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STRONG NUMERICAL SEMIGROUPS

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Abstract. In this paper we study the family of Strong numerical semigroups (ST-semigroup, for short). A numerical semigroup S of multiplicity m is an ST-semigroup if $x+y-m \in S$ for all $\{x,y\} \subset S\backslash\{0\}$ such that $x \not\equiv y \bmod m$. We give a characterization of ST-semigroups based on their minimal system of generators, which allows us to establish an algorithm to know if a numerical semigroup is an ST-semigroup. Besides, we study the family of ST-semigroups with fixed multiplicity and ratio, and we obtain formulas for the embedding dimension, Frobenius number and genus of these semigroups. Moreover, we get an explicit description of the shape of a minimal presentation of an ST-semigroup.

1. Introduction

Let $\mathbb{Z}$ be the set of integers and let $\mathbb{N} = \{z \in \mathbb{Z} \mid z \geq 0\}$. A subset S is a submonoid of $(\mathbb{N},+)$ if $S \subseteq \mathbb{N}$, $0 \in S$ and S is closed under the addition in $\mathbb{N}$. A numerical semigroup S is a submonoid of $(\mathbb{N},+)$ that is cofinite in $\mathbb{N}$, that is, $|\mathbb{N}\backslash S| < +\infty$. If S is a numerical semigroup, then $\mathrm{m}(S) = \min(S\backslash\{0\})$, $\mathrm{g}(S)$ the cardinality of $\mathbb{N}\backslash S$ and $\mathrm{F}(S)$ the greatest integer not belonging to S are relevant invariants of S called multiplicity, genus and Frobenius number of S, respectively.

For a subset $\mathcal{X} \subseteq \mathbb{N}$ we denote by $\langle \mathcal{X} \rangle$ the submonoid of $(\mathbb{N},+)$ generated by $\mathcal{X}$, that is,

$$\langle \mathcal{X} \rangle = \left\{ \sum_{i=1}^{n} \lambda_i x_i \mid n \in \mathbb{N}\backslash\{0\},\ x_i \in \mathcal{X},\ \lambda_i \in \mathbb{N} \text{ for all } i \in \{1,\dots,n\} \right\}.$$

In [14, Lemma 2.1], it is shown that $\langle \mathcal{X} \rangle$ is a numerical semigroup if and only if $\gcd(\mathcal{X}) = 1$.

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The set $\mathcal{X}$ is called a system of generators of $\langle \mathcal{X} \rangle$, and if $\langle \mathcal{X} \rangle \neq \langle \mathcal{X}' \rangle$ for all $\mathcal{X}' \subsetneq \mathcal{X}$, then we say that $\mathcal{X}$ is a minimal system of generators of $\langle \mathcal{X} \rangle$. In [14, Corollary 2.8], it is shown, that every submonoid M of $(\mathbb{N}, +)$ admits a unique minimal system of generators, which is finite. We denote by $\mathrm{msg}(M)$ the minimal system of generators of M and its elements are called minimal generators. The cardinality of $\mathrm{msg}(M)$ is called the embedding dimension of M, denoted by $\mathrm{e}(M)$.

The Frobenius problem is to determine formulas for the Frobenius number and the genus of a numerical semigroup depending of its minimal system of generators. This problem was solved in [18], for numerical semigroups with embedding dimensions two. At present, this problem is open for embedding dimension greater than or equal to three (see [8] for a nice state of the art on this problem).

Throughout this work, m is a positive integer. Given two integers a and b, we write $a \equiv b \pmod{m}$ to denote that m divides $a - b$. If $b \bmod m$ is the remainder of the division of b by m, then $b = b - \lfloor \frac{b}{m} \rfloor m \pmod{m}$, being $\lfloor - \rfloor$ the floor operator.

Following the nomenclature introduced in [10], we have the following.

Definition 1.1. A numerical semigroup S is a Strong numerical semigroup (ST-semigroup, for short) if $x + y - \mathrm{m}(S) \in S$ for all $\{x, y\} \subset S \backslash \{0\}$ such that $x \not\equiv y \pmod{\mathrm{m}(S)}$.

Our main purpose in this work is to study this class of numerical semigroups.

The paper is structured as follows. In Section 2, we give a characterization of an ST-semigroup based on its minimal system of generators. This characterization provides an algorithm to determine whether a numerical semigroup is an ST-semigroup.

We denote by $\mathcal{ST}(m)$ the set of ST-semigroups with multiplicity m. We will show that $\mathcal{ST}(m)$ is a Frobenius pseudo-variety, and applying results of [11], we give an algorithm that allows us to calculate all elements in $\mathcal{ST}(m)$ with a given genus.

In Section 3, we will show, that if A is a finite subset of $\{m, \rightarrow\}$ (the symbol $\rightarrow$ means that every integer greater than or equal to m belongs to the set) such that $\gcd(A \cup \{m\}) = 1$, then there exists the smallest element in $\mathcal{ST}(m)$ that contains A, denoted here by $\mathcal{ST}(m)[A]$. We will give an algorithm to compute the semigroup $\mathcal{ST}(m)[A]$. If $S = \mathcal{ST}(m)[A]$, we will say that A is an $\mathcal{ST}(m)$-system of generators of S, which is minimal if $S \neq \mathcal{ST}(m)[A']$ for all $A' \subsetneq A$. Besides in this section, we will prove that every $S \in \mathcal{ST}(m)$ has a unique minimal $\mathcal{ST}(m)$-system of generators and its cardinality is the $\mathcal{ST}(m)-$rank of S, denoted by $\mathcal{ST}(m)\mathrm{msg}(S)$ and $\mathcal{ST}(m)\mathrm{rank}(S)$, respectively.

In Section 4, we study the elements in $\mathcal{ST}(m)$ with $\mathcal{ST}(m)\mathrm{rank}(S) = 1$; we will see that this set is equal to $\{\mathcal{ST}(m)[\{r\}] \mid r \in \mathbb{N},\ m < r \text{ and } \gcd(m, r) = 1\}$. Furthermore, we will show that the embedding dimension, Frobenius number and genus of $\mathcal{ST}(m)[\{r\}]$ are $m - 1$, $(m - 1)r - (m - 2)m$ and $\frac{(m-1)(r-1)-(m-3)(m-2)}{2}$, respectively.

Given positive integers m and F such that $1 \leq m - 1 \leq F$ and $m \nmid F$, denote by $\mathcal{ST}(m, F) = \{S \in \mathcal{ST}(m) \mid \mathrm{F}(S) = F\}$. In Section 5, we prove that $\mathcal{ST}(m, F)$ is a ratio-covariety. By using the results of [6], we obtain an algorithmic procedure to compute all the elements of $\mathcal{ST}(m, F)$.

From [12] we get an explicit description of the shape of a minimal presentation of an ST-semigroup. Although the techniques used here are purely semigroupistic, these results can be translated to the ring theory. In fact, let S be a numerical semigroup with $\mathrm{msg}(S) = \{n_1, \ldots, n_e\}$ and K a field. The semigroup ring $K[S]$ is the finite type K-algebra associated with S and $K[X] = K[X_1, \ldots, X_e]$ is the polynomial ring in e variables, and the K-algebra epimorphism $\lambda : K[X] \to K[S] \subseteq K[t]$, defined by $X_i \to t^{n_i}$, is an S-graded ring homomorphism of degree zero. Therefore, the prime ideal $P = \ker(\lambda)$ (called the ideal associated to the semigroup) is homogeneous and defines a monomial curve in the e-dimensional affine space over K (see, for example, [3]). In [5], Herzog shows that finding a system of generators for P is equivalent to finding a presentation of S.

2. The Elements in $\mathcal{ST}(m)$ with Fixed Genus

Our aim in this section is to give an algorithm to compute the set of numerical semigroups of $\mathcal{ST}(m)$ with fixed genus g. First, we need some concepts and results.

Proposition 2.1. *Let S be a numerical semigroup. Then, S is an ST-semigroup if and only if $x + y - \mathrm{m}(S) \in S$ for all $\{x, y\} \subset S\backslash\{0\}$ such that $x \neq y$.*

Proof. Necessity. If $x \equiv y \bmod \mathrm{m}(S)$, then there exists $k \in \mathbb{N}\backslash\{0\}$ such that $x = y + k\mathrm{m}(S)$ and thus $x + y - \mathrm{m}(S) = 2y + (k-1)\mathrm{m}(S) \in S$.

In the case of $x \not\equiv y \bmod \mathrm{m}(S)$ the result is trivially true.

Sufficiency. Trivially true. □

For a numerical semigroup S, it is known that its embedding dimension, $\mathrm{e}(S)$, is less than or equal to its multiplicity, $\mathrm{m}(S)$ (see, [14, Proposition 2.10]). We say that S has maximal embedding dimension if $\mathrm{e}(S) = \mathrm{m}(S)$. This class of semigroups has become especially renowned due to the existing applications to commutative algebra via their associated semigroup ring (see for instance, [2–4] and [16]).

Proposition 2.2. ([14, Proposition 3.12]). *Let S be a numerical semigroup. Then, S is a numerical semigroup with maximal embedding dimension if and only if $x + y - \mathrm{m}(S) \in S$ for all $\{x, y\} \subset S\backslash\{0\}$.*

As a consequence of Propositions 2.1 and 2.2, we have the following.

Corollary 2.1. *Every numerical semigroup with maximal embedding dimension is an ST-semigroup.*

The next result can be used to see if a numerical semigroup is an ST-semigroup.

Proposition 2.3. ([10, Proposition 5.1]). *Let S be a numerical semigroup. Then, S is an ST-semigroup if and only if $\{x + y - \mathrm{m}(S), 3x - m(S)\} \subset S$ for all $x, y \in \mathrm{msg}(S)\backslash\{\mathrm{m}(S)\}$ and $x \neq y$.*

Example 2.1. If $S = \langle 4, 5, 7\rangle$, then $\{5 + 7 - 4, 3 \cdot 5 - 4, 3 \cdot 7 - 4\} \subset S$, and using Proposition 2.3, we conclude that S is an ST-semigroup.

A pseudo-Frobenius variety is a non-empty family $\mathcal{P}$ of numerical semigroups that fulfills the following conditions.

(a) $\mathcal{P}$ has a maximum (with respect to the inclusion order).
(b) If $\{S, T\} \subseteq \mathcal{P}$, then $S \cap T \in \mathcal{P}$.
(c) If $S \in \mathcal{P}$ and $S \neq \max(\mathcal{P})$, then $S \cup \{\mathrm{F}(S)\} \in \mathcal{P}$.

From [10, Proposition 2.6 and Corollary 5.4], we obtain the following result.

Proposition 2.4. *The set $\mathcal{ST}(m)$ is a pseudo-Frobenius variety and $\{0, m, \rightarrow\}$ is its maximum.*

A graph G is a pair (V, E), where V is a nonempty set and E is a subset of $\{(v, w) \in V \cdot V \mid v \neq w\}$. The elements of V and E are called vertices and edges of G, respectively. A path of length n connecting the vertices u and v of G is a sequence of distinct edges of the form (v_0, v_1), $(v_1, v_2), \ldots, (v_{n-1}, v_n)$ with $v_0 = u$ and $v_n = v$. A graph G is a tree if there exists a vertex r (known as the root of G) such that for all vertices v that belong to G, there exists a unique path connecting v and r. If (v_0, v_1) is an edge of a tree, then we say that v_0 is a child of v_1.

We define the graph $G(\mathcal{ST}(m))$ as follows: $\mathcal{ST}(m)$ is the set of vertices and $(S, T) \in \mathcal{ST}(m) \cdot \mathcal{ST}(m)$ is an edge if $T = S \cup \{\mathrm{F}(S)\}$.

By applying now Proposition 2.4 and [11, Lemmas 11 and 12; and Theorem 3] we get the following result.

Proposition 2.5. *The graph $G(\mathcal{ST}(m))$ is a tree rooted in $\{0, m, \rightarrow\}$. Moreover, the children of S in $G(\mathcal{ST}(m))$ are the elements of $\{S\backslash\{x\} \mid x \in \mathrm{msg}(S),\ x > \mathrm{F}(S)$ and $S\backslash\{x\} \in \mathcal{ST}(m)\}$.*

Lemma 2.1. ([10, Proposition 5.5]). *Let $S \in \mathcal{ST}(m)$, and $x \in \mathrm{msg}(S)\backslash\{m\}$. Then, $S\backslash\{x\} \in \mathcal{ST}(m)$ if and only if $x + m \notin \{a + b \mid a, b \in \mathrm{msg}(S)\backslash\{m, x\}$, and $a \neq b\} \cup \{3a \mid a \in \mathrm{msg}(S)\backslash\{m, x\}\}$.*

Lemma 2.2. *Let $g \in \mathbb{N}$. Then, the set $\{S \in \mathcal{ST}(m) \mid \mathrm{g}(S) = g\}$ is nonempty if and only if $g \geq m - 1$.*

Proof. If $S \in \mathcal{ST}(m)$ with $\mathrm{g}(S) = g$, then $S \subseteq \{0, m, \rightarrow\}$ and thus $g = \mathrm{g}(S) \geq \mathrm{g}(\{0, m, \rightarrow\}) = m - 1$.

To show that the reverse is true, it is enough to note that $\left\{0, m, 2m, \ldots, \lfloor\frac{g}{m-1}\rfloor m, \lfloor\frac{g}{m-1}\rfloor m + g \bmod (m-1) + 1, \rightarrow\right\} \in \mathcal{ST}(m)$ with genus g. $\square$

Let G be a rooted tree and v one of its vertices. We define the depth of the vertex v as the length of the path that connects v to the root of G, denoted by $d(v)$. Given $k \in \mathbb{N}$, we denote by $N(G, k)$ the set of all vertices with depth k, that is,

$$N(G, k) := \{v \mid d(v) = k\}.$$

The next result has immediate proof.

Lemma 2.3. *With the above notation, we have:*

(a) $N(G(\mathcal{ST}(m)),k) = \{S \in \mathcal{ST}(m) \mid \mathrm{g}(S) = m-1+k\}$,
(b) $N(G(\mathcal{ST}(m)),k+1) = \{S \mid S$ *is a child of an element in* $N(G(\mathcal{ST}(m)),k)\}$.

The following algorithm provides a method for computing all the ST-semigroups with given multiplicity and genus.

Algorithm 2.1.
INPUT: g a positive integer.
OUTPUT: The set $\{S \in \mathcal{ST}(m) \mid \mathrm{g}(S) = g\}$.

(a) If $g < m-1$, then return $\emptyset$.
(b) $A := \{0, m, \rightarrow\}$, $i = m-1$
(c) If $i = g$, then return A.
(d) For each $S \in A$ compute θ_S whose elements are $x \in \mathrm{msg}(S) \backslash \{m\}$ greater than $\mathrm{F}(S)$ and $x + m \notin \{a + b \mid a, b \in \mathrm{msg}(S) \backslash \{m, x\}$, and $a \neq b\} \cup \{3a \mid a \in \mathrm{msg}(S) \backslash \{m, x\}\}$.
(e) $A := \bigcup_{S \in A} \{S \backslash \{x\} \mid x \in \theta_S\}$, $i = i+1$ and go to step (c).

The correctness of this algorithm relies on Proposition 2.5 together with Lemmas 2.1, 2.2 and 2.3.

Example 2.2. Let us compute the set $\{S \in \mathcal{ST}(4) \mid \mathrm{g}(S) = 6\}$.

(a) Start with $A = \{\langle 4, 5, 6, 7\rangle\}$, $i = 3$.
(b) The first loop constructs $\theta_{\langle 4,5,6,7\rangle} = \{5, 6\}$ and then $A = \{\langle 4, 6, 7, 9\rangle, \langle 4, 5, 7\rangle\}$, $i = 4$.
(c) The second loop constructs $\theta_{\langle 4,6,7,9\rangle} = \{6, 7\}$, $\theta_{\langle 4,5,7\rangle} = \{7\}$ and then $A = \{\langle 4, 7, 9, 10\rangle, \langle 4, 6, 9, 11\rangle, \langle 4, 5, 11\rangle\}$, $i = 5$.
(d) The third loop constructs $\theta_{\langle 4,7,9,10\rangle} = \{7, 9, 10\}$, $\theta_{\langle 4,6,9,11\rangle} = \{9\}$, $\theta_{\langle 4,5,11\rangle} = \emptyset$ and then $A = \{\langle 4, 9, 10, 11\rangle, \langle 4, 7, 10, 13\rangle, \langle 4, 7, 9\rangle, \langle 4, 6, 11, 13\rangle\}$, $i = 6$.
(e) Return $\{\langle 4, 9, 10, 11\rangle, \langle 4, 7, 10, 13\rangle, \langle 4, 7, 9\rangle, \langle 4, 6, 11, 13\rangle\}$.

3. $\mathcal{ST}(m)$-System of Generators

By using Proposition 2.4, we get that the intersection of finitely many ST-semigroups in $\mathcal{ST}(m)$ is an ST-semigroup in $\mathcal{ST}(m)$. This result cannot be extended to the intersection of arbitrary families in $\mathcal{ST}(m)$, as the example below shows.

Example 3.1. For each $k \in \{m, \rightarrow\}$, let $S_k = \langle m\rangle \cup \{k, \rightarrow\}$. It is clear that S_k is an ST-semigroup in $\mathcal{ST}(m)$, but $\bigcap_{k \in \{m, \rightarrow\}} S_k = \langle m\rangle$ is not a numerical semigroup.

On the other side, the intersection of arbitrary families of semigroups in $\mathcal{ST}(m)$ is a submonoid of $(\mathbb{N}, +)$. Thus, an $\mathcal{ST}(m)$-monoid is a submonoid of $(\mathbb{N}, +)$ which can be expressed as an intersection of elements in $\mathcal{ST}(m)$. It is easy to prove the following result.

Lemma 3.1. *The intersection of $\mathcal{ST}(m)$-monoids is an $\mathcal{ST}(m)$-monoid.*

In view of this result, given a finite set $A \subseteq \{m, \rightarrow\}$, we can define $\mathcal{ST}(m)[A]$ as the intersection of all $\mathcal{ST}(m)$-monoids containing A.

From Lemma 3.1, we obtain the next result.

Lemma 3.2. *If $A \subseteq \{m, \rightarrow\}$, then $\mathcal{ST}(m)[A]$ is the smallest (with respect to set inclusion) $\mathcal{ST}(m)$-monoids containing A.*

The following result has immediate proof.

Lemma 3.3. *If $A \subseteq \{m, \rightarrow\}$, then $\mathcal{ST}(m)[A]$ is the intersection of all $\mathcal{ST}(m)$-monoid containing A.*

If $M = \mathcal{ST}(m)[A]$, then we say A is an $\mathcal{ST}(m)$-system of generators of M. Moreover, if no proper subset of A generates this $\mathcal{ST}(m)$-monoid, then we say that A is a minimal $\mathcal{ST}(m)$-system of generators of M.

By applying Proposition 2.4 and [11, Corollary 1] we get the following result.

Proposition 3.1. *Every $\mathcal{ST}(m)$-monoid admits a unique minimal $\mathcal{ST}(m)$-system of generators.*

Given M an $\mathcal{ST}(m)$-monoid, denote by $\mathcal{ST}(m)\mathrm{msg}(M)$ its minimal $\mathcal{ST}(m)$-system of generators. Its cardinality is called the $\mathcal{ST}(m) - \mathrm{rank}$ of M and will be denoted by $\mathcal{ST}(m)\mathrm{rank}(M)$.

Proposition 3.2. *Let M be a submonoid of $(\mathbb{N}, +)$ and $m = \min(M \backslash \{0\})$. The following conditions are equivalent.*

1) *M is an $\mathcal{ST}(m)$-monoid.*
2) *$x + y - m \in M$ for all $\{x, y\} \subset M \backslash \{0\}$ such that $x \neq y$.*
3) *$\{a + b - m, 3a - m\} \subseteq M$ for all $\{a, b\} \subseteq \mathrm{msg}(M) \backslash \{m\}$ such that $a \neq b$.*

Proof. 1) implies 2). If M is an $\mathcal{ST}(m)$-monoid, then there exists a family of semigroups in $\mathcal{ST}(m)$, $\{S_i\}_{i \in I}$ with $i \in I$, such that $M = \bigcap_{i \in I} S_i$. If $\{x, y\} \subset M \backslash \{0\}$ such that $x \neq y$, we obtain by Proposition 2.1 that $x + y - m \in S_i$ for all $i \in I$ and thus $x + y - m \in M$.

2) implies 3). Trivial.

3) implies 1) For each $k \in \{m, \rightarrow\}$, let $S_k = M \cup \{k, \rightarrow\}$. It is well known that $S_k \in \mathcal{ST}(m)$ and clear that $M = \bigcap_{k \in \{m, \rightarrow\}} S_k$. □

The following algorithm provides a method for computing the smallest ST-monoid containing a finite set $A \subset \{m, \rightarrow\}$.

Algorithm 3.1.

INPUT: A finite set $A \subset \{m, \rightarrow\}$.

OUTPUT: The set $\mathcal{ST}(m)[A]$.

(a) $M := \langle A \cup \{m\} \rangle$.

(b) $B := \{a + b - m \mid \{a, b\} \subseteq \mathrm{msg}(M) \backslash \{m\}, a \neq b \text{ and } a + b - m \notin M\}$.

(c) $C := \{3\,a - m \mid a \in \operatorname{msg}(M)\backslash\{m\} \text{ and } 3a - m \notin M\}$.
(d) If $B \cup C := \emptyset$, then return M.
(e) $M := \langle \operatorname{msg}(M) \cup B \cup C \rangle$ and go to step (b).

The correctness of this algorithm relies in Proposition 3.2. Now, we give two examples.

Example 3.2. Let us compute the set $\mathcal{ST}(5)[\{7\}]$ using the previous algorithm.

(a) Start with $M = \langle 5, 7 \rangle$.
(b) The first loop does $B = \emptyset$ and $C = \{16\}$, then $M = \langle 5, 7, 16 \rangle$.
(c) The second does $B = \{18\}$ and $C = \emptyset$, then $M = \langle 5, 7, 16, 18 \rangle$.
(d) The third does $B = \emptyset$ and $C = \emptyset$.
(e) Return $\mathcal{ST}(5)[\{7\}] = \langle 5, 7, 16, 18 \rangle$.

Example 3.3. Let us compute the set $\mathcal{ST}(6)[\{8\}]$ using again the previous algorithm.

(a) Start with $M = \langle 6, 8 \rangle$.
(b) The first loop does $B = \emptyset$ and $C = \emptyset$.
(c) Return $\mathcal{ST}(6)[\{8\}] = \langle 6, 8 \rangle$.

Combining the Propositions 2.1 and 3.2, we obtain the next result.

Lemma 3.4. *Let M be an $\mathcal{ST}(m)$-monoid. Then, $M \in \mathcal{ST}(m)$ if and only if M is a numerical semigroup.*

Proposition 3.3. *Let be a finite set $A \subset \{m, \rightarrow\}$. Then, the monoid $\mathcal{ST}(m)[A]$ belongs to $\mathcal{ST}(m)$ if and only if* $\gcd(A \cup \{m\}) = 1$.

Proof. Necessity. Suppose that $\gcd(A \cup \{m\}) \neq 1$. Then, by applying Proposition 3.2, we deduce that $M = (\{m\} + \langle A \rangle) \cup \{0\}$ is an $\mathcal{ST}(m)$-monoid containing A. Besides $\mathcal{ST}(m)[A] \subseteq M$ but is not numerical semigroup and so $\mathcal{ST}(m)[A] \notin \mathcal{ST}(m)$.

Sufficiency. Whereas $A \cup \{0\} \subset \mathcal{ST}(m)[A]$ with $\gcd(A \cup \{m\}) = 1$, we get that $\mathcal{ST}(m)[A]$ is a numerical semigroup. By Lemma 3.4, we can conclude that $\mathcal{ST}(m)[A] \in \mathcal{ST}(m)$. ☐

As a consequence of Proposition 3.3, we obtain the following.

Proposition 3.4. *Let m and r be positive integers such that $2 \leq m \leq r$ and* $\gcd(m, r) = 1$. *Then, $\mathcal{ST}(m)[\{r\}]$ is an element in $\mathcal{ST}(m)$ with $\mathcal{ST}(m)$*rank *one. Moreover, every element in $\mathcal{ST}(m)$ with this $\mathcal{ST}(m) -$* rank *is of this form.*

4. The Elements in $\mathcal{ST}(m)$ with $\mathcal{ST}(m)$rank One

In this section, we study the family of ST-semigroups with multiplicity m and $\mathcal{ST}(m)$rank $= 1$. In particular, we obtain formulas to compute, in terms of m and r, the Frobenius number and genus of these classes of numerical semigroups.

Observe that if $m = 2$ or $m = 3$ and $\gcd(m, r) = 1$ then, by using Proposition 2.3 and Lemma 3.2, we have that $\mathcal{ST}(m)[\{r\}] = \langle m, r \rangle$. Thus, in the sequel, we assume that m and r are positive integers such that $4 \leq m \leq r$ and $\gcd(m, r) = 1$.

Proposition 4.1. *Under the standing notation,* $\mathcal{ST}(m)[\{r\}] = \langle \{m, r\} \cup \{(k+2)r - km \mid k \in \mathbb{N}\backslash\{0\}\}\rangle$.

Proof. Suppose that $T = \langle \{m, r\} \cup \{(k+2)r - km \mid k \in \mathbb{N}\backslash\{0\}\}\rangle$. In order to see that $T \subseteq \mathcal{ST}(m)[\{r\}]$, let us use induction on k to prove that $(k+2)r - km \in \mathcal{ST}(m)[\{r\}]$. For $k = 1$ is clear, that since $\{r, 2r\} \subseteq \mathcal{ST}(m)[\{r\}]$, then by Proposition 2.1, we have that $3r - m \in \mathcal{ST}(m)[\{r\}]$. As by induction hypothesis $\{(k+2)r - km, r\} \subseteq \mathcal{ST}(m)[\{r\}]$ and $(k+2)r - km \neq r$, then using again Proposition 2.1, we get that $(k+2)r - km + r - m = (k+3)r - (k+1)m \in \mathcal{ST}(m)[\{r\}]$.

For the other inclusion, by Lemma 3.2, it is sufficient to prove that $T \in \mathcal{ST}(m)$. In order to see this, we use Proposition 2.3:

(a) if $k, j \in \mathbb{N}\backslash\{0\}$, then $(k+2)r - km + (j+2)r - jm - m = (k+j+3)r - (k+j+1)m + r \in T$,
(b) if $k \in \mathbb{N}\backslash\{0\}$, then $3((k+2)r - km) - m = (3k+2)r - 3km + 4r - 2m + m \in T$.

□

Corollary 4.1. *Under the standing notation,* $\mathcal{ST}(m)[\{r\}] = \langle m, r, 3r - m, 4r - 2m, \ldots, (m-1)r - (m-3)m\rangle$.

Proof. By taking into account Proposition 4.1, it is enough to see that if $k + 2 \geq m$, then $(k+2)r - km$ does not belong to the minimal system of generators $\mathcal{ST}(m)[\{r\}]$.

Suppose $k + 2 = qm + t$ with $q \in \mathbb{N}\backslash\{0\}$ and $t \in \{0, \ldots, m-1\}$. Then, $(k+2)r - km = (qm+t)r - (qm+t-2)m = tr - (t-2)m + q(r-m)m$

We distinguish four cases depending on the value of t.

(a) If $t \geq 3$, then by Proposition 4.1, we get that $\{tr - (t-2)m, q(r-m)m\} \subset \mathcal{ST}(m)[\{r\}]\backslash\{0\}$ and so $(k+2)r - km \notin \mathrm{msg}(\mathcal{ST}(m)[\{r\}])$.
(b) If $t = 2$, then $(k+2)r - km = 2r + q(r-m)m$ and $\{2r, q(r-m)m\} \subseteq \mathcal{ST}(m)[\{r\}]\backslash\{0\}$ and so $(k+2)r - km \notin \mathrm{msg}(\mathcal{ST}(m)[\{r\}])$.
(c) If $t = 1$, then $(k+2)r - km = r + (qr - (qm-1))m$ and thus again $(k+2)r - km \notin \mathrm{msg}(\mathcal{ST}(m)[\{r\}])$.
(d) If $t = 0$, then $(k+2)r - km = (qr - (qm-2))m$ implies that $(k+2)r - km \notin \mathrm{msg}(\mathcal{ST}(m)[\{r\}])$.

□

Let S be a numerical semigroup and let $n \in S\backslash\{0\}$. The Apéry set of S (named so in honor of [1]) with respect to n is $\mathrm{Ap}(S, n) := \{s \in S \mid s - n \notin S\}$.

Lemma 4.1. ([14, Lemma 1.4]). *If S is a numerical semigroup and $n \in S\backslash\{0\}$ then*

$$\mathrm{Ap}(S, n) = \{w(0), w(1), \ldots, w(n-1)\},$$

where $w(i)$ is the least element in S such that $w(i) \equiv i \bmod n$, for all $i \in \{0, \ldots, n-1\}$.

The following result (essentially due to Selmer [17]) shows how $\mathrm{F}(S)$ and $\mathrm{g}(S)$ can be computed by using Apéry sets.

Lemma 4.2. *Let S be a numerical semigroup and $m \in S\backslash\{0\}$. Then,*

(a) $\mathrm{F}(S) = \max(\mathrm{Ap}(S,m)) - m$,

(b) $\mathrm{g}(S) = \frac{1}{m}\left(\sum_{w\in \mathrm{Ap}(S,m)} w\right) - \frac{m-1}{2}$.

Theorem 4.1. *Under the standing notation,*

$$\mathrm{Ap}(\mathcal{ST}(m)[\{r\}],m) = \{0,r,2r,3r-m,4r-2m,\dots,(m-1)r-(m-3)m\}.$$

Proof. In view of Corollary 4.1, we have that $\mathcal{ST}(m)[\{r\}] = \langle m,r,3r-m,4r-2m,\dots(m-1)r-(m-3)m\rangle$. Hence, if $i \in \{1,\dots,m-3\}$, then $2((i+2)r-im) \notin \mathrm{Ap}(\mathcal{ST}(m)[\{r\}],m)$, because by Proposition 4.1, we have that $2((i+2)r-im)-m = (2i+3)r-(2i+1)m+r \in \mathcal{ST}(m)[\{r\}]$. As $\mathcal{ST}(m)[\{r\}] \in \mathcal{ST}(m)$ and applying Proposition 2.1, we deduce that if $i,j \in \{1,\dots,m-3\}$ such that $i \neq j$, then $(i+2)r-im+(j+2)r-jm$ does not belong to $\mathrm{Ap}(\mathcal{ST}(m)[\{r\}],m)$. In the same way, we conclude $3r \notin \mathrm{Ap}(\mathcal{ST}(m)[\{r\}],m)$. Consequently, $\mathrm{Ap}(\mathcal{ST}(m)[\{r\}],m) = \{0,r,2r,3r-m,4r-2m,\dots,(m-1)r-(m-3)m\}$. ☐

Example 4.1. From Corollary 4.1 and Theorem 4.1, we obtain that $\mathcal{ST}(5)[\{7\}] = \langle 5,7,16,18\rangle$ and $\mathrm{Ap}(\mathcal{ST}(5)[\{7\}],5) = \{0,7,14,16,18\}$.

Corollary 4.2. *Under the standing notation, then the following conditions hold:*

(a) $S = \mathcal{ST}(m)[\{r\}]$ *is a numerical semigroup with* $\mathrm{e}(S) = m-1$,

(b) $\mathrm{F}(S) = (m-1)r-(m-2)m$,

(c) $\mathrm{g}(S) = \frac{(m-1)(r-1)-(m-3)(m-2)}{2}$.

Proof. (a) From Corollary 4.1 and Theorem 4.1, it is straightforward to see that $\mathrm{msg}(\mathcal{ST}(m)[\{r\}) = \{m,r,3r-m,4r-2m,\dots,(m-1)r-(m-3)m\}$ and so $\mathrm{e}(S) = m-1$. (b) and (3) This is a consequence of Theorem 4.1 and Lemma 4.2. ☐

Example 4.2. If $S = \mathcal{ST}(5)[\{7\}]$, then by Corollary 4.2, we have that $\mathrm{e}(S) = 4$, $\mathrm{F}(S) = 13$ and $\mathrm{g}(S) = 9$.

5. The Elements in $\mathcal{ST}(m)$ with Fixed Frobenius Number

If S is a numerical semigroup with multiplicity $m \geq 2$ and Frobenius number F, then $m-1 \leq F$ and $m \nmid F$. The particular case, $F = m-1$ implies that $S = \{0, F+1, \rightarrow\}$.

Given positive integers m and F such that $2 \leq m \leq F$ and $m \nmid F$, denote by $\mathcal{ST}(m,F) = \{S \in \mathcal{ST}(m) \mid \mathrm{F}(S) = F\}$.

Our next goal is to give an algorithm to compute the elements in $\mathcal{ST}(m,F)$.

Let S be a numerical semigroup such that $S \neq \mathbb{N}$, the ratio of S is defined by $\mathrm{r}(S) := \min\{s \in S \mid \mathrm{m}(S) \nmid s\}$.

A ratio-covariety is a nonempty family $\mathcal{R}$ of numerical semigroups fulfilling the following conditions:

(a) $\mathcal{R}$ has a minimum (with respect to the inclusion order).

(b) If $\{S,T\} \subseteq \mathcal{R}$, then $S \cap T \in \mathcal{R}$.

(c) If $S \in \mathcal{R}$ and $S \neq \min(\mathcal{R})$, then $S\backslash\{\mathrm{r}(S)\} \in \mathcal{R}$.

The next result is well known, and its proof is not difficult.

Lemma 5.1. *Let S and T be numerical semigroups and $x \in S\backslash\{0\}$. Then,*

(a) *$S \cap T$ is a numerical semigroup and* $\mathrm{F}(S \cap T) = \max\{\mathrm{F}(S), \mathrm{F}(T)\}$,

(b) *$S\backslash\{x\}$ is a numerical semigroup if and only if $x \in \mathrm{msg}(S)$.*

Proposition 5.1. *The set $\mathcal{ST}(m, F)$ is a ratio-covariety and its minimum is $\triangle(m, F) = \langle m\rangle \cup \{F + 1, \rightarrow\}$.*

Proof. Using Proposition 2.1, we deduce that $\triangle(m, F) \in \mathcal{ST}(m, F)$. It is clear, for all $S \in \mathcal{ST}(m, F)$, we have that $\triangle(m, F) \subseteq S$ and so $\min(\mathcal{ST}(m, F)) = \triangle(m, F)$. If $\{S, T\} \subseteq \mathcal{ST}(m, F)$, then by Proposition 2.1 and Lemma 5.1, we obtain $S \cap T \in \mathcal{ST}(F, m)$.

In order to conclude the proof, we will see that if $S \in \mathcal{ST}(m, F)$ and $S \neq \triangle(m, F)$ then $S\backslash\{\mathrm{r}(S)\} \in \mathcal{ST}(m, F)$. In fact, if $S \neq \triangle(m, F)$, then $\mathrm{r}(S) < F$ and by Lemma 5.1, we have that $S\backslash\{\mathrm{r}(S)\}$ is a numerical semigroup with $\mathrm{m}(S\backslash\{\mathrm{r}(S)\}) = m$ and $\mathrm{F}(S\backslash\{\mathrm{r}(S)\}) = F$. If $\{x, y\} \subseteq S\backslash\{\mathrm{r}(S)\}$ and $x \neq y$, since $S \in \mathcal{ST}(m, F)$ we get that $x + y - m \in S$ with $x + y - m \neq \mathrm{r}(S)$. Hence $x + y - m \in S\backslash\{\mathrm{r}(S)\}$ and thus $S\backslash\{\mathrm{r}(S)\} \in \mathcal{ST}(m, F)$. ☐

We define the graph $G(\mathcal{ST}(m, F))$ as the graph whose vertices are elements of $\mathcal{ST}(m, F)$ and $(S, T) \in \mathcal{ST}(m, F) \cdot \mathcal{ST}(m, F)$ is an edge if $T = S\backslash\{\mathrm{r}(S)\}$.

As a consequence of Proposition 5.1 and [6, Proposition 2.4], we deduce the following.

Proposition 5.2. *The graph $G(\mathcal{ST}(m, F))$ is a tree with root equal to $\triangle(m, F)$.*

We now characterize the children of an arbitrary vertex of $G(\mathcal{ST}(m, F))$. First, we need to introduce some concepts and results.

Let S be a numerical semigroup. An integer $x \notin S$ is called a special gap of S if $S \cup \{x\}$ is again a numerical semigroup. We denote by $SG(S)$ the set of special gaps of S.

By applying Proposition 5.1 and [6, Proposition 4], we obtain the following.

Proposition 5.3. *Let $S \in \mathcal{ST}(m, F)$. Then, the set formed by the children of S in the tree $G(\mathcal{ST}(m, F))$ is $\{S \cup \{x\} \mid x \in SG(S),\ \mathrm{m}(S) < x < \mathrm{r}(S)$ and $S \cup \{x\} \in \mathcal{ST}(m, F)\}$.*

By using Proposition 2.3, we deduce the next result.

Lemma 5.2. *Let $S \in \mathcal{ST}(m, F)$ and $a \in SG(S)$ such that $m < a < \mathrm{r}(S)$. Then, $S \cup \{a\} \in \mathcal{ST}(m, F)$ if and only if $a \neq F$ and $\{a + b - m,\ 3a - m\} \subset S \cup \{a\}$ for all $b \in \mathrm{msg}(S \cup \{a\})\backslash\{a, m\}$.*

By combining Proposition 5.3 with Lemma 5.2, we have the following.

Proposition 5.4. *If $S \in \mathcal{ST}(m,F)$, then the children of S in the tree $G(\mathcal{ST}(m,F))$ are $S \cup \{x\}$ such that $x \in SG(S)$, $m < x < \mathrm{r}(S)$, $x \neq F$ and $\{x+y-m, 3x-m\} \subset S \cup \{x\}$ for all $y \in \mathrm{msg}(S \cup \{x\}) \backslash \{x,m\}$.*

Remark 5.1. Let S be a numerical semigroup with $n \in S \backslash \{0\}$. If we know its $\mathrm{Ap}(S,n) = \{w(0), w(1), \dots, w(n-1)\}$, then

(a) we can conclude that an integer z belongs to S if and only if $z \geq w(z \bmod n)$,
(b) by using [7, Remark 1 and Remark 2] we can compute the set $SG(S)$ and $\mathrm{Ap}(S \cup \{x\}, n)$ with $x \in SG(S)$.

We are already able to present the algorithm stated at the beginning of this section.

Algorithm 5.1.
INPUT: m and F such that $2 \leq m \leq F$ and $m \nmid F$.
OUTPUT: The set $\mathcal{ST}(m,F)$.

(a) Compute $\mathrm{Ap}(\triangle(m,F), m)$.
(b) Does $\mathcal{ST}(m,F) = \{\triangle(m,F)\}$ and $\mathcal{B} = \{\triangle(m,F)\}$.
(c) For each $S \in \mathcal{B}$ compute Λ_S whose elements are $x \in SG(S)$, $m < x < \mathrm{r}(S)$, $x \neq F$ and $\{x+b-m, 3x-m\} \subseteq S \cup \{x\}$ for all $b \in \mathrm{msg}(S \cup \{x\}) \backslash \{x,m\}$.
(d) If $\bigcup_{S \in \mathcal{B}} \Lambda_S = \emptyset$, then return $\mathcal{ST}(m,F)$.
(e) Does $C = \bigcup_{S \in \mathcal{B}} \{S \cup \{x\} \mid x \in \Lambda_S\}$.
(f) Does $\mathcal{ST}(m,F) = \mathcal{ST}(m,F) \cup C$ and $\mathcal{B} = C$.
(g) For all $S \in \mathcal{B}$ compute $\mathrm{Ap}(S,m)$ and go to (c).

Example 5.1. Let us compute the set of elements in $\mathcal{ST}(3,7)$ by applying Algorithm 5.1.

(a) Start calculating $\mathrm{Ap}(\triangle(3,7),3) = \{0,8,10\}$.
(b) Does $\mathcal{ST}(3,7) = \{\triangle(3,7)\}$ and $\mathcal{B} = \{\triangle(3,7)\}$.
(c) The first loop constructs $\Lambda_{\triangle(3,7)} = \{5\}$ then $C = \{\triangle(3,7) \cup \{5\}\}$.
(d) Does $\mathcal{ST}(3,7) = \{\triangle(3,7), \triangle(3,7) \cup \{5\}\}$, $\mathcal{B} = \{\triangle(3,7) \cup \{5\}\}$ and thus $\mathrm{Ap}(\triangle(3,7) \cup \{5\}, 3) = \{0,5,10\}$.
(e) The second loop constructs $\Lambda_{(\triangle(3,7) \cup \{5\}} = \emptyset$.
(f) Return $\mathcal{ST}(3,7) = \{\triangle(3,7), (\triangle(3,7) \cup \{5\}\}$.

6. Minimal Presentation of an ST-Semigroup

Let S be a numerical semigroup with $\mathrm{msg}(S) = \{n_1 < n_2 < \cdots < n_e\}$. The free monoid on the set $\{x_1, x_2, \dots, x_e\}$ is defined as $F[x_1, x_2, \dots, x_e] = \{\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_e x_e \mid \lambda_i \in \mathbb{N} \text{ for all } i \in \{1, \dots, e\}\}$, then the map

$$\varphi : F[x_1, x_2, \dots, x_e] \to S$$

is a monoid epimorphism defined by $\varphi(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_e x_e) = \lambda_1 n_1 + n_2 + \cdots + \lambda_e n_e$.

It is well known (see [15]) that if σ is the kernel congruence of φ, that is, $(x,y) \in \sigma$ if and only if $\varphi(x) = \varphi(y)$, then S is isomorphic to $\frac{F[x_1, x_2, \dots, x_e]}{\sigma}$. In [9] is shown that the congruence σ is finitely generated. That is, there exists

$\rho = \{(\alpha_1, \beta_1), \ldots, (\alpha_t, \beta_t)\} \subseteq F[x_1, x_2, \ldots, x_e] \times F[x_1, x_2, \ldots, x_e]$ such that σ is the smallest congruence on $F[x_1, x_2, \ldots, x_e]$ containing ρ.

We say that ρ is a presentation for S and it is a minimal presentation for S if no proper subset of ρ generates σ. In [13] we can see that the concepts of minimal presentation with respect to set inclusion and presentation with lowest cardinality coincide. In addition, in [13], a method is given to compute a minimal presentation of a numerical semigroup.

Our aim in this section is to get an explicit description of the shape of a minimal presentation for an ST-semigroup. The next result is deduced from [10, Proposition 5.7] and describes the Apéry set of $S \in \mathcal{ST}(m)$.

Proposition 6.1. *If $S \in \mathcal{ST}(m)$, then $\mathrm{Ap}(S, m) = \{w(0), w(1), \ldots, w(m-1)\}$, where $w(i)$ is the least element of the set $\{0\} \cup (\mathrm{msg}(S) \backslash \{m\}) \cup (2(\mathrm{msg}(S) \backslash \{m\})$ that is congruent to i modulo m.*

Example 6.1. From Example 3.2, we know that $S = \langle 5, 7, 16, 18 \rangle$ is an ST-semigroup. Since $\{0\} \cup (\mathrm{msg}(S) \backslash \{5\}) \cup (2(\mathrm{msg}(S) \backslash \{5\}) = \{0\} \cup \{7, 16, 18\} \cup \{14, 32, 36\}$, then by using Proposition 6.1, we have that $\mathrm{Ap}(S, 5) = \{w(0) = 0, w(1) = 16, w(2) = 7, w(3) = 18, w(4) = 14\}$.

As a consequence of Proposition 6.1, we have the next result.

Corollary 6.1. *Under the standing notation, we have the following.*

(a) *If $S \in \mathcal{ST}(m)$, then $\frac{m+1}{2} \leq \mathrm{e}(S) \leq m$.*
(b) *If S is an ST-semigroup with $\mathrm{msg}(S) = \{n_1 < \cdots < n_e\}$, then there exists $A \subseteq \{n_2, \ldots, n_e\}$ such that $\mathrm{Ap}(S, n) = \{0, n_2, \ldots, n_e\} \cup 2A$.*

Let S be a numerical semigroup with $\mathrm{msg}(S) = \{n_1 < \cdots < n_e\}$, we say that $n \in S$ has unique expression if there exists only one element $(\lambda_1, \ldots, \lambda_e) \in N^e$ such that $n = \lambda_1 n_1 + \cdots + \lambda_e n_e$.

The following result can be deduced from [12, Theorem 1].

Proposition 6.2. *Let S be a numerical semigroup with $\mathrm{msg}(S) = \{n_1 < \cdots < n_e\}$ and all the elements in $\mathrm{Ap}(S, n_1)$ have a unique expression. Let $T = \{(\lambda_2, \ldots, \lambda_e) \in N^{e-1} \mid \lambda_2 n_2 + \cdots + \lambda_e n_e \notin \mathrm{Ap}(S, n_1)\}$ and $\mathrm{Minimals}(T) = \{(\lambda_{12}, \ldots, \lambda_{1e}), \ldots, (\lambda_{t2}, \ldots, \lambda_{te})\}$ with respect to the usual order of $\mathbb{N}^{e-1}$. For each $i \in \{1, \ldots, t\}$ we define $(\alpha_{i1}, \ldots, \alpha_{ie}) \in \mathbb{N}^e$ such that $\lambda_{i2} n_2 + \cdots + \lambda_{ie} n_e = \alpha_{i1} n_1 + \cdots + \alpha_{ie} n_e$ and $\alpha_{i1} \neq 0$.*

Then, $\rho = \{(\lambda_{i2} x_2 + \cdots + \lambda_{ie} x_e, \alpha_{i1} x_1 + \cdots + \alpha_{ie} x_e) \mid i \in \{1, \ldots, t\}\}$ is a minimal presentation of S.

Lemma 6.1. *If $S \in \mathcal{ST}(m)$, then all the elements of $\mathrm{Ap}(S, m)$ have a unique expression.*

Proof. Suppose that $\mathrm{msg}(S) = \{m = n_1 < \cdots < n_e\}$. From Corollary 6.1, we have that $\mathrm{Ap}(S, n_1) = \{0, n_2, \ldots, n_e\} \cup 2A$ for some $A \subseteq \{n_2, \ldots, n_e\}$. Clearly, every element in $\mathrm{msg}(S)$ has a unique expression. To conclude the proof, we check that if $n_k \in A$, then $2n_k$ has also a unique expression. Assume to the contrary that $2n_k = n_i + n_j + s$ for some $i, j \in \{2, \dots, e\}$ and $s \in S$. If $i \neq j$ by Proposition 2.1, we obtain that $n_i + n_j - n_1 \in S$, and thus $2n_k - n_1 \in S$, contradicting that $2n_k \in \mathrm{Ap}(S, n_1)$.

Hence, we consider $2n_k = 2n_i + s$ for some $i \in \{2, \dots, e\} \setminus \{k\}$ and $s \in S$. If $s \neq 2n_i$, we obtain again $2n_i + s - n_1 \in S$ and so $2n_k - n_1 \in S$, which is impossible. If $s = 2n_i$, then $2n_k = 4n_i$ and thus $n_k = 2n_i$, contradicting that $n_k \in \mathrm{msg}(S)$. □

By using Proposition 2.1, we obtain the following.

Lemma 6.2. *Let S be an ST-semigroup with $\mathrm{msg}(S) = \{n_1 < \cdots < n_e\}$, and let $\mathrm{Ap}(S, n_1) = \{0, n_2, \dots, n_e\} \cup 2A$ for some $A \subseteq \{n_2, \dots, n_e\}$. Let $T = \{(\lambda_2, \dots, \lambda_e) \in N^{e-1} \mid \lambda_2 n_2 + \cdots + \lambda_e n_e \notin \mathrm{Ap}(S, n_1)\}$. For each $i \in \{2, \dots, e\}$ we define $C_i = (C_{i2}, \dots, C_{ie}) \in \mathbb{N}^{e-1}$ such that $C_{ii} = 1$ and $C_{ij} = 0$ for all $i \in \{2, \dots, e\} \setminus \{i\}$. Then,*

$$\begin{aligned} \mathrm{Minimals}(T) = &\{C_i + C_j \mid i, j \in \{2, \dots, e\} \text{ and i} \neq \text{j}\} \\ &\cup \{2C_i \mid n_i \in \{n_2, \dots, n_e\} \setminus \{A\}\} \cup \{3C_i \mid n_i \in A\}. \end{aligned}$$

In the conditions of Lemma 6.2, we will use the following notation.

(a) If $i, j \in \{2, \dots, e\}$ and $i < j$, then let $(\lambda_{ij}^1, \dots, \lambda_{ij}^e) \in \mathbb{N}^e$ such that $n_i + n_j = \lambda_{ij}^1 n_1 + \cdots + \lambda_{ij}^e n_e$ and $\lambda_{ij}^1 \neq 0$.
(b) If $n_i \in \{n_2 < \cdots < n_e\} \setminus \{A\}$, then let $(\lambda_i^1, \dots, \lambda_i^e) \in \mathbb{N}^e$ such that $2n_i = \lambda_i^1 n_1 + \cdots + \lambda_i^e n_e$ and $\lambda_i^1 \neq 0$.
(c) If $n_i \in A$, then let $(\lambda_i^1, \dots, \lambda_i^e) \in \mathbb{N}^e$ such that $3n_i = \lambda_i^1 n_1 + \cdots + \lambda_i^e n_e$ and $\lambda_i^1 \neq 0$.

As a consequence of Proposition 6.2 and Lemmas 6.1 and 6.2, we obtain the following.

Theorem 6.1. *Let S be an ST-semigroup with $\mathrm{msg}(S) = \{n_1 < \cdots < n_e\}$, and let $\mathrm{Ap}(S, n_1) = \{0, n_2, \dots, n_e\} \cup 2A$ for some $A \subseteq \{n_2, \dots, n_e\}$. Then, a minimal presentation of S is*

$$\begin{aligned} \rho = &\Big\{(x_i + x_j, \lambda_{ij}^1 x_1 + \cdots + \lambda_{ij}^e x_e) \mid i, j \in \{2, \dots, e\} \text{ and i} < \text{j}\Big\} \\ &\cup \Big\{(2x_i, \lambda_i^1 x_1 + \cdots + \lambda_i^e x_e) \mid n_i \in \{n_2 < \cdots < n_e\} \setminus \{A\}\Big\} \\ &\cup \Big\{3x_i,\ \lambda_i^1 x_1 + \cdots + \lambda_i^e x_e \mid n_i \in A\Big\}. \end{aligned}$$

By using the previous theorem we obtain the next result.

Corollary 6.2. *If S is an ST-semigroup, then the cardinality of a minimal presentation of S is $\frac{\mathrm{e}(S)(\mathrm{e}(S)-1)}{2}$.*

Example 6.2. From Example 6.1, we know that $S = \langle 5, 7, 16, 18\rangle$ is an ST-semigroup and $\mathrm{Ap}(S, 5) = \{0, 7, 16, 18, 2 \cdot 7\}$. Then applying Thereom 6.1, we have that a minimal presentation of S is

$$\rho =\{(x_2 + x_3, x_1 + x_4), (x_2 + x_4, 5x_1), (x_3 + x_4, 4x_1 + 2x_2),$$
$$(2x_3, 5x_1 + x_2), (2x_4,\ 4x_1 + x_3), (3x_2, x_1 + x_3)\}.$$

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