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PROPERTIES OF (C,r)-HANKEL OPERATORS AND (R,r)-HANKEL OPERATORS ON HILBERT SPACES

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Abstract. We introduce the operators which are generalizations of Hankel-type operators, called the (C,r)-Hankel operator and (R,r)-Hankel operator on general Hilbert spaces. Our main result is to obtain characterizations for a bounded operator on general Hilbert spaces to be a (C,r)-Hankel operator (or (R,r)-Hankel operator). We also discuss some algebraic properties like boundedness (for $|r| \neq 1$) of these operators and the relationship between them. Moreover, some characterizations for the commutativity of these operators are explored.

1. Introduction

The notion of Hankel matrices made its first appearance in 1861 when Hankel began the study of finite matrices with entries being a function of the sum of the coordinates only [6], the Hilbert matrix being the most prominent example of the same [2]. Hankel, Kronecker, Nehari and Hartman are the most celebrated names in this area for their contribution towards the most classical results about Hankel operators. For a pivot study on Hankel operators, one can refer [3, 5, 13].

Since inception, a lot of research has been done on this class of matrices, corresponding operators and associated variants due to their high scores of applications in the fields of perturbation theory, interpolation process, rational approximation, probability, moment problems, theory of systems and control etc. (refer [11–13]). The rapid development of this domain has led to numerous generalizations both in terms of twists in the operator form as well as the space of play. To adduce a few, Hankel operators,

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slant Hankel operators, essentially Hankel operators, λ-Hankel operators, weighted Hankel operators, small Hankel operators, slant little-Hankel operators, essentially slant-Hankel operators, kth-order slant Hankel operators etc. have been studied on different spaces like Hardy spaces, Bergmann spaces, Fock spaces, weighted Fock spaces, Harmonic Dirichlet spaces and so on [1, 3, 4, 8–10, 14] and references therein.

Recently, Mirotin et al. introduced the idea of μ-Hankel operators on Hilbert spaces in the following way and discussed this class on Hardy space in particular [10]: Let μ be a complex number, $\alpha = (\alpha_n)_{n\geq 0}$ be a sequence of complex numbers, H and H' be separable Hilbert spaces. The operator $A_{\mu,\alpha} : H \to H'$ is called μ-Hankel operator if for some orthonormal bases $(e_k)_{k\geq 0} \subset H$ and $(e'_j)_{j\geq 0} \subset H'$, the matrix $(a_{jk})_{k,j\geq 0}$ of this operator consists of elements of the form $a_{jk} = \mu^k \alpha_{j+k}$. All these developments motivated the authors to define two new classes of operators on general Hilbert spaces that are closely related to Hankel operators in the sense that these classes result in Hankel-type operators if alternate columns of one or alternate rows of the other are deleted. Interesting results are established to derive the connection between these classes, over and above the discussion of their algebraic properties. Characterizations are obtained for which these operators commute. It is also proved that these classes neither contain any Fredholm operator nor unitary operator.

We begin with the following preliminaries.

A bounded linear operator T on a Hilbert space H is said to be Hilbert-Schmidt operator if the Hilbert-Schmidt norm $\|T\|^2_{HS} = \sum_n \|T(u_n)\|^2 < +$ for an orthonormal basis $(u_n)_{n\in\mathbb{N}_0}$ of H, where $\|\cdot\|$ represents the norm of H. A bounded operator T on H is said to be a Fredholm operator if Range of T is closed, dimension of kernel T and dimension of kernel T^* are finite. In this case, index of T is defined as

$$\operatorname{index} T = \dim\ker T - \dim\ker T^*.$$

A bounded operator T on H is said to be isometry if $T^*T = I_H$, and unitary if T is bijective and $T^*T = TT^* = I_H$, where I_H denotes the identity operator on H. Throughout the paper, we restrict the symbols H_1 and H_2 for any separable Hilbert spaces. If $H_1 = H_2$, then it is denoted by H. We denote by $(u_i)_{i\in\mathbb{N}_0}$ and $(v_i)_{i\in\mathbb{N}_0}$, the orthonormal bases for H_1 and H_2, respectively. The symbols U_1 and U_2 denote the right shift operators on H_1 and H_2, respectively and are defined as $U_1(u_i) = u_{i+1}$ and $U_2(v_i) = v_{i+1}$ for all $i \in \mathbb{N}_0$. The symbols $\mathbb{C}$, $\mathbb{Z}$ and $\mathbb{N}_0$ denote the set of all complex numbers, integers and non-negative integers, respectively.

2. The (C, r)-Hankel Operator and (R, r)-Hankel Operator

We now introduce (C, r)-Hankel operators and (R, r)-Hankel operators on general Hilbert spaces as under.

Definition 2.1. Let r be a non-zero complex number and $(\alpha_n)_{n\in\mathbb{N}_0}$ be a sequence of complex numbers. Then the operator (C, r)-Hankel operator, $C_{r,\alpha}$ from a Hilbert space H_1 to Hilbert space H_2 is defined as

$$C_{r,\alpha}(u_i) = \sum_{j=0}^{+\infty} r^i \alpha_{i+2j} v_j, \quad \text{for all } i \in \mathbb{N}_0,$$

where $(u_i)_{i\in\mathbb{N}_0}$ and $(v_i)_{i\in\mathbb{N}_0}$ are orthonormal bases for H_1 and H_2, respectively.

For $i,j \in \mathbb{N}_0$, the $(i,j)^{\text{th}}$-entry of the matrix representation of $C_{r,\alpha}$ with respect to the orthonormal bases is $C_{i,j}$, where

$$C_{i,j} = \langle C_{r,\alpha}(u_j), v_i\rangle = \left\langle \sum_{l=0}^{+\infty} r^j \alpha_{j+2l} v_l, v_i \right\rangle = \sum_{l=0}^{+\infty} r^j \alpha_{j+2l} \langle v_l, v_i\rangle = r^j \alpha_{j+2i},$$

and hence, the corresponding matrix is given as:

$$[C_{r,\alpha}] = \begin{bmatrix} \alpha_0 & r\alpha_1 & r^2\alpha_2 & r^3\alpha_3 & r^4\alpha_4 & \cdots \\ \alpha_2 & r\alpha_3 & r^2\alpha_4 & r^3\alpha_5 & r^4\alpha_6 & \cdots \\ \alpha_4 & r\alpha_5 & r^2\alpha_6 & r^3\alpha_7 & r^4\alpha_8 & \cdots \\ \alpha_6 & r\alpha_7 & r^2\alpha_8 & r^3\alpha_9 & r^4\alpha_{10} & \cdots \\ \alpha_8 & r\alpha_9 & r^2\alpha_{10} & r^3\alpha_{11} & r^4\alpha_{12} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Definition 2.2. Let r be a non-zero complex number and $(\alpha_n)_{n\in\mathbb{N}_0}$ be a sequence of complex numbers. Then the operator (R,r)-Hankel operator, $R_{r,\alpha}$ from a Hilbert space H_1 to Hilbert space H_2 is defined as

$$R_{r,\alpha}(u_i) = \sum_{j=0}^{+\infty} r^i \alpha_{2i+j} v_j, \quad \text{for all } i \in \mathbb{N}_0,$$

where $(u_i)_{i\in\mathbb{N}_0}$ and $(v_i)_{i\in\mathbb{N}_0}$ are orthonormal bases for H_1 and H_2, respectively.

Observe that for $i,j \in \mathbb{N}_0$, if $R_{i,j}$ is the $(i,j)^{\text{th}}$-entry of the matrix representation of $R_{r,\alpha}$ with respect to the orthonormal bases, then

$$R_{i,j} = \langle R_{r,\alpha}(u_j), v_i\rangle = r^j \alpha_{2j+i},$$

and the corresponding matrix is given as:

$$[R_{r,\alpha}] = \begin{bmatrix} \alpha_0 & r\alpha_2 & r^2\alpha_4 & r^3\alpha_6 & r^4\alpha_8 & \cdots \\ \alpha_1 & r\alpha_3 & r^2\alpha_5 & r^3\alpha_7 & r^4\alpha_9 & \cdots \\ \alpha_2 & r\alpha_4 & r^2\alpha_6 & r^3\alpha_8 & r^4\alpha_{10} & \cdots \\ \alpha_3 & r\alpha_5 & r^2\alpha_7 & r^3\alpha_9 & r^4\alpha_{11} & \cdots \\ \alpha_4 & r\alpha_6 & r^2\alpha_8 & r^3\alpha_{10} & r^4\alpha_{12} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Note 2.1. **(A)** A (C,r)-Hankel operator becomes r^2-Hankel operator if its alternate columns are deleted and a (R,r)-Hankel operator becomes r-Hankel operator if its alternate rows are deleted.

(B) For every non-zero complex number r and complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$, the (C,r)-Hankel operator, $C_{r,\alpha}$ and (R,r)-Hankel operator, $R_{r,\alpha}$ may not correspond to bounded linear operators.

Example 2.1. Take $r = 1+i$, $\alpha_n = \frac{1}{\sqrt{n+1}}$, for all $n \in \mathbb{N}_0$, and $x = \sum_{n=0}^{+\infty} \frac{1}{(1+i)^n} u_n \in H$. Then,

$$\|x\|^2 = \sum_{n=0}^{+\infty} |x_n|^2 = \sum_{n=0}^{+\infty} \left| \frac{1}{(1+i)^n} \right|^2$$

is finite whereas

$$\|C_{r,\alpha}(x)\|^2 = \sum_{j=0}^{+\infty} \left| \sum_{n=0}^{+\infty} \frac{1}{(1+i)^n} r^n \alpha_{n+2j} \right|^2 = \sum_{j=0}^{+\infty} \left| \sum_{n=0}^{+\infty} \frac{1}{\sqrt{n+2j+1}} \right|^2 \to +\infty$$

and

$$\|R_{r,\alpha}(x)\|^2 = \sum_{j=0}^{+\infty} \left| \sum_{n=0}^{+\infty} \frac{1}{(1+i)^n} r^n \alpha_{2n+j} \right|^2 = \sum_{j=0}^{+\infty} \left| \sum_{n=0}^{+\infty} \frac{1}{\sqrt{2n+j+1}} \right|^2 \to +\infty.$$

3. Boundedness of (C,r)-Hankel Operators and (R,r)-Hankel Operators

In this section, we study conditions under which these operators become bounded. Characterizations of these operators are also derived.

Theorem 3.1. *Let r be a non-zero complex number such that $|r| < 1$ and $(\alpha_n)_{n\in\mathbb{N}_0}$ be a complex sequence. Then the following hold.*

(A) *The operator $C_{r,\alpha} : H_1 \to H_2$ is bounded if and only if $\sum_{n\in\mathbb{N}_0} |\beta_n|^2 < +\infty$ where*

$$\beta_n = \begin{cases} \alpha_n, & \text{if } n \text{ is even,} \\ r\alpha_n, & \text{if } n \text{ is odd.} \end{cases} \tag{3.1}$$

(B) *The operator $R_{r,\alpha} : H_1 \to H_2$ is bounded if and only if $\sum_{n=0}^{+\infty} |\alpha_n|^2 < +\infty$.*

Proof. **(A)** Let $|r| < 1$. If $C_{r,\alpha}$ is bounded, then there exists a positive constant C such that $\|C_{r,\alpha}(x)\|^2 \le C\|x\|^2$ for every $x \in H_1$. Take in particular $x = u_0$, we get $\sum_{n\in\mathbb{N}_0} |\alpha_{2n}|^2 = \|C_{r,\alpha}(u_0)\|^2 \le C\|u_0\|^2 = C$. Again, taking $x = u_1$, it follows that $|r|^2 \sum_{n\in\mathbb{N}_0} |\alpha_{2n+1}|^2 = \|C_{r,\alpha}(u_1)\|^2 \le C\|u_1\|^2 = C$. Therefore, $\sum_{n\in\mathbb{N}_0} |\beta_n|^2 = \sum_{n\in\mathbb{N}_0} |\alpha_{2n}|^2 + |r|^2 \sum_{n\in\mathbb{N}_0} |\alpha_{2n+1}|^2$ is finite.

Conversely, suppose that (3.1) holds. Consider

$$\begin{aligned} \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} |C_{i,j}|^2 &= \sum_{n=0}^{+\infty} |\alpha_{2n}|^2 \left(1 + |r^2|^2 + |r^4|^2 + \cdots + |r^{2n}|^2\right) + \sum_{n=0}^{+\infty} |\alpha_{2n+1}|^2 \\ &\quad \times \left(|r|^2 + |r^3|^2 + |r^5|^2 + \cdots + |r^{2n+1}|^2\right) \\ &= \sum_{n=0}^{+\infty} |\alpha_{2n}|^2 \left(1 + |r|^4 + |r|^8 + \cdots + |r|^{4n}\right) + \sum_{n=0}^{+\infty} |\alpha_{2n+1}|^2 \Big(|r|^2 + |r|^6 \end{aligned}$$

$$
\begin{aligned}
&+ |r|^{10} + \cdots + |r|^{2(2n+1)}\Big) \\
&= \sum_{n=0}^{+\infty} |\alpha_{2n}|^2 \Big(1 + |r|^4 + (|r|^4)^2 + \cdots + (|r|^4)^n\Big) + |r|^2 \sum_{n=0}^{+\infty} |\alpha_{2n+1}|^2 \Big(1 + |r|^4 \\
&+ (|r|^4)^2 + \cdots + (|r|^4)^n\Big) \\
&= \left(\sum_{n=0}^{+\infty} |\alpha_{2n}|^2 + |r|^2 \sum_{n=0}^{+\infty} |\alpha_{2n+1}|^2\right) (1 + |r|^4 + \Big(|r|^4)^2 + \cdots + (|r|^4)^n\Big) \\
&= \left(\sum_{n=0}^{+\infty} |\alpha_{2n}|^2 + |r|^2 \sum_{n=0}^{+\infty} |\alpha_{2n+1}|^2\right) \left(\frac{1 - |r|^{4n}}{1 - |r|^4}\right) \\
&\leq \left(\frac{1}{1 - |r|^4}\right) \left(\sum_{n=0}^{+\infty} |\alpha_{2n}|^2 + |r|^2 \sum_{n=0}^{+\infty} |\alpha_{2n+1}|^2\right) \\
&= \left(\frac{1}{1 - |r|^4}\right) \left(\sum_{n=0}^{+\infty} |\beta_n|^2\right).
\end{aligned}
$$

Using (3.1), it follows that $\sum_{i=0}^{+\infty} \sum_{j=0}^{+\infty} |C_{i,j}|^2 < +\infty$. Therefore, the operator $C_{r,\alpha}$ is Hilbert-Schmidt and hence bounded.

(B) Let $|r| < 1$ and $R_{r,\alpha}$ be bounded, then there exists a positive constant C such that $\|R_{r,\alpha}(x)\|^2 \leq C\|x\|^2$ for every $x \in H_1$. Taking in particular $x = u_0$, we get $\sum_{n\in\mathbb{N}_0} |\alpha_n|^2 = \|C_{r,\alpha}(u_0)\|^2 \leq C\|u_0\|^2 = C$.

Conversely, suppose that $\sum_{n=0}^{+\infty} |\alpha_n|^2 < +\infty$. Consider

$$
\begin{aligned}
\sum_{i=0}^{+\infty} \sum_{j=0}^{+\infty} |R_{i,j}|^2 &= \sum_{n=0}^{+\infty} |\alpha_{2n}|^2 \Big(1 + |r|^2 + |r^2|^2 + \cdots + |r^n|^2\Big) + \sum_{n=0}^{+\infty} |\alpha_{2n+1}|^2 \\
&\quad \times \Big(1 + |r|^2 + |r^2|^2 + \cdots + |r^n|^2\Big) \\
&= \left(\sum_{n=0}^{+\infty} |\alpha_{2n}|^2 + \sum_{n=0}^{+\infty} |\alpha_{2n+1}|^2\right) \Big(1 + |r|^2 + |r|^4 + \cdots + |r|^{2n}\Big) \\
&= \left(\sum_{n=0}^{+\infty} |\alpha_{2n}|^2 + \sum_{n=0}^{+\infty} |\alpha_{2n+1}|^2\right) \Big(1 + |r|^2 + (|r|^2)^2 + \cdots + (|r|^2)^n\Big) \\
&= \left(\sum_{n=0}^{+\infty} |\alpha_{2n}|^2 + \sum_{n=0}^{+\infty} |\alpha_{2n+1}|^2\right) \left(\frac{1 - |r|^{2n}}{1 - |r|^2}\right) \\
&\leq \left(\frac{1}{1 - |r|^2}\right) \left(\sum_{n=0}^{+\infty} |\alpha_{2n}|^2 + \sum_{n=0}^{+\infty} |\alpha_{2n+1}|^2\right) \\
&= \left(\frac{1}{1 - |r|^2}\right) \left(\sum_{n=0}^{+\infty} |\alpha_n|^2\right).
\end{aligned}
$$

Using $\sum_{n=0}^{+\infty} |\alpha_n|^2 < +\infty$, it follows that $\sum_{i=0}^{+\infty} \sum_{j=0}^{+\infty} |R_{i,j}|^2 < +\infty$. Therefore, the operator $R_{r,\alpha}$ is Hilbert-Schmidt and hence bounded. □

The next theorem gives characterizations of bounded linear (C, r)-Hankel and (R, r)-Hankel operators in terms of operator equations involving shift operator.

Theorem 3.2. *Let U_1 and U_2 be the right shift operators on H_1 and H_2, respectively. Let r be a non-zero complex number. Then the following hold.*

- **(A)** *A bounded operator $T : H_1 \to H_2$ is a (C, r)-Hankel operator for some complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$ if and only if $TU_1^2 = r^2U_2^*T$.*
- **(B)** *A bounded operator $T : H_1 \to H_2$ is a (R, r)-Hankel operator for some complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$ if and only if $TU_1^2 = r^2(U_2^4)^*T$ and $T_{i,1} = rT_{i+2,0}$ for all $i \in \mathbb{N}_0$, where $(T_{i,j})$ represents matrix representation of T with respect to orthonormal bases of H_1 and H_2, respectively.*

Proof. **(A)** Suppose $T : H_1 \to H_2$ is a (C, r)-Hankel operator for some complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$. For each $i, j \in \mathbb{N}_0$,

$$\langle TU_1^2(u_i), v_j\rangle = \langle T(u_{i+2}), v_j\rangle = r^{i+2}\alpha_{i+2+2j}$$

and

$$\langle r^2U_2^*T(u_i), v_j\rangle = r^2\langle T(u_i), U_2(v_j)\rangle = r^2\langle T(u_i), v_{j+1}\rangle = r^2r^i\alpha_{i+2+2j} = r^{i+2}\alpha_{i+2+2j}.$$

Using the boundedness of T, it follows that $TU_1^2 = r^2U_2^*T$.

Conversely, let $TU_1^2 = r^2U_2^*T$. We define a complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$ as follows:

$$\alpha_n = \begin{cases} \langle T(u_0), v_{n/2}\rangle, & \text{if } n \text{ is even,} \\ (1/r)\langle T(u_1), v_{(n-1)/2}\rangle, & \text{elsewhere.} \end{cases} \tag{3.2}$$

Then, for all non-negative integers i, j such that $i \geq 2$,

$$\begin{aligned} \langle T(u_i), v_j\rangle &= \langle TU_1^2(u_{i-2}), v_j\rangle = \langle r^2U_2^*T(u_{i-2}), v_j\rangle = r^2\langle T(u_{i-2}), U_2(v_j)\rangle \\ &= r^2\langle T(u_{i-2}), v_{j+1}\rangle = \cdots = r^4\langle T(u_{i-4}), v_{j+2}\rangle = \cdots = \\ &= \begin{cases} r^i\langle T(u_0), v_{j+i/2}\rangle, & \text{if } i \text{ is even,} \\ r^{i-1}\langle T(u_1), v_{j+(i-1)/2}\rangle, & \text{if } i \text{ is odd,} \end{cases} \\ &= \begin{cases} r^i\alpha_{2j+i}, & \text{if } i \text{ is even,} \\ r^{i-1}r\alpha_{2j+i}, & \text{if } i \text{ is odd,} \end{cases} \\ &= r^i\alpha_{2j+i}. \end{aligned}$$

Hence, $T = C_{r,\alpha}$ for the sequence $(\alpha_n)_{n\in\mathbb{N}_0}$ defined in (3.2).

(B) Suppose $T : H_1 \to H_2$ is a (R, r)-Hankel operator for some complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$. Clearly, $T_{i,1} = rT_{i+2,0}$ for all $i \in \mathbb{N}_0$. Now, for each $i, j \in \mathbb{N}_0$,

$$\langle TU_1^2(u_i), v_j\rangle = \langle T(u_{i+2}), v_j\rangle = r^{i+2}\alpha_{2i+4+j}$$

and

$$\langle r^2(U_2^4)^*T(u_i), v_j\rangle = r^2\langle T(u_i), U_2^4(v_j)\rangle = r^2\langle T(u_i), v_{j+4}\rangle = r^2r^i\alpha_{2i+j+4} = r^{i+2}\alpha_{2i+4+j}.$$

Using the boundedness of T, it follows that $TU_1^2 = r^2(U_2^4)^*T$.

Conversely, suppose that $TU_1^2 = r^2(U_2^4)^*T$ and

$$T_{i,1} = rT_{i+2,0}, \quad \text{for all } i \in \mathbb{N}_0, \tag{3.3}$$

where $(T_{i,j})$ represents matrix representation of the operator T. For each $n \in \mathbb{N}_0$, let

$$\alpha_n = \begin{cases} \langle T(u_0), v_n\rangle, & \text{if } n \text{ is even},\\ \langle T(u_0), v_1\rangle, & \text{if } n = 1,\\ (1/r)\langle T(u_1), v_{n-2}\rangle, & \text{elsewhere}. \end{cases} \tag{3.4}$$

Then $(\alpha_n)_{n\in\mathbb{N}_0}$ is a sequence in the complex plane. Using (3.3) and (3.4), for all non-negative integers i, j such that $i \geq 2$, evaluating

$$\begin{aligned}
\langle T(u_i), v_j\rangle &= \langle TU_1^2(u_{i-2}), v_j\rangle = \langle r^2(U_2^4)^*T(u_{i-2}), v_j\rangle = r^2\langle T(u_{i-2}), U_2^4(v_j)\rangle\\
&= r^2\langle T(u_{i-2}), v_{j+4}\rangle = \cdots = r^4\langle T(u_{i-4}), v_{j+8}\rangle = \cdots =\\
&= \begin{cases} r^i\langle T(u_0), v_{j+2i}\rangle, & \text{if } i \text{ is even},\\ r^{i-1}\langle T(u_1), v_{j+2(i-1)}\rangle, & \text{if } i \text{ is odd}, \end{cases}\\
&= \begin{cases} r^i\alpha_{j+2i}, & \text{if } i, j \text{ both are even},\\ r^{i-1}\langle T(u_1), v_{j+2i-2)}\rangle, & \text{if } i \text{ is even and } j \text{ is odd},\\ r^{i-1}r\alpha_{j+2i}, & \text{and } i, j \text{ both are odd},\\ r^{i-1}r\langle T(u_0), v_{j+2i}\rangle, & \text{if } i \text{ is odd and } j \text{ is even}, \end{cases}\\
&= r^i\alpha_{j+2i}.
\end{aligned}$$

Hence, $T = R_{r,\alpha}$ for complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$. □

Proposition 3.1. *Let r be a non-zero complex number and $(\alpha_n)_{n\in\mathbb{N}_0} \subset \mathbb{C}$ be a sequence. Then the adjoint of bounded (C,r)-Hankel operator, $C_{r,\alpha} : H_1 \to H_2$ is the (R,s)-Hankel operator, $R_{s,\beta}$ from H_2 to H_1, where $s = \frac{1}{r^2}$ and $\beta_n = \overline{r^n}\overline{\alpha_n}$ for each $n \in \mathbb{N}_0$.*

Proof. Let $i, j \in \mathbb{N}_0$. Evaluating

$$\langle C_{r,\alpha}^*(v_j), u_i\rangle = \langle v_j, C_{r,\alpha}(u_i)\rangle = \overline{\langle C_{r,\alpha}(u_i), v_j\rangle} = \overline{r^i}\overline{\alpha_{i+2j}} = \overline{r}^i\overline{\alpha_{i+2j}}$$

and

$$\langle R_{s,\beta}(v_j), u_i\rangle = s^j\beta_{i+2j} = \left(\frac{1}{\overline{r^2}}\right)^j \overline{r^{i+2j}}\overline{\alpha_{i+2j}} = \overline{r}^i\overline{\alpha_{i+2j}}.$$

Hence, $C_{r,\alpha}^* = R_{s,\beta}$, where $s = \frac{1}{r^2}$ and $\beta_n = \overline{r^n}\overline{\alpha_n}$ for each $n \in \mathbb{N}_0$. □

Theorem 3.3. *Let U_1 and U_2 be the right shift operators on H_1 and H_2, respectively. Let r be a non-zero complex number. Then a bounded operator $T : H_1 \to H_2$ is a (R,r)-Hankel operator for some complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$ if and only if $TU_1 = r(U_2^*)^2T$.*

Proof. Suppose that T is a (R,r)-Hankel operator for some complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$. Using Proposition 3.1, it follows that $T = R_{r,\alpha} = C_{s,\beta}^*$, where $C_{s,\beta} : H_2 \to H_1$ is (C,s)-Hankel operator, $s = \left(\frac{1}{\overline{r}}\right)^{\frac{1}{2}}$ and $\beta_n = \left(\frac{1}{\overline{r^n}}\right)^{\frac{1}{2}}\overline{\alpha_n}$ for each $n \in \mathbb{N}_0$. Now, Theorem 3.2 gives $C_{s,\beta}U_2^2 = s^2 U_1^* C_{s,\beta}$. Taking adjoint on both sides, it follows that $(U_2^*)^2 C_{s,\beta}^* = \overline{s}^2 C_{s,\beta}^* U_1$. That is, $TU_1 = r(U_2^*)^2 T$.

Conversely, if an operator T is such that $TU_1 = r(U_2^*)^2 T$, then, by reversing the steps above and by using Theorem 3.2 and Proposition 3.1, we can conclude that T is a (R, r)-Hankel operator for some complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$. ☐

Corollary 3.1. *The kernel of* (R, r)*-Hankel operator is an invariant subspace of shift operator.*

Proposition 3.2. *For a non-zero complex number* $r \in \mathbb{C}$*, if an operator* T *is a* (C, r)*-Hankel operator as well as* (R, r)*-Hankel operator on* H *for some complex sequence* $(\alpha_n)_{n\in\mathbb{N}_0}$*, then* U^*T *is* r*-Toeplitz operator on* H*, where* U *is the right shift operator on* H*.*

Proof. Suppose that T is a (C, r)-Hankel operator as well as (R, r)-Hankel operator on H for some complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$. Since T is (C, r)-Hankel operator, therefore, by using Theorem 3.2, it follows that

$$TU^2 = r^2 U^* T. \tag{3.5}$$

Also, T is (R, r)-Hankel operator, therefore, Proposition 3.3 gives

$$TU = r(U^*)^2 T. \tag{3.6}$$

Using (3.5) and (3.6), we obtain that

$$r^2 U^* T = TU^2 = (TU)U = r(U^*)^2 TU = rU^*(U^*T)U.$$

This implies that $U^*(U^*T)U = r(U^*T)$ which means that U^*T is r-Toeplitz operator [7] on H. ☐

In Theorem 3.1, boundedness conditions of these operators for the case $|r| < 1$ have been discussed. We discuss boundedness of these operators for $|r| > 1$ in the next result.

Theorem 3.4. *Let* r *be a non-zero complex number such that* $|r| > 1$ *and* $(\alpha_n)_{n\in\mathbb{N}_0}$ *be a complex sequence. Then the following hold.*

(A) *The operator* $C_{r,\alpha} : H_1 \to H_2$ *is bounded if and only if*

$$\sum_{n=0}^{+\infty} |r|^{2n} |\alpha_n|^2 < +\infty.$$

(B) *Then the operator* $R_{r,\alpha} : H_1 \to H_2$ *is bounded if and only if*

$$\sum_{n=0}^{+\infty} |\gamma_n|^2 < +\infty,$$

where

$$\gamma_n = \begin{cases} \left(\frac{1}{r^n}\right)^{\frac{1}{2}} \overline{\alpha_n}, & \text{if } n \text{ is even,} \\ \left(\frac{1}{r^{n+1}}\right)^{\frac{1}{2}} \overline{\alpha_n}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $|r| > 1$ and $(\alpha_n)_{n\in\mathbb{N}_0}$ be a complex sequence.

(A) Let $s = \frac{1}{r^2}$ and $(\beta_n)_{n\in\mathbb{N}_0}$ be a sequence, where $\beta_n = \overline{r^n}\,\overline{\alpha_n}$ for each $n \in \mathbb{N}_0$. The operator $C_{r,\alpha}$ is bounded if and only if $C^*_{r,\alpha}$ is bounded. Using Proposition 3.1, it follows that the operator $C^*_{r,\alpha}$ is bounded if and only if $R_{s,\beta}$ is bounded. Since $|s| < 1$, therefore, using Theorem 3.1 (B), it is concluded that $R_{s,\beta}$ is bounded if and only if $\sum_{n=0}^{+\infty} |\beta_n|^2 < +\infty$, that is, $\sum_{n=0}^{+\infty} |r|^{2n}|\alpha_n|^2 < +\infty$.

(B) Let $s = \left(\frac{1}{r}\right)^{\frac{1}{2}}$ and $\beta_n = \left(\frac{1}{r^n}\right)^{\frac{1}{2}} \overline{\alpha_n}$ for each $n \in \mathbb{N}_0$. Since $|r| > 1$, so $|s| < 1$. The operator $R_{r,\alpha}$ is bounded if and only if $R^*_{r,\alpha}$ is bounded. Using Proposition 3.1, it follows that the operator $R^*_{r,\alpha}$ is bounded if and only if $C_{s,\beta}$ is bounded. Since $|s| < 1$, therefore, using Theorem 3.1 (A), it gives $C_{s,\beta}$ is bounded if and only if $\sum_{n=0}^{+\infty} |\gamma_n|^2 < +\infty$, where

$$\gamma_n = \begin{cases} \beta_n, & \text{if } n \text{ is even,} \\ s\beta_n, & \text{if } n \text{ is odd.} \end{cases}$$

That is,

$$\gamma_n = \begin{cases} \left(\frac{1}{r^n}\right)^{\frac{1}{2}} \overline{\alpha_n}, & \text{if } n \text{ is even,} \\ \left(\frac{1}{r^{n+1}}\right)^{\frac{1}{2}} \overline{\alpha_n}, & \text{if } n \text{ is odd.} \end{cases}$$

□

For $r \in \mathbb{C}\backslash\{0\}$, let $\mathcal{C}_r(H_1, H_2)$ and $\mathcal{R}_r(H_1, H_2)$ denote the classes of all bounded (C, r)-Hankel operators and (R, r)-Hankel operators, respectively defined from H_1 to H_2. They are denoted by $\mathcal{C}_r(H)$ and $\mathcal{R}_r(H)$ if $H_1 = H_2$. It can easily be seen that the classes $\mathcal{C}_r(H_1, H_2)$ and $\mathcal{R}_r(H_1, H_2)$ are weakly closed and hence strongly closed, vector subspaces of the space $\mathcal{B}(H_1, H_2)$, where $\mathcal{B}(H_1, H_2)$ is the class of all bounded linear operators from H_1 to H_2.

Proposition 3.3. *Let $r \in \mathbb{C}\backslash\{0\}$. Then there does not exist any Fredholm operator in the classes $\mathcal{C}_r(H_1, H_2)$ and $\mathcal{R}_r(H_1, H_2)$.*

Proof. Suppose that there exist a Fredholm (C, r)-Hankel operator, $C_{r,\alpha}$ in $\mathcal{C}_r(H_1, H_2)$ for some complex sequence $(\alpha_n)_{n\in\mathbb{N}_0}$, whose index is n. Using Theorem 3.2 (A), it follows that $C_{r,\alpha}U_1^2 = r^2U_2^*C_{r,\alpha}$, where U_1 and U_2 are right shift operators on H_1 and H_2, respectively. Since $C_{r,\alpha}$ is Fredholm of index n, this implies that $C_{r,\alpha}U_1^2$ is Fredholm of index $n - 2$. On the other hand, $r^2U_2^*C_{r,\alpha}$ is Fredholm of index $n + 1$. This means that $n - 2 = n + 1$ which is a contradiction. Hence, there does not exist any Fredholm operator in the class $\mathcal{C}_r(H_1, H_2)$.

Similarly, using Theorem 3.2 (B), one can obtain that there does not exist any Fredholm operator in the class $\mathcal{R}_r(H_1, H_2)$. □

4. Commutativity of (C, r)-Hankel Operators and (R, r)-Hankel Operators

This section is devoted to explore the characterizations for commutativity of operators in $\mathcal{C}_r(H)$ and $\mathcal{R}_r(H)$.

Theorem 4.1. *Let r and s be non-zero complex numbers and $(\alpha_n)_{n\in\mathbb{N}_0}$ and $(\beta_n)_{n\in\mathbb{N}_0}$ be two complex sequences. Then the following hold.*

(A) *The bounded operators $C_{r,\alpha}$ and $C_{s,\beta}$ on Hilbert space H commute if and only if*

$$\sum_{j=0}^{+\infty} s^i\beta_{i+2j}r^j\alpha_{j+2k} = \sum_{j=0}^{+\infty} r^i\alpha_{i+2j}s^j\beta_{j+2k},$$

for all $i, k \in \mathbb{N}_0$, provided the series converge.

(B) *The bounded operators $C_{r,\alpha}$ and $R_{s,\beta}$ on H commute if and only if*

$$\sum_{j=0}^{+\infty} s^i\beta_{2i+j}r^j\alpha_{j+2k} = \sum_{j=0}^{+\infty} r^i\alpha_{i+2j}s^j\beta_{2j+k},$$

for all $i, k \in \mathbb{N}_0$, provided the series converge.

Proof. **(A)** For each $i \in \mathbb{N}_0$, consider

$$\begin{aligned}C_{r,\alpha}C_{s,\beta}(u_i) &= C_{r,\alpha}\left(\sum_{j=0}^{+\infty} s^i\beta_{i+2j}u_j\right) = \left(\sum_{j=0}^{+\infty} s^i\beta_{i+2j}C_{r,\alpha}(u_j)\right)\\ &= \left(\sum_{j=0}^{+\infty} s^i\beta_{i+2j}\left(\sum_{k=0}^{+\infty} r^j\alpha_{j+2k}u_k\right)\right)\\ &= \left(\sum_{k=0}^{+\infty}\left(\sum_{j=0}^{+\infty} s^i\beta_{i+2j}r^j\alpha_{j+2k}\right)u_k\right). \qquad (4.1)\end{aligned}$$

Similarly, we obtain that

$$C_{s,\beta}C_{r,\alpha}(u_i) = \left(\sum_{k=0}^{+\infty}\left(\sum_{j=0}^{+\infty} r^i\alpha_{i+2j}s^j\beta_{j+2k}\right)u_k\right). \qquad (4.2)$$

Since $(u_i)_{i\in\mathbb{N}_0}$ is an orthonormal basis for H, therefore, using (4.1) and (4.2), it follows that the bounded operators $C_{r,\alpha}$ and $C_{s,\beta}$ commute if and only if

$$\sum_{j=0}^{+\infty} s^i\beta_{i+2j}r^j\alpha_{j+2k} = \sum_{j=0}^{+\infty} r^i\alpha_{i+2j}s^j\beta_{j+2k},$$

for all $i, k \in \mathbb{N}_0$.

(B) For each $i \in \mathbb{N}_0$, evaluate

$$\begin{aligned}C_{r,\alpha}R_{s,\beta}(u_i) &= C_{r,\alpha}\left(\sum_{j=0}^{+\infty} s^i\beta_{2i+j}u_j\right) = \left(\sum_{j=0}^{+\infty} s^i\beta_{2i+j}C_{r,\alpha}(u_j)\right)\\ &= \left(\sum_{j=0}^{+\infty} s^i\beta_{2i+j}\left(\sum_{k=0}^{+\infty} r^j\alpha_{j+2k}u_k\right)\right)\\ &= \left(\sum_{k=0}^{+\infty}\left(\sum_{j=0}^{+\infty} s^i\beta_{2i+j}r^j\alpha_{j+2k}\right)u_k\right). \qquad (4.3)\end{aligned}$$

Similarly, it is obtained that

$$R_{s,\beta}C_{r,\alpha}(u_i) = \left(\sum_{k=0}^{+\infty}\left(\sum_{j=0}^{+\infty} r^i\alpha_{i+2j}s^j\beta_{2j+k}\right)u_k\right). \tag{4.4}$$

Using (4.3) and (4.4), it follows that the bounded operators $C_{r,\alpha}$ and $R_{s,\beta}$ commute if and only if

$$\sum_{j=0}^{+\infty} s^i\beta_{2i+j}r^j\alpha_{j+2k} = \sum_{j=0}^{+\infty} r^i\alpha_{i+2j}s^j\beta_{2j+k},$$

for all $i,k \in \mathbb{N}_0$. □

The following example demonstrates commuting operators in $\mathcal{C}_r(H)$.

Example 4.1. If $r = s = \frac{i}{2}$, $\alpha(n) = (\frac{i}{2})^n$ and $\beta(n) = \frac{i^n}{2^{n+1}}$ for all $n \in \mathbb{N}_0$, then one can easily see that the operators $C_{r,\alpha}$ and $C_{s,\beta}$ are bounded (using Theorem 3.1) and they satisfy the following expression:

$$\sum_{j=0}^{+\infty} s^i\beta_{i+2j}r^j\alpha_{j+2k} = \sum_{j=0}^{+\infty} r^i\alpha_{i+2j}s^j\beta_{j+2k},$$

for all $i,k \in \mathbb{N}_0$. Hence, the operators $C_{r,\alpha}$ and $C_{s,\beta}$ commute on H.

Let $\mathscr{C}_{0,0}$ denote the set of all complex sequences whose only finitely many terms are non-zero.

Theorem 4.2. *Let $r,s \in \mathbb{C}\backslash\{0\}$ and $\alpha,\beta \in \mathscr{C}_{0,0}$ be non-zero sequences, where $\alpha = (\alpha_j)_{j\in\mathbb{N}_0}$ and $\beta = (\beta_j)_{j\in\mathbb{N}_0}$. Let n and m be the largest non-negative integers such that*

$$\alpha_n \neq 0 \quad \textit{and} \quad \beta_m \neq 0.$$

Then the operators $R_{r,\alpha}$ and $R_{s,\beta}$ on Hilbert space H commute if and only if $n = m$, $r = s$ and there exists $\lambda \in \mathbb{C}$ such that $\beta_j = \lambda\alpha_j$ for all $j \in \mathbb{N}_0$.

Proof. Let the operators $R_{r,\alpha}$ and $R_{s,\beta}$ commute. That is,

$$R_{r,\alpha}R_{s,\beta}(x) = R_{s,\beta}R_{r,\alpha}(x), \tag{4.5}$$

for all $x \in H$. Two cases arise.

Case 1. If $n = m$. Let $n = 2p + r_1$ where $p \in \mathbb{N}_0$ and $r_1 = 0$ or 1. In particular, take $x = u_p$ in (4.5), we have

$$R_{r,\alpha}R_{s,\beta}(u_p) = R_{s,\beta}R_{r,\alpha}(u_p). \tag{4.6}$$

Subcase 1. If $r_1 = 0$. Consider

$$\begin{aligned} R_{r,\alpha}R_{s,\beta}(u_p) &= R_{r,\alpha}\left(\sum_{j=0}^{+\infty} s^p\beta_{2p+j}u_j\right) = s^p\beta_{2p}R_{r,\alpha}(u_0) = s^p\beta_{2p}\left(\sum_{j=0}^{+\infty}\alpha_j u_j\right) \\ &= s^p\beta_{2p}\left(\sum_{j=0}^{n}\alpha_j u_j\right) = \sum_{j=0}^{n}(s^p\beta_{2p}\alpha_j)u_j. \end{aligned} \tag{4.7}$$

Similarly, we can obtain that

$$R_{s,\beta}R_{r,\alpha}(u_p) = \sum_{j=0}^{n}(r^p\alpha_{2p}\beta_j)u_j. \tag{4.8}$$

Since $(u_j)_{j\in\mathbb{N}_0}$ is an orthonormal basis of H, therefore, using (4.6), (4.7) and (4.8), we get $s^p\beta_{2p}\alpha_j = r^p\alpha_{2p}\beta_j$ for all $0 \le j \le n$. Let $\lambda = \frac{\beta_n}{\alpha_n}$. This implies that $r^p = s^p$ and $\beta_j = \lambda\alpha_j$ for all $0 \le j \le n$.

Subcase 2. If $r_1 = 1$. Consider

$$\begin{aligned} R_{r,\alpha}R_{s,\beta}(u_p) &= R_{r,\alpha}\left(\sum_{j=0}^{+\infty} s^p\beta_{2p+j}u_j\right) = R_{r,\alpha}\left(\sum_{j=0}^{1} s^p\beta_{2p+j}u_j\right) \\ &= s^p\beta_{2p}\left(\sum_{j=0}^{+\infty}\alpha_j u_j\right) + s^p\beta_{2p+1}\left(\sum_{j=0}^{+\infty} r\alpha_{2+j}u_j\right) \\ &= s^p\beta_{2p}\left(\sum_{j=0}^{n}\alpha_j u_j\right) + s^p\beta_{2p+1}\left(\sum_{j=0}^{n-2} r\alpha_{2+j}u_j\right) \\ &= \left(\sum_{j=0}^{n-2} s^p\left(\beta_{2p}\alpha_j + \beta_{2p+1}r\alpha_{2+j}\right)u_j\right) + s^p\beta_{2p}\alpha_{n-1}u_{n-1} + s^p\beta_{2p}\alpha_n u_n. \end{aligned} \tag{4.9}$$

Similarly, we can obtain that

$$R_{s,\beta}R_{r,\alpha}(u_p) = \left(\sum_{j=0}^{n-2} r^p\left(\alpha_{2p}\beta_j + \alpha_{2p+1}s\beta_{2+j}\right)u_j\right) + r^p\alpha_{2p}\beta_{n-1}u_{n-1} + r^p\alpha_{2p}\beta_n u_n. \tag{4.10}$$

Again using the fact that the set $(u_j)_{j\in\mathbb{N}_0}$ is an orthonormal basis of H, therefore, using (4.6), (4.9) and (4.10), we get $s^p\beta_{2p}\alpha_n = r^p\alpha_{2p}\beta_n, s^p\beta_{2p}\alpha_{n-1} = r^p\alpha_{2p}\beta_{n-1}$ and $s^p\left(\beta_{2p}\alpha_j + \beta_{2p+1}r\alpha_{2+j}\right) = r^p\left(\alpha_{2p}\beta_j + \alpha_{2p+1}s\beta_{2+j}\right)$ for each $0 \le j \le n-2$. Let $\lambda = \frac{\beta_n}{\alpha_n}$. On solving successively, it follows that $s^p = r^p$ and $\beta_j = \lambda\alpha_j$ for all $0 \le j \le n$.

Using (4.5) at $x = u_1$, together with $s^p = r^p$ and $\beta_j = \frac{\beta_n}{\alpha_n}\alpha_j$ for all $0 \le j \le n$, one can obtain $s = r$ in both the subcases.

Case 2. If $n \ne m$. Without loss of generality, we can assume that $n > m$. Let $n = 2p + r_1$ and $m = 2q + r_2$, where $p, q \in \mathbb{N}_0$ and $r_1, r_2 \in \{0, 1\}$. In this case, we claim that the operators $R_{r,\alpha}$ and $R_{s,\beta}$ do not commute. Assume on the contrary that $R_{r,\alpha}$ and $R_{s,\beta}$ commute.

Subcase 1. If $m = 0$. Using (4.5), we get $R_{r,\alpha}R_{s,\beta}(u_0) = R_{s,\beta}R_{r,\alpha}(u_0)$ which gives $\beta_0\left(\sum_{j=0}^{n}\alpha_j u_j\right) = \beta_0\alpha_0 u_0$. On comparing the coefficients of u_n, it follows that $\beta_0\alpha_n = 0$, which is not possible as $\beta_0 \ne 0$ and $\alpha_n \ne 0$.

Subcase 2. If $m = 1$. Again using (4.5) for $x = u_0$, we get $\beta_0\left(\sum_{j=0}^{n}\alpha_j u_j\right) + \beta_1\left(\sum_{j=0}^{n-2} r\alpha_{j+2}u_j\right) = \sum_{j=0}^{1}\alpha_0\beta_j u_j$. On comparing the coefficients of u_n and u_{n-2}, we get $\beta_0\alpha_n = 0$ and $\beta_0\alpha_{n-2} + \beta_1 r\alpha_n = 0$, which is not possible as $\beta_1 \ne 0$ and $\alpha_n \ne 0$.

Subcase 3. If $m = 2q$. Take $x = u_q$ in (4.5), we get

$$s^q \beta_{2q} \sum_{j=0}^{n} \alpha_j u_j = \sum_{j=0}^{\min(n-m,q)} \sum_{k=0}^{m-2j} r^q s^j \alpha_{m+j} \beta_{2j+k} u_k.$$

On comparing the coefficients of u_n, it follows that $s^q \beta_{2q} \alpha_n = 0$. It follows that $\alpha_n = 0$, which is not true.

Subcase 4. If $m = 2q + 1$. Take $x = u_q$ in (4.5), we get $s^q \beta_{2q} \sum_{j=0}^{n} \alpha_j u_j + s^q \beta_{2q+1} \sum_{j=0}^{n-2} \alpha_{j+2} u_j = \sum_{j=0}^{\min(n-2q,q)} \sum_{k=0}^{m-2j} r^q s^j \alpha_{2q+j} \beta_{2j+k} u_k$. On comparing the coefficients of u_n, it follows that $s^q \beta_{2q} \alpha_n = 0$. It follows that $\alpha_n = 0$, which is not true.

Hence, from all the subcases, it follows that the operators $R_{r,\alpha}$ and $R_{s,\beta}$ can not commute. ☐

As a consequence of this result and by using Proposition 3.1, we get the following result.

Corollary 4.1. *Let $r, s \in \mathbb{C} \backslash \{0\}$ and $\alpha, \beta \in \mathscr{C}_{0,0}$ be non-zero sequences, where $\alpha = (\alpha_j)_{j \in \mathbb{N}_0}$ and $\beta = (\beta_j)_{j \in \mathbb{N}_0}$. Let n and m be the largest non-negative integers such that*

$$\alpha_n \neq 0 \quad \text{and} \quad \beta_m \neq 0.$$

Then the operators $C_{r,\alpha}$ and $C_{s,\beta}$ on Hilbert space H commute if and only if $n = m$, $r^2 = s^2$ and there exists $\lambda \in \mathbb{C}$ such that $s^j \beta_j = \lambda r^j \alpha_j$ for all $j \in \mathbb{N}_0$.

Now, we show that the class $\mathscr{C}_r(H)$ and hence, $\mathscr{R}_r(H)$ does not contain any unitary operator.

Proposition 4.1. *The class $\mathscr{C}_r(H)$ does not contain any unitary operator for any non-zero $r \in \mathbb{C}$.*

Proof. Suppose there exists unitary operator $C_{r,\alpha}$ in $\mathscr{C}_r(H)$ for some complex sequence $(\alpha_n)_{n \in \mathbb{N}_0}$. This implies that

$$\|C_{r,\alpha}(x)\|^2 = \|x\|^2 = \|C^*_{r,\alpha}(x)\|^2, \tag{4.11}$$

for all $x \in H$.

Case 1. If $|r| < 1$. For $x = u_0$ in (4.11), we get

$$\sum_{j=0}^{+\infty} |\alpha_{2j}|^2 = 1. \tag{4.12}$$

Now take $x = u_2$ in (4.11), we get

$$\sum_{j=0}^{+\infty} |r|^4 |\alpha_{2j+2}|^2 = 1. \tag{4.13}$$

On solving (4.12) and (4.13), we obtain that $|\alpha_0|^2 = 1 - \frac{1}{|r|^4} < 0$, a contradiction.

Case 2. If $|r| > 1$. Using Proposition 3.1, it follows that $C^*_{r,\alpha} = R_{s,\beta}$, where $s = \frac{1}{r^2}$ and $\beta_n = \overline{r^n}\overline{\alpha_n}$ for each $n \in \mathbb{N}_0$. For $x = u_0$ in (4.11), we get

$$\sum_{j=0}^{+\infty} |\beta_j|^2 = 1. \tag{4.14}$$

Now take $x = u_1$ in (4.11), we get

$$\sum_{j=0}^{+\infty} |s|^2 |\beta_{j+2}|^2 = 1. \tag{4.15}$$

On solving (4.14) and (4.15), it follows that $|\beta_0|^2 + |\beta_1|^2 = 1 - \frac{1}{|s|^2} < 0$ (a contradiction), since $|r| > 1$ implies $|s| < 1$.

Case 3. If $|r| = 1$. For each $i \in \mathbb{N}_0$, take $x = u_i$ in (4.11), we get $\sum_{j=0}^{+\infty} |\alpha_{2j}|^2 = 1, \sum_{j=0}^{+\infty} |r|^2 |\alpha_{2j+1}|^2 = 1, \sum_{j=0}^{+\infty} |r|^4 |\alpha_{2j+2}|^2 = 1, \ldots$ On solving these equations, we get $\alpha_i = 0$ for all $i \in \mathbb{N}_0$, a contradiction.

Hence, there does not exist any unitary operator in the class $\mathscr{C}_r(H)$ for any non-zero complex number r. □

As a consequence of this result, we get the following result.

Corollary 4.2. *Let r be a non-zero complex number, then the following hold.*

- **(A)** *If $|r| < 1$, then the class $\mathscr{C}_r(H)$ does not contain any isometry.*
- **(B)** *If $|r| > 1$, then the class $\mathscr{R}_r(H)$ does not contain any isometry.*

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