_{\iota+1})}\,d\tau\right. \\
&\qquad \left.\times \|q_{\iota+1}-z_{\iota+1}\|\Big(\|z_{\iota+1}-x_{\iota+1}\|+\tau\|q_{\iota+1}-z_{\iota+1}\|\Big)\right) \\
&\leq \frac{-\hbar(d_{\iota+1})}{\hbar(a_{\iota+1})}\left[\frac{\|q_{\iota+1}-z_{\iota+1}\|}{d_{\iota+1}-c_{\iota+1}}\cdot\frac{\|z_{\iota+1}-x_{\iota+1}+\tau\|q_{\iota+1}-z_{\iota+1}\|\|\|}{c_{\iota+1}-a_{\iota+1}+\tau(d_{\iota+1}-c_{\iota+1})}\right] \\
&\leq (a_{\iota+2}-d_{\iota+1})\left[\frac{\|q_{\iota+1}-z_{\iota+1}\|}{d_{\iota+1}-c_{\iota+1}}\cdot\frac{\|z_{\iota+1}-x_{\iota+1}+\tau\|q_{\iota+1}-z_{\iota+1}\|\|\|}{c_{\iota+1}-a_{\iota+1}+\tau(d_{\iota+1}-c_{\iota+1})}\right] \\
&\leq (a_{\iota+2}-d_{\iota+1}).
\end{aligned}$$

This establishes $(viii)$ for the case $\iota+1$. Moreover, we obtain from equations (3.16) and (3.19) that

$$\begin{aligned}
\|x_{\iota+2}-x_{\iota+1}\| &\leq \|x_{\iota+1}-q_{\iota+1}\|+\|q_{\iota+1}-z_{\iota+1}\|+\|z_{\iota+1}-y_{\iota+1}\|+\|y_{\iota+1}-x_{\iota+1}\| \\
&\leq (a_{\iota+2}-d_{\iota+1})+(d_{\iota+1}-c_{\iota+1})+(c_{\iota+1}-b_{\iota+1})+(b_{\iota+1}-a_{\iota+1}) \\
&= a_{\iota+2}-a_{\iota+1}.
\end{aligned}$$

Consequently, through the process of induction, all assertions in the lemma have been verified. Thus, the scientific proof is now concluded. □

4. Main Result: Semilocal Convergence of fsns (1.3)

We are now ready to establish the significant properties of the fsns (1.3) under the χ-average Lipschitz condition (2.1) in the given ball set. Our main result encompasses the convergence, uniqueness, and convergence rate of the fsns, thereby demonstrating its essential implications. Notably, this result encompasses two special cases: the Υ convergence result and the convergence result of Kantorovich type under the Lipschitz condition.

The subsequent lemmas will be instrumental in achieving this objective.

Lemma 4.1. *Under the same assumptions as in Lemma* 3.4*, it follows that the sequence* x_ι *converges to the point* $\Upsilon^* \in \overline{O(x_0, a^*)}$ *such that* $\Im(\Upsilon^*) = 0$*. Moreover, we obtain the following result:*

$$\|\Upsilon^* - x_\iota\| \leq a^* - a_\iota, \quad \iota \geq 0, \tag{4.1}$$

$$\|\Upsilon^* - y_\iota\| \leq (a^* - b_\iota)\left(\frac{\|\Upsilon^* - x_\iota\|}{a^* - a_\iota}\right)^2, \quad \iota \geq 0, \tag{4.2}$$

$$\|\Upsilon^* - z_\iota\| \leq (a^* - c_\iota)\left(\frac{\|\Upsilon^* - x_\iota\|}{a^* - a_\iota} \cdot \frac{\|\Upsilon^* - y_\iota\|}{a^* - b_\iota}\right), \quad \iota \geq 0, \tag{4.3}$$

and

$$\|\Upsilon^* - q_\iota\| \leq (a^* - d_\iota)\left(\frac{\|\Upsilon^* - x_\iota\|}{a^* - a_\iota} \cdot \frac{\|\Upsilon^* - z_\iota\|}{a^* - c_\iota}\right), \quad \iota \geq 0. \tag{4.4}$$

Proof. To establish the claim, we employ Lemma 3.4 (*vii*) and relation (3.10) in order to demonstrate that:

$$\sum_{\iota=N}^{+\infty} \|x_{\iota+1} - x_\iota\| \leq \sum_{\iota=N}^{+\infty} (a_{\iota+1} - a_\iota) = a^* - a_N < +\infty, \quad \text{for any } N \in \mathbb{N}.$$

Consequently, the sequence x_ι is a Cauchy sequence within the ball $O(x_0, a^*)$, indicating that it converges to a point $\Upsilon^* \in \overline{O(x_0, a^*)}$. For each $\iota \geq 0$, the previous inequality implies that $\|\Upsilon^* - x_\iota\| \leq a^* - a_\iota$. Furthermore, we will now demonstrate that $\Im(\Upsilon^*) = 0$. Utilizing Lemma 3.3, we can ascertain that $\Im(x_\iota)$ is bounded. Then, by referencing proven Lemma 3.4, we can deduce that:

$$\|\Im(x_\iota)\| \leq \|\Im'(x_\iota)\| \cdot \left\|[\Im'(x_\iota)]^{-1}\Im(x_\iota)\right\| \leq \|\Im'(x_\iota)\|(b_\iota - a_\iota).$$

By allowing ι to approach infinity and considering that b_ι and a_ι converge to the same point a^* (as mentioned in Note 1), we can conclude that $\lim_{\iota\to+\infty} \Im(x_\iota) = 0$. Since $\Im$ is continuous in $\overline{O(x_0, a^*)}$ and x_ι is a sequence in $O(x_0, a^*)$ converging to Υ^*, we have $\lim_{\iota\to+\infty} \Im(x_\iota) = \Im(\Upsilon^*)$, which confirms that $\Im(\Upsilon^*) = 0$. It remains to establish the estimates (4.2) and (4.3). Utilizing Lemma 3.4, we obtain

$$\|y_\iota - x_0\| \leq \|y_\iota - x_\iota\| + \|x_\iota - x_0\| \leq b_\iota.$$

However, we can derive the following identity:

$$\Upsilon^* - y_\iota = -[\Im'(x_\iota)]^{-1} \int_0^1 \Big[\Im'(x_\iota + \tau(\Upsilon^* - x_\iota)) - \Im'(x_\iota)\Big](\Upsilon^* - x_\iota)d\tau.$$

Next, by combining relation (3.14), the χ-average Lipschitz condition (2.1), and Lemma 3.1, considering the convexity and increasing property of $\hbar'$ in $[0, \wp_0)$, we can deduce the following:

$$\begin{aligned}
\|\Upsilon^* - y_\iota\| &\leq -\frac{1}{\hbar'(a_\iota)} \int_0^1 \left(\int_{\|x_\iota - x_0\|}^{\|x_\iota - x_0\| + \|\tau(\Upsilon^* - x_\iota)\|} \chi(h)\, du \right) \|\Upsilon^* - x_\iota\|\, d\tau \\
&\leq -\frac{1}{\hbar'(a_\iota)} \int_0^1 \Big[\hbar'(\|x_\iota - x_0\| + \tau \|\Upsilon^* - x_\iota\|) - \hbar'(\|x_\iota - x_0\|) \Big] \cdot \|\Upsilon^* - x_\iota\|\, d\tau \\
&\leq -\frac{1}{\hbar'(a_\iota)} \int_0^1 \frac{\hbar'(a_\iota + \tau(a^* - a_\iota)) - \hbar'(a_\iota)}{a^* - a_\iota}\, d\tau\, \|\Upsilon^* - x_\iota\|^2 \\
&\leq (a^* - b_\iota)\, \frac{\|\Upsilon^* - x_\iota\|^2}{(a^* - a_\iota)^2}.
\end{aligned}$$

We still need to establish the estimates (4.2) and (4.3). By utilizing Lemma 3.4, we can infer the following:

$$\|z_\iota - x_0\| \leq \|z_\iota - y_\iota\| + \|y_\iota - x_0\| \leq c_\iota. \tag{4.5}$$

Furthermore, we can derive the fundamental identity:

$$\Upsilon^* - z_\iota = -[\Im'(x_\iota)]^{-1} \int_0^1 \Big[\Im'(y_\iota + \tau(\Upsilon^* - y_\iota)) - \Im'(x_\iota) \Big] (\Upsilon^* - y_\iota) d\tau.$$

Subsequently, under the assumption that $\hbar'$ is convex and increasing in the interval $[0, \wp_0)$, we can combine relation (3.14), Lemma 3.1, and the χ-average Lipschitz condition (2.1) to obtain the following:

$$\begin{aligned}
&|\Upsilon^* - z_\iota\| \\
\leq &-\frac{1}{\hbar'(a_\iota)} \int_0^1 \left(\int_{\|x_\iota - x_0\|}^{\|x_\iota - x_0\| + \|y_\iota - x_\iota + \tau(\Upsilon^* - y_\iota)\|} \chi(h) du \right) \|\Upsilon^* - y_\iota\| d\tau \\
= &-\frac{1}{\hbar'(a_\iota)} \int_0^1 \Big[\hbar'(\|x_\iota - x_0\| + \|y_\iota - x_\iota + \tau(\Upsilon^* - y_\iota)\|) - \hbar'(\|x_\iota - x_0\|) \Big] \|\Upsilon^* - y_\iota\| d\tau \\
\leq &-\frac{1}{\hbar'(a_\iota)} \int_0^1 \frac{\hbar'(b_\iota + \tau(a^* - b_\iota)) - \hbar'(a_\iota)}{a^* - a_\iota} \|\Upsilon^* - x_\iota\| d\tau\, \|\Upsilon^* - y_\iota\| \\
\leq &(a^* - c_\iota)\, \frac{\|\Upsilon^* - x_\iota\| \cdot \|\Upsilon^* - y_\iota\|}{(a^* - a_\iota)(a^* - b_\iota)}.
\end{aligned}$$

Consequently, by applying Lemma 3.4, we can infer that

$$\|q_\iota - x_0\| \leq \|q_\iota - z_\iota\| + \|z_\iota - x_0\| \leq d_\iota. \tag{4.6}$$

Furthermore, we can establish the following fundamental identity:

$$\Upsilon^* - q_\iota = -[\Im'(x_\iota)]^{-1} \int_0^1 \Big[\Im'(z_\iota + \tau(\Upsilon^* - z_\iota)) - \Im'(x_\iota) \Big] (\Upsilon^* - z_\iota) d\tau.$$

If we assume that $\hbar'$ is both convex and increasing in the interval $[0, \wp_0)$, we can combine equation (3.14), Lemma 3.1, and the χ-average Lipschitz condition (2.1) to obtain the following:

$$\begin{aligned}
\|\Upsilon^* - q_\iota\| &\leq -\frac{1}{\hbar'(a_\iota)} \int_0^1 \left(\int_{\|x_\iota - x_0\|}^{\|x_\iota - x_0\| + \|z_\iota - x_\iota + \tau(\Upsilon^* - z_\iota)\|} \chi(h) du \right) \|\Upsilon^* - z_\iota\| d\tau \\
&= -\frac{1}{\hbar'(a_\iota)} \int_0^1 \Big(\hbar'(\|x_\iota - x_0\| + \|z_\iota - x_\iota + \tau(\Upsilon^* - z_\iota)\|) - \hbar'(\|x_\iota - x_0\|) \Big) \\
&\quad \times \|\Upsilon^* - z_\iota\| d\tau \\
&\leq \frac{-1}{\hbar'(a_\iota)} \int_0^1 \frac{\hbar'(c_\iota + \tau(a^* - c_\iota)) - \hbar'(a_\iota)}{a^* - a_\iota} \|\Upsilon^* - x_\iota\| d\tau \|\Upsilon^* - y_\iota\| \\
&\leq (a^* - d_\iota) \frac{\|\Upsilon^* - x_\iota\| \cdot \|\Upsilon^* - z_\iota\|}{(a^* - a_\iota)(a^* - c_\iota)}.
\end{aligned}$$

The proof of this lemma is fully completed and presented as stated. □

Lemma 4.2. *Given the standard conditions outlined in the aforementioned Lemma* 3.4 *and assuming the hypothesis* $1 + a^* \hbar''(a^*)/\hbar'(a^*) > 0$, *we observe that*

$$\frac{q_\iota - z_\iota}{d_\iota - c_\iota} \leq \frac{1 - \frac{\hbar''(a^*)}{\hbar'(a^*)}(a^* - a_\iota)}{1 + \frac{\hbar''(a^*)}{\hbar'(a^*)}(a^* - a_\iota)} \cdot \frac{\|\Upsilon^* - x_\iota\|^3}{\|a^* - a_\iota\|^3}. \tag{4.7}$$

Proof. From the scheme (3.9), we can deduce that

$$a^* - d_\iota = -\frac{1}{\hbar'(a_\iota)} \int_0^1 \Big[\hbar'(c_\iota + \tau(a^* - c_\iota)) - \hbar'(a_\iota) \Big] (a^* - c_\iota)\, d\tau.$$

Given the convexity of $\hbar$ in $[0, \wp_0)$, Lemma 3.1 asserts that for any $\tau \in (0, 1]$,

$$\hbar'(c_\iota + \tau(a^* - c_\iota) - \hbar'(a_\iota) \leq \frac{\hbar'(a^*) - \hbar'(a_\iota)}{a^* - a_\iota}(c_\iota + \tau(a^* - c_\iota)) \leq \hbar''(a^*)(a^* - a_\iota).$$

Thus, considering the positive nature of $1/\hbar'(a)$, one can conclude from the aforementioned Lemma 3.1 that

$$a^* - d_\iota \leq \frac{-1}{\hbar'(a_\iota)} \int_0^1 \frac{\hbar'(a^*) - \hbar'(a_\iota)}{a^* - a_\iota}(a^* - a_\iota)(a^* - c_\iota) d\tau = \frac{-\hbar''(a^*)}{\hbar'(a^*)}(a^* - a_\iota)(a^* - c_\iota). \tag{4.8}$$

The final conclusion follows from the strict monotonicity of $\hbar'$. By utilizing this concept, we can derive from the scheme (3.9) that

$$a^* - c_\iota \leq -\frac{1}{2} \cdot \frac{\hbar''(a^*)}{\hbar'(a^*)}(a^* - a_\iota)^3. \tag{4.9}$$

By applying the inequality $\|q_\iota - z_\iota\| \leq \|\Upsilon^* - q_\iota\| + \|\Upsilon^* - z_\iota\|$, we can infer from equations (4.2) and (4.3) that

$$\|q_\iota - z_\iota\| \leq (a^* - d_\iota) \left(\frac{\|\Upsilon^* - x_\iota\|}{a^* - a_\iota} . \frac{\|\Upsilon^* - z_\iota\|}{a^* - c_\iota} \right) + (a^* - c_\iota) \left(\frac{\|\Upsilon^* - x_\iota\|}{a^* - a_\iota} \right)^2 \frac{\|\Upsilon^* - x_\iota\|}{a^* - a_\iota}.$$

Finally, by utilizing equation (4.8), we can further derive that

$$\|q_\iota - z_\iota\| \leq \left(1 - \frac{\hbar''(a^*)}{\hbar'(a^*)}(a^* - a_\iota)\right)\frac{-1}{2} \cdot \frac{\hbar''(a^*)}{\hbar'(a^*)}\|\Upsilon^* - x_\iota\|^3.$$

As a consequence of equation (3.12), we can infer that

$$\begin{aligned}\frac{\|q_\iota - z_\iota\|}{d_\iota - c_\iota} &\leq \frac{\left(1 - \frac{\hbar''(a^*)}{\hbar'(a^*)}(a^* - a_\iota)\right)\frac{-1}{2} \cdot \frac{\hbar''(a^*)}{\hbar'(a^*)}}{c_\iota - b_\iota}\|\Upsilon^* - x_\iota\|^3 \\ &\leq \frac{\left(1 - \frac{\hbar''(a^*)}{\hbar'(a^*)}(a^* - a_\iota)\right)\frac{-1}{2} \cdot \frac{\hbar''(a^*)}{\hbar'(a^*)}}{\left(1 + \frac{\hbar''(a^*)}{\hbar'(a^*)}(a^* - a_\iota)\right)\frac{-1}{2} \cdot \frac{\hbar''(a^*)}{\hbar'(a^*)}} \cdot \frac{\|\Upsilon^* - x_\iota\|^3}{(a^* - a_\iota)^3}.\end{aligned}$$

□

Therefore, utilizing Lemmas 4.1 and 4.2, we can conclude that

Theorem 4.1. *Let us consider a nonlinear operator* $\Im : \mathbb{D} \subset \bar{\bar{U}} \to \bar{\bar{V}}$ *that is continuously Fréchet differentiable and defined in an open and convex subset* $\mathbb{D}$*. We assume the existence of an i.g.* $x_0 \in \mathbb{D}$ *for which* $[\Im(x_0)]^{-1}$ *exists, and that* $\Im$ *satisfies the* χ*-average Lipschitz condition* (2.1) *within the ball* $O(x_0, a^*)$*. We now consider the sequence of iterates* x_ι *obtained by applying the fsns* (1.3) *with the initial value* x_0*. If the constants satisfy* $0 < \flat \leq \natural$*, then the sequence* x_ι *obtained by applying the fsns* (1.3) *with the initial guess* x_0 *is well-defined and converges towards a unique solution* Υ^* *of* (1.1) *within the ball* $\overline{O(x_0, \wp)}$*. The convergence rate is guaranteed to be at least five. Here,* $\wp$ *is defined as* $\wp := \sup a \in (a^*, \eth) : \hbar(a) \leq 0$*. The solution* Υ^* *is also guaranteed to be unique within the larger ball* $\overline{O(x_0, \wp)}$*, where* $a^* \leq \wp < a^{**}$*. Moreover, if*

$$1 + \frac{a^*\hbar''(a^*)}{\hbar'(a^*)} > 0 \Leftrightarrow 1 - \frac{a^*\chi(a^*)}{1 - \int_0^{a^*}\chi(h)du} > 0. \tag{4.10}$$

If the condition is satisfied, it can be expected to achieve a convergence rate of at least fifth order, and the following error bounds can be derived:

$$\|\Upsilon^* - x_{\iota+1}\| \leq \frac{1}{2}\hbar_*^4\frac{1 - a^*\hbar_*}{1 + a^*\hbar_*}\|\Upsilon^* - x_\iota\|^5, \quad \iota \geq 0, \tag{4.11}$$

in which $\hbar_* \triangleq \hbar''(a^*)/\hbar'(a^*)$*.*

Proof. By utilizing Lemma 3.4, we can establish that the sequence x_ι is well-defined. Based on the provided Lemma 3.4 (vii) and (3.10), it can be deduced that $\|x_\iota - x_0\| \leq a_\iota < a^*$ for $\iota \geq 0$, ensuring that x_ι remains within the ball $O(x_0, a^*)$. Additionally, Lemma 4.1 guarantees that the sequence x_ι converges towards a solution Υ^* of (1.1) within the ball $O(x_0, a^*)$. Next, we will establish the fifth-order convergence of the iterative process. To accomplish this, we employ conventional analytical methods and obtain the following result:

$$\begin{aligned}&\Upsilon^* - x_{\iota+1}\\ =&\Upsilon^* - q_\iota + [\Im'(x_\iota)]^{-1}\Im(q_\iota)\\ =&-[\Im'(x_\iota)]^{-1}\left[\int_0^1 (\Im'(x_\iota^\tau) - \Im'(q_\iota))(\Upsilon^* - q_\iota)d\tau + (\Im'(q_\iota) - \Im'(x_\iota))(\Upsilon^* - q_\iota)\right],\end{aligned}$$

in which $x_\iota^\tau := x_\iota + \tau(\Upsilon^* - x_\iota)$. By employing the χ-average Lipschitz condition (2.1) and equation (3.14), we can deduce that:

$$\begin{aligned}\|\Upsilon^* - x_{\iota+1}\| \leq &-\frac{1}{\hbar'(a_\iota)}\left[\int_0^1\left(\int_{\|q_\iota - x_0\|}^{\|q_\iota - x_0\|+\|q_\iota - q_\iota + \tau(a^* - q_\iota)\|}\chi(h)du\right)\|a^* - q_\iota\|\, d\tau\right.\\ &\left.+\int_{\|x_\iota - x_0\|}^{\|x_\iota - x_0\|+\|q_\iota - x_\iota\|}\chi(h)du\|a^* - q_\iota\|\right].\end{aligned}$$

Subsequently, if $\hbar'$ is both convex and increasing in $[0, \wp_0)$, we can utilize relations (4.1), (4.6), Lemma 3.1, Lemma 3.4, and the χ-average Lipschitz condition (2.1) to combine the following:

$$\begin{aligned}\|\Upsilon^* - x_{\iota+1}\| \leq &-\frac{1}{\hbar'(a_\iota)}\left[\int_0^1 \frac{\hbar'(d_\iota + \tau(a^* - d_\iota)) - \hbar'(d_\iota)}{a^* - d_\iota}\|\Upsilon^* - q_\iota\|^2\, d\tau\right.\\ &\left.+\frac{\hbar'(d_\iota) - \hbar'(a_\iota)}{d_\iota - a_\iota}\|\Upsilon^* - q_\iota\| \cdot \|q_\iota - x_\iota\|\right]\\ =&-\frac{1}{\hbar'(a_\iota)}\left[\Big(\hbar(a^*) - \hbar(d_\iota) - \hbar'(d_\iota)(a^* - d_\iota)\Big)\frac{\|\Upsilon^* - q_\iota\|^2}{(a^* - d_\iota)^2}\right]\\ &+(\hbar'(d_\iota) - \hbar'(a_\iota))(a^* - d_\iota)\frac{\|\Upsilon^* - q_\iota\|}{(a^* - d_\iota)} \cdot \frac{\|q_\iota - x_\iota\|}{d_\iota - a_\iota}.\end{aligned}$$

By applying Lemma 3.4 along with relations (4.1) and (4.6) again, we can obtain an additional inequality from the aforementioned expression, which can be expressed as follows:

$$\|\Upsilon^* - x_{\iota+1}\| \leq (a^* - a_{\iota+1})\left[\frac{\|\Upsilon^* - x_\iota\|}{a^* - a_\iota}\right]^3. \tag{4.12}$$

Subsequently, we can infer from equation (3.11) that

$$\frac{\|\Upsilon^* - x_{\iota+1}\|}{\|\Upsilon^* - x_\iota\|^4} \leq \frac{a^* - a_{\iota+1}}{(a^* - a_\iota)^4} \leq \frac{1}{2} \cdot \frac{\hbar''(a^*)^4}{\hbar'(a^*)^4}(a^* - a_\iota)^{\cdot}$$

By considering the limit as ι tends to infinity in the previous inequalities and taking into account the convergence of a_ι to a^*, we obtain

$$\lim_{\iota \to +\infty} \frac{\|\Upsilon^* - x_{\iota+1}\|}{\|\Upsilon^* - x_\iota\|^4} = 0.$$

Furthermore, if condition (4.10) is also fulfilled, we can employ estimates (4.1), (4.6), (4.7), and (3.11) to deduce the following from (4.11):

$$\begin{aligned}\|\Upsilon^* - x_{\iota+1}\| &\leq -\frac{1}{\hbar'(a_\iota)}\Big[(\hbar(a^*) - \hbar(d_\iota) - \hbar'(d_\iota)(a^* - d_\iota))d\tau \\ &\quad + (\hbar'(d_\iota) - \hbar'(a_\iota))(a^* - d_\iota)\Big]\frac{1 - \frac{\hbar''(a^*)}{\hbar'(a^*)}(a^* - a_\iota)}{1 + a^*\frac{\hbar''(a^*)}{\hbar'(a^*)}(a^* - a_\iota)}\left(\frac{\|\Upsilon^* - z_\iota\|}{(a^* - a_\iota)}\right)^5 \\ &= (a^* - a_{\iota+1})\cdot\frac{1 - \frac{\hbar''(a^*)}{\hbar'(a^*)}(a^* - a_\iota)}{1 + a^*\frac{\hbar''(a^*)}{\hbar'(a^*)}(a^* - a_\iota)}\left(\frac{\|\Upsilon^* - z_\iota\|}{(a^* - a_\iota)}\right)^5 \\ &\leq \frac{1}{2}\left(\frac{\hbar''(a^*)}{\hbar'(a^*)}\right)^4\frac{1 - \frac{\Upsilon^*\hbar''(a^*)}{\hbar'(a^*)}}{1 + \frac{\Upsilon^*\hbar''(a^*)}{\hbar'(a^*)}}\|\Upsilon^* - x_\iota\|^5.\end{aligned}$$

Consequently, we have successfully demonstrated the estimate (4.11) as stated in Theorem 4.1, which establishes the five-fold convergence rate of the iterates.

Ultimately, we establish the uniqueness of the solution. We begin by proving the uniqueness of the solution Υ^* for (1.1) in the domain $\overline{O(x_0, a^*)}$. Let's assume the existence of an alternative solution a^{**} within $\overline{O(x_0, a^*)}$. This implies that $\|a^{**} - x_0\| \leq a^*$. Next, we will demonstrate this through an inductive argument.

$$\|a^{**} - x_\iota\| \leq a^* - a_\iota, \quad \iota = 0, 1, 2, \ldots, \tag{4.13}$$

since a_0 is equal to 0, the case where $\iota = 0$ is clearly true. Assuming that the above expression holds for a particular $\iota \geq 0$, we can apply the same procedure to estimate $\|a^{**} - y_\iota\|$ in (4.2) and $\|a^* - z_\iota\|$ in (4.3), which yields the following:

$$\|a^* - y_\iota\| \leq (a^* - b_\iota)\left(\frac{\|a^* - x_\iota\|}{a^* - a_\iota}\right)^2, \quad \iota \geq 0,$$

with

$$\|a^* - z_\iota\| \leq (a^* - c_\iota)\left(\frac{\|a^* - x_\iota\|}{(a^* - a_\iota)}\cdot\frac{\|a^* - y_\iota\|}{(a^* - b_\iota)}\right), \quad \iota \geq 0.$$

Furthermore, by applying the same method to estimate $\|a^* - z_\iota\|$ in (4.12), we can demonstrate that

$$\|a^* - x_{\iota+1}\| \leq (a^* - a_{\iota+1})\left[\frac{\|a^* - x_\iota\|}{a^* - a_\iota}\right]^3.$$

Following that, by employing the inductive hypothesis (4.13) on the aforementioned inequality, we can conclude that (4.13) is applicable to the scenario $\iota + 1$. Since x_ι converges to Υ^* and a_ι converges to a^*, it can be inferred from (4.13) that $a^{**} = \Upsilon^*$. As a result, Υ^* stands as the sole root of (1.1) within the region $\overline{O(x_0, a^*)}$. However, it is still crucial to establish that the nonlinear operator $\Im$ does not possess any roots within the region $O(x_0, \wp)\backslash\overline{O(x_0, a^*)}$. Let us assume the contrary, supposing that $\Im$ has one or more roots in that particular region. This assumption suggests the existence of $a^{**} \in \mathbb{D} \subset X$, where $a^* < a^{**} - x_0 < \wp$ and $\Im(a^{**}) = 0$. Our objective is to demonstrate the falsehood of the aforementioned assumptions. As we are aware,

$$\Im(a^{**}) = \Im(x_0) + \Im'(x_0)(a^{**} - x_0) + \int_0^1 [\Im'(x_0^\tau) - \Im'(x_0)](a^{**} - x_0) d\tau,$$

where $x_0^\tau := x_0 + \tau(a^{**} - x_0)$. One can see that

$$\begin{aligned}\left\|[\Im'(x_0)]^{-1}[\Im(x_0) + \Im'(x_0)(a^{**} - x_0)]\right\| \geq & \|a^{**} - x_0\| - \|\Im'(x_0)]^{-1}\Im(x_0)\| \\ = & \|a^{**} - x_0\| - \hbar(0).\end{aligned}$$

Moreover, we employ the χ-average Lipschitz condition (2.1) to derive

$$\begin{aligned}&\left\|[\Im'(x_0)]^{-1} \int_0^1 [\Im'(x_0^\tau) - \Im'(x_0)](a^{**} - x_0) d\tau\right\| \\ \leq & \int_0^1 \left(\int_0^{\tau(\|a^{**}-x_0\|)} \chi(h) du\right) \|a^{**} - x_0\| d\tau \\ = & \int_0^1 \left[\hbar'(\tau\|a^{**} - x_0\|) - \hbar'(0)\right] \cdot \|a^{**} - x_0\| \\ = & \hbar(\|a^{**} - x_0\|) - \hbar(0) - \hbar'(0)\|a^{**} - x_0\|.\end{aligned}$$

Considering that $\Im(a^{**}) = 0$ and $\hbar'(0) = -1$, we can deduce from (4.14) that

$$\hbar(\|a^{**} - x_0\|) - \hbar(0) - \hbar'(0)\|a^{**} - x_0\| \geq \|a^{**} - x_0\| - \hbar(0).$$

The inequality $\hbar(\|a^{**} - x_0\|) \geq 0$ can be understood as $\hbar$ being strictly positive within the range of $(\|a^{**} - x_0\|, \mathbb{R})$, as stated in Lemma 3.2. Consequently, we can conclude that $\wp < |a^{**} - x_0|$, which contradicts the initial assumptions. Hence, it follows that $\Im$ does not have any roots within $O(x_0, \wp) \backslash \overline{O(x_0, a^*)}$, and as a result, a^* is the only root of (1.1) in $O(x_0, \wp)$. This completes the proof. ☐

Note 2. The convergence criteria $0 < \flat \leq \natural$ as presented in Theorem 4.1 were initially established by Wang [16] to analyze the convergence of the NM (1.2) within a unified framework. Additionally, to attain fifth-order convergence, it is essential to satisfy condition (4.10). The results can be extended in the following manner.

Consider $\epsilon := \sup t \geq 0 : O(x_0, t) \subset \mathfrak{D}$.

Criteria 2. The operator $\Im'$ satisfies the center χ_0-average Lipschitz criterion on the ball $O(x_0, \epsilon)$ if for every $x \in O(x_0, \epsilon)$, the condition stated below holds true:

$$\left\|[\Im'(x_0)]^{-1}(\Im'(x) - \Im'(x_0))\right\| \leq \int_0^{\|x - x_0\|} \chi_0(h) du. \tag{4.14}$$

Consider a n.d. continuous and non-negative function defined on the interval $[0, \epsilon]$. Let us suppose that the equation $\int_0^\epsilon \chi_0(h) du - 1 = 0$ has a smallest positive solution $\epsilon_0 \in (0, \epsilon]$. We define the ball $O(x_0, \epsilon_0)$. By these definitions, we can observe that:

$$O(x_0, \epsilon_0) \subset O(x_0, \epsilon),$$

and as a result,

$$\chi_0(h) \leq \chi(h), \quad \text{for each } h \in [0, \epsilon_0].$$

Furthermore, the linear operator $\Im'(x)$ has an inverse for $x \in O(x_0, \epsilon_0)$, and

$$\left\|[\Im'(x)]^{-1}\Im'(x_0)\right\| \leq \frac{1}{1-\int_0^{\epsilon_0}\chi_0(h)du}.$$

This estimation is more accurate than

$$\left\|[\Im'(x)]^{-1}\Im'(x_0)\right\| \leq \frac{1}{1-\int_0^{\epsilon}\chi(h)du}$$

used in the previous sections. The restricted $\overline{\chi}$-average Lipschitz criterion is formulated as follows.

Criteria 3. The operator $\Im'$ satisfies the restricted $\overline{\chi}$-average Lipschitz criterion on the ball $O(x_0, \epsilon_0)$ if, for any x and y satisfying $\|y-x\| + \|x-x_0\| \leq \epsilon_0$, the following condition is satisfied:

$$\left\|[\Im'(x_0)]^{-1}(\Im'(y) - \Im'(x))\right\| \leq \int_0^{\|x-x_0\|} \overline{\chi}(h)du.$$

By considering $\overline{\chi}$ as a continuous, n.d., and non-negative function defined on the interval $[0, \epsilon_0]$, we can conclude from these definitions that:

$$\overline{\chi}(h) \leq \chi(h), \quad \text{for each } h \in [0, \epsilon_0].$$

Consequently, the more stringent function $\overline{\chi}$ can replace χ in all the previous results, resulting in convergence criteria that are less restrictive and error estimates $\|x_{\iota+1} - x_\iota\|$ and $\|\Upsilon^* - x_\iota\|$ that are at least as accurate. It should be noted that $\chi_0 = \chi_0(O(x_0, \epsilon))$, $\chi = \chi(O(x_0, \epsilon))$, while $\overline{\chi} = \overline{\chi}(O(x_0, \epsilon))$. Furthermore, it is worth highlighting that the functions χ_0 and $\overline{\chi}$ are specific cases derived from the original function χ, without the need for additional conditions to achieve these improvements. Moreover, it is possible to provide a more precise characterization of the uniqueness region of the solution. Now, we turn our attention to the new result.

Theorem 4.2. *Let us consider the following assumptions: there exists a solution* $a^{**} \in O(x_0, \wp_1)$ *of the equation* $\Im(x) = 0$ *for some* $\wp_1 > 0$, *the condition* (4.14) *is satisfied on the ball* $O(x_0, \wp_1)$, *and there exists* $\wp_2 \geq \rho_1$ *such that for* $\vartheta_\tau = (1-\tau)$ $\|\Upsilon^* - x_0\| + \tau\|a^{**} - x_0\|$

$$\int_0^1 \int_0^{(1-\tau)\wp_1 + \tau\rho_2} \chi_0(h)dud\tau < 1. \tag{4.15}$$

Let us define the set $D_1 = O(x_0, \epsilon_0) \cap \overline{O}(x_0, \wp_2)$. *Then, within the set* D_1, *the equation* $\Im(x) = 0$ *possesses a single solution given by* Υ^*.

Proof. Let us define the linear operator $S = \int_0^1 \Im'(\Upsilon^* + \tau(a^{**} - \Upsilon^*))d\tau$, where $a^{**} \in D_1$ and $\Im(a^{**}) = 0$. By this definition, and considering conditions (4.14) and (4.15), it follows that:

$$\left\|[\Im'(x_0)]^{-1}(S - \Im'(x_0))\right\| \leq \int_0^1 \int_0^{\vartheta_\tau} \chi_0(h)dud\tau < 1,$$

since $\vartheta_\tau < (1-\tau)\wp_1 + \tau\rho_2$. Thus, S is invertible and

$$a^{**} - a^* = S^{-1}(\Im(a^{**}) - \Im(a^*)) = S^{-1}(0) - 0.$$

Based on the aforementioned analysis, we can conclude that $a^{**} = a^*$. Thus, we have successfully completed the proof. □

Remark 4.1. (i) The limit point a^* in Theorem 4.1 can be replaced by $\wp$.

(ii) If all the assumptions of Theorem 4.1 hold, we can choose $\wp_1 = a^*$ and $a^{**} = a^*$ in Theorem 4.2.

Exploring Special Cases in the Study. Subsequently, by applying Theorem 4.1, we will establish several corollaries by considering different variations of the positive function χ. To begin with, let's examine the scenario where χ is a positive constant. In this scenario, the χ-average Lipschitz condition (2.1) can be simplified to the following Lipschitz condition.

Corollary 4.1. *Consider a nonlinear operator $\Im : \mathbb{D} \subset \bar{\bar{U}} \to \bar{\bar{V}}$, which is continuously Fréchet differentiable in a convex subset, including $\mathbb{D}$. Let $x_0 \in \mathbb{D}$ be an i.g. such that $[\Im'(x_0)]^{-1}$ exists. Furthermore, assume that $\Im$ satisfies the Lipschitz condition:*

$$\left\|[\Im'(x_0)]^{-1}(\Im'(y) - \Im'(x))\right\| \leq \varkappa\|y - x\|, \quad x, y \in O(x_0, \wp_0), \tag{4.16}$$

in which $\wp_0 = 1/\chi$. The expression of the m.f. $\hbar$ introduced in equation (3.4) *can be written as follows:*

$$\hbar(a) = \flat - a + \frac{\chi}{2}a^2, \quad a \in [0, \eth].$$

The value of $\eth$ can be computed using equation (3.3) *as $\eth = 2/\chi$. The constant $\natural$, defined in equation* (3.6)*, can be expressed as $\natural = 1/(2\varkappa)$. Moreover, Lemma* 3.2 *suggests that if $\chi\flat \leq 1/2$, then the roots of $\hbar$ within the intervals $(0, 1/\chi)$ and $(1/\chi, 2/\chi)$ are as follows:*

$$a^* = \frac{1 - \sqrt{1 - 2\varkappa\flat}}{\chi} \quad \textit{and} \quad a^{**} = \frac{1 + \sqrt{1 - 2\varkappa\flat}}{\chi}. \tag{4.17}$$

Let $\{x_\iota\}$ represent the iterates generated by the fsns (1.3) *with the i.g. x_0. Under the assumption that $0 < \chi\flat \leq 1/2$, the iterates $\{x_\iota\}$ are well-defined and converge Q-cubically to a unique solution$a^* \in O(x_0, a^*)$ of equation* (1.1)*, where $a^* < \wp < a^{**}$ and the values of a^* and a^{**} are given by* (4.18)*. Furthermore, if $0 < \chi\flat \leq 3/8$, the convergence order is at least five, and the following error bound holds:*

$$\|\Upsilon^* - x_{\iota+1}\| \leq \frac{1}{2} \cdot \frac{\chi^4}{(1 - 2\varkappa\flat)^2} \cdot \frac{1}{2\sqrt{1 - 2\varkappa\flat} - 1}\|\Upsilon^* - x_\iota\|^5, \quad \iota \geq 0. \tag{4.18}$$

Next, suppose that $\Upsilon > 0$. We introduce the positive function χ, which is defined as follows:

$$\chi(h) := \frac{2\Upsilon}{(1 - \Upsilon u)^3}, \quad h \in \left[0, \frac{1}{\Upsilon}\right). \tag{4.19}$$

Corollary 4.2. *Suppose that $\Im : \mathbb{D} \subset \bar{\bar{U}} \to \bar{\bar{V}}$ is a nonlinear operator that is continuously Fréchet differentiable in a convex subset $\mathbb{D}$, which is also open. Let*

$x_0 \in \mathbb{D}$ be an i.g. satisfying the conditions: $[\Im(x_0)]^{-1}$ exists and $\Im$ satisfies the following condition:

(4.20)
$$\left\|[\Im'(x_0)]^{-1}(\Im'(y)-\Im'(x))\right\| \leq \frac{1}{(1-\Upsilon\|x-x_0\|-\|y-x\|)^2}-\frac{1}{(1-\Upsilon\|x-x_0\|)^2}.$$

The expression of the m.f. $\hbar$ defined by equation (3.4) can now be written as follows:

$$\hbar(a)=\flat-a+\frac{\Upsilon t^2}{1-\Upsilon t}, \quad a\in\left[0,\frac{1}{\Upsilon}\right].$$

The value of $\wp_0$ can be determined using equation (3) as $\wp_0=(1-\frac{1}{\sqrt{2}})\frac{1}{\Upsilon}$. The constant $\natural$, defined in equation (3.6), can be expressed as $\natural=\frac{0.1715728}{\Upsilon}$, respectively. Letting $\theta:=\flat\Upsilon\leq 0.1715728$, the roots of $\hbar$ are given by $a^*=\frac{1+\theta-\sqrt{(1+\theta)^2-8\theta}}{4\Upsilon}$ and $a^{**}=\frac{1+\theta+\sqrt{(1+\theta)^2-8\theta}}{4\Upsilon}$, respectively. The constant $\hbar^*:=\frac{\hbar''(a^*)}{\hbar'(a^*)}$ given in Theorem 4.1, now has a specific form: $\hbar^*=-\frac{32\Upsilon}{\sqrt{(1+\theta)^2-8\theta}.(3-\theta+\sqrt{(1+\theta)^2-8\theta)^2})}$. The sequence of iterates $\{x_\iota\}$ is generated by the fsns method given in equation (1.3) with the i.g. x_0. Assuming that $0<\theta\leq 0.1715728$, the iterates x_ι are well-defined and converge Q-cubically to a unique solution a^* of (1.1). The solution $a^*\in O(x_0,a^*)$, where $a^*<\wp<a^{**}$ and a^* and a^{**} are the bounds of the solution. Furthermore, if $0<\theta\leq\frac{1}{6}\left(17-\frac{49}{(937-48\sqrt{330})^{1/3}}-(937-48\sqrt{330})^{1/3}\right)$, the convergence order is guaranteed to be at least five. In this case, the following error bound can be established:

$$\|\Upsilon^*-x_{\iota+1}\|\leq\frac{l}{2}(\hbar^*)^4\|\Upsilon^*-x_\iota\|^5, \quad \iota\geq 0, \tag{4.21}$$

where $l:=-\frac{7-\theta^3+\sqrt{1-6\theta+\theta^2}+\theta^2(9+\sqrt{1-6\theta+\theta^2})-3\theta(5+2\sqrt{1-6\theta+\theta^2})}{1+\theta^3-9\sqrt{1-6\theta+\theta^2}-\theta^2(9+\sqrt{1-6\theta+\theta^2})+\theta(23+6\sqrt{1-6\theta+\theta^2})}$.

5. Numerical Illustration: Application and Results

In this part, we will demonstrate the practical application of the s.c. results derived in the previous part.

Example 5.1. Consider $\bar{\bar{U}}=C[0,1]$, which denotes the space of continuous functions (c.f.) defined on the interval $[0,1]$. The norm is given by

$$\|x\|=\max_{b\in[0,1]}|x(b)|.$$

Let $\mho=\bar{\bar{U}}[0,1]$, which represents the function $\Im$ defined on $\mho$.

$$\Im(x)(b)=x(b)-2\lambda\int_0^1\gamma(b,a)x(a)^3da. \tag{5.1}$$

TABLE 1. Analysis of Domains of Uniqueness and Existence of Solution for fsns

λ	Ball of convergence	
	Existence $\overline{O(x_0, a^*)}$	Uniqueness $\overline{O(x_0, a^{**})}$
1	$\overline{O(0.25, 0.00409923)}$	$\overline{O(0.25, 1.26672)}$
0.5	$\overline{O(0.25, 0.00200157)}$	$\overline{O(0.25, 2.60217)}$
0.25	$\overline{O(0.25, 0.00098834)}$	$\overline{O(0.25, 5.26984)}$
0.125	$\overline{O(0.25, 0.00049118)}$	$\overline{O(0.25, 10.6037)}$
0.0625	$\overline{O(0.25, 0.000244864)}$	$\overline{O(0.25, 21.2706)}$

The symbol γ represents the Green's function kernel defined on the interval $[0,1]\times[0,1]$, which can be expressed as follows:

$$\gamma(b,a) = \begin{cases} (1-b)a, & a \le b, \\ b(1-a), & b \le a. \end{cases}$$

We consider b belonging to the interval $[0,1]$ and λ as a real number. The function $x \in C[0,1]$ is the variable to be determined. Consequently, the obtained result is as follows:

$$\Im'(x)y(b) = y(b) - 6\lambda \int_0^1 \gamma(b,a)x(a)^2 h(a)da, \quad y \in \phi. \tag{5.2}$$

We proceed by defining S as the maximum value of the integral $\int_0^1 |\gamma(b,a)|da$ over the interval $[0,1]$, which yields $S = 1/8$. Furthermore, considering the initial approximate solution $x_0(a) = 0.25$, for any x and y belonging to the set ϕ, we can determine the corresponding values of

$$\flat = \left\| [\Im'(x_0)]^{-1}\Im(x_0) \right\| \le \frac{0.0039063 \cdot |\lambda|}{1 - 0.046875 \cdot |\lambda|}. \tag{5.3}$$

By utilizing the definition of χ-average and referring to corollary (4.1), we can derive the expression $\chi = \frac{3}{2}|\lambda| \cdot \frac{1}{1-0.046875\times|\lambda|}$. Since $\flat < \natural$, the convergence criterion is satisfied. This theorem allows us to conclude that the fsns sequence (1.3) generated with the i.g. x_0 converges to the zero of $\Im$. Table 1 presents the domain of uniqueness and existence of the solution for different values of λ, namely $\lambda = 0.0625, 0.125, 0.25, 0.5, 1$.

Table 2 presents the convergence criteria $\chi\beta < 3/8$ and the corresponding error bounds for various values of λ. A comparison with the s.c. of the two-step method described in reference [9] reveals that the novel convergence criteria demonstrated in this study are stronger.

6. CONCLUSIONS

In conclusion, this study successfully investigated the s.c. for multistep fsns (1.3) for solving nonlinear equations in a b.s. environment using unique majorant and χ-average Lipschitz conditions (2.1). The proposed scheme provides a more flexible

Table 2. Analysis of Convergence Criteria and Error Bounds for fsns

λ	$\chi\flat < 3/8$	Error bound
1	0.006449	3.18912
0.5	0.00153602	0.17556
0.25	0.000374952	0.0103885
0.125	0.0000926362	0.00063303
0.0625	0.0000230232	0.0000390852

and versatile way of addressing a broader class of issues by reducing the rigorous Lipschitz and Hölder continuity requirements. The convergence analysis in Theorem 4.1 gave rigorous guarantees on the solution's existence and uniqueness, as well as computable error limitations. The provided majorant function with average Lipschitz conditions has been shown to be an excellent tool for characterizing the scheme's convergence behaviour. The derived radii of convergence balls allow us to specify the range of initial values for which convergence is guaranteed, increasing the scheme's practical applicability. The numerical experiments performed on several benchmark issues indicated that the suggested system outperformed existing methods that relied exclusively on Lipschitz or Hölder continuity assumptions.

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References

[1] I. K. Argyros, *The Theory and Applications of Iteration Methods*, Second Edition, Engineering Series, CRC Press, Taylor and Francis, Abingdon, UK, 2022.

[2] I. K. Argyros, P. Jidesh and S. George, *On the local convergence of Newton-like methods with fourth and fifth-order of convergence under hypotheses only on the first Fréchet derivative*, Novi Sad J. Math. **47**(1) (2017), 1–15. https://doi.org/10.30755/NSJOM.02360

[3] T. Bittencourt and O. P. Ferreira, *Kantorovich's theorem on Newton's method under majorant condition in Riemannian manifolds*, J. Glob. Optim. **68**(2) (2017), 387–411. https://doi.org/10.1007/s10898-016-0472-y

[4] O. P. Ferreira and B. F. Svaiter, *Kantorovich's majorants principle for Newton's method*, Comput. Optim. Appl. **42**(2) (2009), 213–229. https://doi.org/10.1007/s10589-007-9082-4

[5] O. P. Ferreira, *A robust semi-local convergence analysis of Newton's method for cone inclusion problems in Banach spaces under affine invariant majorant condition*, J. Comput. Appl. Math. **279** (2015), 318–335. https://doi.org/10.1016/j.cam.2014.11.019

[6] O. P. Ferreira and B. F. Svaiter, *A robust Kantorovich's theorem on the inexact Newton method with relative residual error tolerance*, J. Complex. **28**(3) (2012), 346–363. https://doi.org/10.1016/j.jco.2012.02.002

[7] J. B. Hiriart-Urruty and C. Lemaréchal, *Convex Analysis and Minimization Algorithms I: Fundamentals* (Vol. 305), Springer science & Business Media, 1993.

[8] Y. Ling and X. Xu, *On the semilocal convergence behavior for Halley's method*, Comput. Optim. Appl. **58**(3) (2014), 597–618. https://doi.org/10.1007/s10589-014-9641-4
[9] Y. Ling, J. Liang and W. Lin, *On semilocal convergence analysis for two-step Newton method under generalized Lipschitz conditions in Banach spaces*, Numer. Algorithms **90** (2022), 577–606. https://doi.org/10.1007/s11075-021-01199-2
[10] S. Regmi, C. I. Argyros, I. K. Argyros and S. George, *Efficient fifth convergence order methods for solving equations*, Trans. Math. Program. Appl. **9**(1) (2021), 23–34.
[11] A. A. Ruiz Magrenan and I. K. Argyros, *Two-step Newton methods*, J. Complex. **30**(4) (2014), 533–553. https://doi.org/10.1016/j.jco.2014.05.002
[12] J. O. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, Society for Industrial and Applied Mathematics, 2000.
[13] A. Saxena, I. K. Argyros, J. P. Jaiswal, C. Argyros and K. R. Pardasani, *On the local convergence of two-step Newton type method in Banach spaces under generalized Lipschitz conditions*, Mathematics **9** (2021), Article ID 669. https://doi.org/10.3390/math9060669
[14] A. Saxena, J. P. Jaiswal, K. R. Pardasani and I. K. Argyros, *Convergence criteria of a three-step scheme under the generalized Lipschitz condition in Banach spaces*, Mathematics **10** (2022), Article ID 3946. https://doi.org/10.3390/math10213946
[15] X. Wang, *Convergence of Newton's method and uniqueness of the solution of equations in Banach space*, IMA J. Numer. Anal. **20** (2000), 123–134. https://doi.org/10.1093/imanum/20.1.123
[16] X. Wang, *Convergence of Newton's method and inverse function theorem in Banach space*, Math. Comput. **68** (1999) (225), 169–186. https://doi.org/10.1090/S0025-5718-99-00999-0

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