Kragujevac Journal of Mathematics
Volume 43(1) (2019), Pages 15–21.

REMARKS ON THE DEGREE KIRCHHOFF INDEX

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Abstract. Let G be a simple connected graph with n vertices and m edges, with normalized Laplacian eigenvalues $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_{n-1} > \rho_n = 0$. The degree Kirchhoff index $Kf^*(G)$ is defined as $Kf^*(G) = 2m\sum_{i=1}^{n-1} \frac{1}{\rho_i}$. In this paper we obtain lower and upper bounds for $Kf^*(G)$.

1. Introduction

Let $G = (V, E)$, $V = \{1, 2, \dots, n\}$, be a simple connected graph with n, $n \geq 3$, vertices and m edges. Denote with $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta > 0$, a sequence of vertex degrees of G, $\mathbf{A}$ the adjacency matrix of graph, and by $\mathbf{D} = \operatorname{diag}(d_1, d_2, \dots, d_n)$ the diagonal matrix of its vertex degrees. Matrix $\mathbf{L} = \mathbf{D} - \mathbf{A}$ is the Laplacian matrix of G. Eigenvalues of $\mathbf{L}$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$, form the so-called Laplacian spectrum of graph G. Provided that the graph G has no isolated vertices, the normalized Laplacian matrix is defined as $\mathbf{L}^* = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2} = \mathbf{I} - \mathbf{D}^{-1/2}\mathbf{A}\mathbf{D}^{-1/2}$. Its eigenvalues, $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_{n-1} > \rho_n = 0$, represent normalized Laplacian eigenvalues of G. The following identities are valid for ρ_i, $i = 1, 2, \dots, n-1$, (see [5])

$$\sum_{i=1}^{n-1} \rho_i = n \quad \text{and} \quad \sum_{i=1}^{n-1} \rho_i^2 = n + 2R_{-1},$$

where

$$R_{-1} = \sum_{i \sim j} \frac{1}{d_i d_j},$$

is the general Randić index (also called branching index) introduced in [19]. A symbol $i \sim j$ denotes that vertices i and j are adjacent.

Key words and phrases. Degree Kirchhoff index, Laplacian eigenvalues (of graph), vertex degree.
2010 *Mathematics Subject Classification.* Primary: 05C12. Secondary: 05C50.
Received: May 30, 2017.
Accepted: August 30, 2017.

In [9], Klein and Randić, introduced the notion of resistance distance, r_{ij}, defined as the resistance between the nodes i and j in an electrical network corresponding to the graph G in which all edges are replaced by unit resistors. The sum of resistance distances of all pairs of vertices of a graph G is named as the Kirchhoff index, i.e.,

$$Kf(G) = \sum_{i<j} r_{ij}.$$

Eventually, it was shown [7] (see also [8]) that the resistance distance can be expressed in terms of Laplacian matrix and its spectrum. Namely, it was proven that

$$Kf(G) = n \sum_{i=1}^{n-1} \frac{1}{\mu_i}.$$

The Kirchhoff index spawned a family of resistance indices, such as degree Kirchhoff index, multiplicative-degree-Kirchhoff index and additive-degree Kirchhoff index (see for example [2, 16, 17]). The degree Kirchhoff index, introduced in [4], is defined as

$$Kf^*(G) = \sum_{i<j} d_i d_j r_{ij}.$$

In analogy with the Kirchhoff index, the degree Kirchhoff index can also be represented as

$$Kf^*(G) = 2m \sum_{i=1}^{n-1} \frac{1}{\rho_i}.$$

The graph invariants $Kf(G)$ and $Kf^*(G)$ are currently much studied in the mathematical and mathematico-chemical literature; see for example [11] and the references cited therein.

In this paper, inspired by the inequality for $Kf(G)$ reported in [18], we determine an upper bound for $Kf^*(G)$ which depends on n, m and k, where k is an arbitrary real number such that $\rho_{n-1} \geq k > 0$. We also prove one general inequality that sets up a lower bound for $Kf^*(G)$ in terms of n, m and s, where s is an arbitrary real number such that $\rho_1 \geq s \geq \rho_{n-1}$. For some particular values of s we obtain some lower bounds for $Kf^*(G)$ reported in the literature.

2. Preliminaries

In this section we recall some results reported in the literature on lower and upper bounds for $Kf(G)$ and $Kf^*(G)$, and an analytic inequality for real number sequences which will be used in the subsequent considerations.

The following upper bound for $Kf(G)$ that depends on parameters n, m and k, where k is an arbitrary real number with the property $\mu_{n-1} \geq k > 0$, was established in [18]

$$Kf(G) \leq \frac{(n+k)(n-1) - 2m}{k}, \tag{2.1}$$

with equality if and only if $k = n$ and $G \cong K_n$, or $k = 1$ and $G \cong K_{1,n-1}$, or $k = \frac{n}{2}$ and $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $k = n - 2$ and $G \cong K_n - e$.

In [15] (see also [1, 6, 8, 12]) the following inequality was proven

$$Kf^*(G) \geq \frac{2m(n-1)^2}{n}, \tag{2.2}$$

with equality if and only if $G \cong K_n$.

In [6] the following lower bound for $Kf^*(G)$ in terms of parameters n, m, and Δ was determined

$$Kf^*(G) \geq 2m \left(\frac{\Delta}{\Delta+1} + \frac{(n-2)^2}{n-1-\frac{1}{\Delta}} \right), \tag{2.3}$$

with equality if and only if $G \cong K_n$.

In [3] it was proven that

$$Kf^*(G) \geq 2m \left(\frac{1}{P} + \frac{(n-2)^2}{n-P} \right), \tag{2.4}$$

where

$$P = 1 + \sqrt{\frac{2R_{-1}}{n(n-1)}}.$$

Equality in (2.4) is attained if and only if $G \cong K_n$.

In [10] the following upper bound for $Kf^*(G)$ was established

$$Kf^*(G) \leq 2m \frac{(n-1)(\rho_1 + \rho_{n-1}) - n}{\rho_1 \rho_{n-1}}. \tag{2.5}$$

Let $a_{k_1}, a_{k_1+1}, \ldots, a_{n-k_2}$ and $b_{k_1}, b_{k_1+1}, \ldots, b_{n-k_2}$ be two non-negative real number sequences of similar monotonicity, and $p_{k_1}, p_{k_1+1}, \ldots, p_{n-k_2}$ be positive real number sequence, $1 \leq k_1 \leq n - k_2$, $0 \leq k_2 \leq n-1$. Then the following inequality holds (see, for example, [13, 14])

$$\sum_{i=k_1}^{n-k_2} p_i \sum_{i=k_1}^{n-k_2} p_i a_i b_i \geq \sum_{i=k_1}^{n-k_2} p_i a_i \sum_{i=k_1}^{n-k_2} p_i b_i. \tag{2.6}$$

3. Main Results

In the following theorem we establish an upper bound for $Kf^*(G)$ in terms of parameters n, m and k, $\rho_{n-1} \geq k > 0$. The inequality proven in this theorem is analogous to the inequality (2.1) for $Kf(G)$.

Theorem 3.1. *Let G be a simple connected graph with n, $n \geq 3$, vertices and m edges. Then, for any real k with the property $\rho_{n-1} \geq k > 0$, holds*

$$Kf^*(G) \leq m \frac{n-2+k(n-1)}{k}, \tag{3.1}$$

with equality if and only if $k = \frac{n}{n-1}$ and $G \cong K_n$, or $k = 1$ and $G \cong K_{r,n-r}$, $1 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor$, or $k = \frac{3}{n-1}$ and $G \cong C_n$.

Proof. Consider the function

$$f(x) = \frac{(n-1)(x+\rho_{n-1})-n}{x}, \quad x > 0.$$

This is an increasing function for $x > 0$. Since $0 < x = \rho_1 \leq 2$, we have $f(x) = f(\rho_1) \leq f(2)$, and according to (2.5) we get

$$Kf^*(G) \leq 2m\frac{(n-1)(2+\rho_{n-1})-n}{2\rho_{n-1}} = m\frac{(n-1)\rho_{n-1}+n-2}{\rho_{n-1}}. \tag{3.2}$$

Now, consider the function

$$g(x) = \frac{n-2}{x}, \quad x > 0.$$

This is a decreasing function for $x > 0$. Since $x = \rho_{n-1} \geq k > 0$, then $g(\rho_{n-1}) \leq g(k)$, and therefore from (3.2) we get

$$Kf^*(G) \leq m\frac{n-2+k(n-1)}{k},$$

which completes the proof. □

Corollary 3.1. *Let G be a simple connected planar graph with n, $n \geq 3$, vertices and m edges. Then, for any real k with the property $\rho_{n-1} \geq k > 0$, holds*

$$Kf^*(G) \leq \frac{3(n-2)(n-2+(n-1)k)}{k}, \tag{3.3}$$

with equality if and only if $k = \frac{3}{2}$ and $G \cong K_3$, or $k = \frac{4}{3}$ and $G \cong K_4$.

Proof. Since for simple connected planar graphs holds $m \leq 3(n-2)$, the inequality (3.3) directly follows from (3.1). □

In the next theorem we determine a lower bound for the degree Kirchhoff index in terms of parameters n, m and s, where s is a real number with the property $\rho_1 \geq s \geq \rho_{n-1}$.

Theorem 3.2. *Let G be a simple connected graph with n, $n \geq 3$, vertices and m edges. Then, for any real s with the property $\rho_1 \geq s \geq \rho_{n-1}$, holds*

$$Kf^*(G) \geq 2m\left(\frac{1}{s}+\frac{(n-2)^2}{n-s}\right), \tag{3.4}$$

with equality if and only if $s = \frac{n}{n-1}$ and $G \cong K_n$, or $s = 2$ and $G \cong K_{r,n-r}$, $1 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. For $k_1 = 1$, $k_2 = 2$, $p_i = \rho_i$, $a_i = b_i = \frac{1}{\rho_i}$, $i = 1, 2, \dots, n-2$, the inequality (2.6) becomes

$$\sum_{i=1}^{n-2} \rho_i \sum_{i=1}^{n-2} \frac{1}{\rho_i} \geq \left(\sum_{i=1}^{n-2} 1\right)^2,$$

i.e.

$$(n-\rho_{n-1})\left(\sum_{i=1}^{n-1}\frac{1}{\rho_i}-\frac{1}{\rho_{n-1}}\right)\geq (n-2)^2,$$

wherefrom follows

$$Kf^*(G)\geq 2m\left(\frac{1}{\rho_{n-1}}+\frac{(n-2)^2}{n-\rho_{n-1}}\right), \tag{3.5}$$

with equality if and only if $\rho_1=\rho_2=\cdots=\rho_{n-2}$.

For $k_1=2$, $k_2=1$, $p_i=\rho_i$, $a_i=b_i=\frac{1}{\rho_i}$, $i=2,3,\ldots,n-1$, the inequality (2.6) transforms into

$$\sum_{i=2}^{n-1}\rho_i\sum_{i=2}^{n-1}\frac{1}{\rho_i}\geq\left(\sum_{i=2}^{n-1}1\right)^2,$$

i.e.,

$$\left(\sum_{i=1}^{n-1}\rho_i-\rho_1\right)\left(\sum_{i=1}^{n-1}\frac{1}{\rho_i}-\frac{1}{\rho_1}\right)\geq (n-2)^2,$$

wherefrom follows

$$Kf^*(G)\geq 2m\left(\frac{1}{\rho_1}+\frac{(n-2)^2}{n-\rho_1}\right), \tag{3.6}$$

with equality if and only if $\rho_2=\rho_3=\cdots=\rho_{n-1}$.

Consider the function

$$f(x)=\frac{1}{x}+\frac{(n-2)^2}{n-x}.$$

This is an increasing function for $x\geq\frac{n}{n-1}$, monotone decreasing for $x\leq\frac{n}{n-1}$ and has a minimum for $x=\frac{n}{n-1}$. Then for any real s, such that $\rho_1\geq s\geq\frac{n}{n-1}$, from (3.6) follows

$$Kf^*(G)\geq 2mf(\rho_1)\geq 2mf(s)=2m\left(\frac{1}{s}+\frac{(n-2)^2}{n-s}\right).$$

Also, for any real s with the property $\frac{n}{n-1}\geq s\geq\rho_{n-1}$, from (3.5) follows

$$Kf^*(G)\geq 2mf(\rho_{n-1})\geq 2mf(s)=2m\left(\frac{1}{s}+\frac{(n-2)^2}{n-s}\right).$$

Finally, from the last two inequalities we obtain (3.4). □

According to (3.5) and (3.6) we have the following corollary of Theorem 3.2.

Corollary 3.2. *Let G be a simple connected graph with n, $n\geq 3$, vertices and m edges. Then*

$$Kf^*(G)\geq 2m\max\left\{\frac{1}{\rho_1}+\frac{(n-2)^2}{n-\rho_1},\frac{1}{\rho_{n-1}}+\frac{(n-2)^2}{n-\rho_{n-1}}\right\},$$

with equality if and only if $G\cong K_n$, or $G\cong K_{r,n-r}$, $1\leq r\leq\left\lfloor\frac{n}{2}\right\rfloor$.

Remark 3.1. In [3] the following inequalities were proven

$$\rho_1 \geq P \geq \frac{\Delta+1}{\Delta} \geq \frac{n}{n-1} \geq \rho_{n-1}.$$

Now, for $s = \frac{n}{n-1}$ from (3.4) the inequality (2.2) is obtained, for $s = \frac{\Delta+1}{\Delta}$ we get (2.3), and for $s = P$ it follows (2.4).

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