SOME INEQUALITIES FOR THE POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract. Let $P(z)$ be a polynomial of degree $n$ which has no zeros in $|z| < 1$, then it was proved by Liman, Mohapatra and Shah \cite{11} that

$$
\left| zD_\alpha P(z) + n\beta \left( \frac{|\alpha| - 1}{2} \right) P(z) \right|
\leq \frac{n}{2} \left\{ |\alpha + \beta \left( \frac{|\alpha| - 1}{2} \right)| + |z + \beta \left( \frac{|\alpha| - 1}{2} \right)| \right\} \max_{|z|=1} |P(z)|
- \frac{n}{2} \left\{ |\alpha + \beta \left( \frac{|\alpha| - 1}{2} \right)| - |z + \beta \left( \frac{|\alpha| - 1}{2} \right)| \right\} \min_{|z|=1} |P(z)|,
$$

for any $\beta$ with $|\beta| \leq 1$ and $|z| = 1$. In this paper we generalize the above inequality and our result also generalizes certain well known polynomial inequalities.

1. Introduction

Let $\mathcal{P}_n$ denote the class of all complex polynomials of degree at most $n$. If $P \in \mathcal{P}_n$, then according to Bernstein theorem \cite{5}, we have

$$
\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|.
$$

Bernstein proved it in 1912. Later, in 1930 he \cite{6} revisited his inequality and proved the following result from which inequality (1.1) can be deduced for $Q(z) = Mz^n$, where $M = \max_{|z|=1} |P(z)|$.

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**Theorem 1.1.** Let $P(z)$ and $Q(z)$ be two polynomials with degree of $P(z)$ not exceeding that of $Q(z)$. If $P(z)$ has all its zeros in $|z| \leq 1$ and 

$$|P(z)| \leq |Q(z)|, \quad \text{for } |z| = 1,$$

then

$$|P'(z)| \leq |Q'(z)|, \quad \text{for } |z| = 1 \quad (1.2)$$

More generally, it was proved by Malik and Vong [12] that for any $\beta$ with $|\beta| \leq 1$, inequality (1.2) can be replaced by

$$\left| z P'(z) + \frac{n \beta}{2} P(z) \right| \leq |z Q'(z) + \frac{n \beta}{2} Q(z)|, \quad \text{for } |z| = 1. \quad (1.3)$$

By restricting the zeros of a polynomial, the maximum value may be smaller. Indeed, if $P \in \mathcal{P}_n$ has no zero inside the unit circle $|z| < 1$, then inequality (1.1) can be replaced by

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \quad (1.4)$$

Inequality (1.4) was conjectured by Erdős and later proved verified by Lax [10]. This result was further improved by Aziz and Dawood [2] who, under the same hypothesis, proved that

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=1} |P(z)| \right\}. \quad (1.5)$$

Jain [8] generalized the inequality (1.4) and proved that if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every real or complex number $\beta$ with $|\beta| \leq 1$, $|z| = 1$,

$$\left| z P'(z) + \frac{n \beta}{2} P(z) \right| \leq \frac{n}{2} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=1} |P(z)| \right\} \left( 1 + \frac{\beta}{2} - \frac{1}{2} \right)^2 \max_{|z|=1} |P(z)|. \quad (1.6)$$

As a refinement of (1.5), Deewan and Hans [7] proved the following.

**Theorem 1.2.** If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every real or complex number $\beta$ with $|\beta| \leq 1$

$$\left| z P'(z) + \frac{n \beta}{2} P(z) \right| \leq \frac{n}{2} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=1} |P(z)| \right\} \left( 1 + \frac{\beta}{2} - \frac{1}{2} \right)^2 \max_{|z|=1} |P(z)|. \quad (1.6)$$

Let $D_\alpha P(z)$ be an operator that carries $n^{th}$ degree polynomial $P(z)$ to the polynomial

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z), \quad \alpha \in \mathbb{C},$$

of degree at most $(n - 1)$. $D_\alpha P(z)$ generalizes the ordinary derivative $P'(z)$ in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z).$$
Aziz was among the first to extend these results to polar derivatives. It is proved by Aziz [1] that for $P \in \mathcal{P}_n$ having no zeros in $|z| < 1$ and $|\alpha| \geq 1$, 
\[ |D_{\alpha}P(z)| \leq \frac{n}{2}(|\alpha z^{n-1}| + 1) \max_{|z|=1} |P(z)|, \quad \text{for } |z| \geq 1. \]

As an extension of (1.1) for the polar derivative Aziz and Shah [4] proved the following.

**Theorem 1.4.** If $P(z)$ is a polynomial of degree $n$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ 
\[ |D_{\alpha}P(z)| \leq n|\alpha z^{n-1}| \max_{|z|=1} |P(z)|, \quad \text{for } |z| \geq 1. \]

Liman et al. [11] extended (1.3) to the polar derivative and proved the following result.

**Theorem 1.4.** Let $Q(z)$ be a polynomial of degree $n$ having all its zeros $|z| \leq 1$ and $P(z)$ be a polynomial of degree at most $n$. If $|P(z)| \leq |Q(z)|$ for $|z| = 1$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq 1$, $|\beta| \leq 1$, 
\[ |zD_{\alpha}P(z) + n\beta \left(\frac{|\alpha| - 1}{2}\right) P(z)| \leq |zD_{\alpha}Q(z) + n\beta \left(\frac{|\alpha| - 1}{2}\right) Q(z)|, \quad \text{for } |z| \geq 1. \]

2. **Main Result**

In this paper, we first prove the following result which is generalization of Theorem 1.4 and also obtain some compact generalization for polar derivative.

**Theorem 2.1.** Let $Q(z)$ be a polynomial of degree $n$ having all its zeros $|z| \leq k$, $k \geq 1$ and $P(z)$ be a polynomial of degree at most $n$. If $|P(z)| \leq |Q(z)|$ for $|z| = k$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geq k$, $|\beta| \leq 1$, 
\[ |zD_{\alpha}P(z) + n\beta \left(\frac{|\alpha| - k}{1 + k^n}\right) P(z)| \leq |zD_{\alpha}Q(z) + n\beta \left(\frac{|\alpha| - k}{1 + k^n}\right) Q(z)|, \quad \text{for } |z| \geq k. \]

**Remark 2.1.** For $k = 1$, Theorem 2.1 reduces to the Theorem 1.4.

Dividing both sides of (2.1) by $|\alpha|$ and letting $|\alpha| \to \infty$ we get following generalization of (1.3).

**Corollary 2.1.** Let $Q(z)$ be a polynomial of degree $n$ having all its zeros $|z| \leq k$, $k \geq 1$ and $P(z)$ be a polynomial of degree at most $n$. If $|P(z)| \leq |Q(z)|$ for $|z| = k$, then $\beta \in \mathbb{C}$ with $|\beta| \leq 1$
\[ |z'P(z) + \frac{n\beta}{1 + k^n} P(z)| \leq |z'Q(z) + \frac{n\beta}{1 + k^n} Q(z)|, \quad \text{for } |z| \geq k. \]

By applying Theorem 2.1 to the polynomials $P(z)$ and $Q(z) = M(z/k^n)$, where $M = \max_{|z|=k} |P(z)|$, we get the following result.
Corollary 2.2. If \( P(z) \) is a polynomial of degree \( n \), then for any \( \alpha, \beta \), with \( |\alpha| \geq k, |\beta| \leq 1 \) and \( |z| \geq k \)

\[
|zD_\alpha P(z) + n\beta \left( \frac{|\alpha| - k}{1 + k^n} \right) P(z)| \leq n\frac{|z|^n}{k^n} \left| \alpha + \beta \left( \frac{|\alpha| - k}{1 + k^n} \right) \right| M.
\]

By applying Theorem 2.1 to the polynomials \( P(z) \) and \( Q(z) = m \frac{z^n}{k^n} \), where \( m = \min_{|z|=k} |P(z)| \), we get the following result.

Corollary 2.3. If \( P(z) \) is a polynomial of degree \( n \) having all its zeros in \( |z| \leq k \), \( k \geq 1 \), then for any \( \alpha, \beta \) with \( |\alpha| \geq k, |\beta| \leq 1 \) and \( |z| \geq k \)

\[
|zD_\alpha P(z) + n\beta \left( \frac{|\alpha| - k}{1 + k^n} \right) P(z)| \geq n\frac{|z|^n}{k^n} \left| \alpha + \beta \left( \frac{|\alpha| - k}{1 + k^n} \right) \right| m.
\]

Theorem 2.2. Let \( Q(z) \) be a polynomial of degree \( n \) having all its zeros \( |z| \leq k \), \( k \geq 1 \) and \( P(z) \) be a polynomial of degree at most \( n \). If \( |P(z)| \leq |Q(z)| \) for \( |z| = k \), then for all \( \alpha \in \mathbb{C} \) with \( |\alpha| \geq k \)

\[
|zD_\alpha P(z)| + n \left( \frac{|\alpha| - k}{1 + k^n} \right) |Q(z)| \leq |zD_\alpha Q(z)| + n \left( \frac{|\alpha| - k}{1 + k^n} \right) |P(z)|.
\]

Dividing both sides of (2.2) by \( \alpha \) and letting \( |\alpha| \rightarrow \infty \), we get the following result.

Corollary 2.4. Let \( Q(z) \) be a polynomial of degree \( n \) having all its zeros \( |z| \leq k \), \( k \geq 1 \) and \( P(z) \) be a polynomial of degree at most \( n \). If \( |P(z)| \leq |Q(z)| \) for \( |z| = k \), then for \( |z| = 1 \)

\[
\left| \frac{P'(z)}{n} \right| + \left| \frac{Q(z)}{1 + k^n} \right| \leq \left| \frac{Q'(z)}{n} \right| + \left| \frac{P(z)}{1 + k^n} \right|.
\]

Theorem 2.3. If \( P(z) \) is a polynomial of degree \( n \) which does not vanish in \( |z| < k \), \( k \geq 1 \) then for all \( \alpha, \beta \in \mathbb{C} \) with \( |\alpha| \geq k, |\beta| \leq 1 \) and for \( |z| \geq k \)

\[
|zD_\alpha P(z) + n\beta \left( \frac{|\alpha| - k}{1 + k^n} \right) P(z)| \leq \frac{n}{2} \left\{ \left| z^n \right| \alpha + \beta \left( \frac{|\alpha| - k}{1 + k^n} \right) + k^n \right| z + \beta \left( \frac{|\alpha| - k}{1 + k^n} \right) \} \max_{|z|=1} |P(z)|
\]

\[
- \frac{n}{2} \left\{ \left| z^n \right| \alpha + \beta \left( \frac{|\alpha| - k}{1 + k^n} \right) - \left| z + \beta \left( \frac{|\alpha| - k}{1 + k^n} \right) \right| \right\} m,
\]

where \( m = \min_{|z|=k} |P(z)| \).

Dividing both sides of (2.3) by \( |\alpha| \) and letting \( |\alpha| \rightarrow \infty \), we get the following generalization of a result due to Dewan and Hans [7].
**Corollary 2.5.** If \( P(z) \) is a polynomial of degree \( n \) which does not vanish in \( |z| < k \), \( k \geq 1 \), then for all \( \beta \in \mathbb{C} \) with \( |\beta| \leq 1 \) and for \( |z| \geq k \)
\[
\left| zP(z) + \frac{n\beta}{1 + k^n} P(z) \right| \leq \frac{n}{2} \left\{ |z|^n \left| 1 + \frac{\beta}{1 + k^n} \right| + k^n \left| \frac{\beta}{1 + k^n} \right| \right\} \max_{|z|=1} |P(z)|
- \frac{n}{2} \left\{ \frac{|z|^n}{k^n} \left| 1 + \frac{\beta}{1 + k^n} \right| - \left| \frac{\beta}{1 + k^n} \right| \right\} m,
\]
where \( m = \min_{|z|=k} |P(z)|. \)

3. **Lemma**

For the proofs of these theorems we need the following lemmas. The first lemma which we need is due to Laguerre (see [9, page 38]).

**Lemma 3.1.** If all the zeros of an \( n \)th degree polynomial \( P(z) \) lie in a circular region \( C \) and \( w \) is any zero of \( D_\alpha P(z) \), then at most one of the points \( w \) and \( \alpha \) may lie outside \( C \).

**Lemma 3.2.** Let \( A \) and \( B \) be any two complex numbers, then the following holds.

(i) If \( |A| \geq |B| \) and \( B \neq 0 \), then \( A \neq vB \) for all complex numbers \( v \) with \( |v| < 1 \).

(ii) Conversely, if \( A \neq vB \) for all complex number \( v \) with \( |v| < 1 \), then \( |A| \geq |B| \).

Lemma 3.2 is due to Xin Li [13].

**Lemma 3.3.** If \( P(z) \) is a polynomial of degree \( n \), then for \( k \geq 1 \)
\[
\max_{|z|=k} |P(z)| \leq k^n \max_{|z|=1} |P(z)|.
\]

Lemma 3.3 is simple consequence of maximum modulus theorem.

**Lemma 3.4.** If the polynomial \( P(z) \) has all its zeros in \( |z| \leq k \), \( k \geq 1 \), then for every \( \alpha \in \mathbb{C} \) with \( |\alpha| \geq k \)
\[
n(|\alpha| - k)|P(z)| \leq (1 + k^n)|D_\alpha P(z)|.
\]

Lemma 3.4 is due to Aziz and Rather [3].

**Lemma 3.5.** If \( P(z) \) is a polynomial of degree \( n \), then for any \( \alpha \) with \( |\alpha| \geq k \), \( |\beta| \leq 1 \) and \( |z| = k \)
\[
\left| zD_\alpha P(z) + n\beta \left( \frac{|\alpha| - k}{1 + k^n} \right) P(z) \right| + \left| zD_\alpha q(z) + n\beta \left( \frac{|\alpha| - k}{1 + k^n} \right) q(z) \right|
\leq n \left\{ |z|^n \alpha + \beta \left( \frac{|\alpha| - k}{1 + k^n} \right) + k^n \beta \left( \frac{|\alpha| - k}{1 + k^n} \right) \right\} \max_{|z|=1} |P(z)|,
\]
where \( q(z) = \left( \frac{z}{k} \right)^n \frac{P(k^2)}{n}. \)
Proof. Let \( M = \max_{|z|=k} |P(z)| \). An application of Rouche’s Theorem shows that all the zeros of the polynomial \( G(z) = k^n P(z) + \lambda M z^n \) lie in \( |z| < k \), \( k \geq 1 \) for every \( \lambda \) with \( |\lambda| > 1 \). If \( H(z) = \left( \frac{1}{k} \right)^n \bar{G} \left( \frac{1}{k} \bar{z} \right) = k^n Q(z) + \bar{\lambda} M k^n \), then \( |G(z)| = |H(z)| \) for \( |z| = k \) and hence for any \( \gamma \) with \( |\gamma| < 1 \) the polynomial \( \gamma H(z) + G(z) \) has all its zeros in \( |z| < k \), \( k \geq 1 \). By applying Lemma 3.4, we have for any \( \alpha \) with \( |\alpha| \geq k \)

\[
(1 + k^n)z(\gamma D_\alpha H(z) + D_\alpha G(z)) \geq n(|\alpha| - k)|\gamma H(z) + G(z)|.
\]

Since \( \gamma H(z) + G(z) \neq 0 \) for \( |z| \geq k \), \( k \geq 1 \), so the right hand side is non zero. Thus, by using (i) of Lemma 3.2 we have for all \( \beta \) satisfying \( |\beta| < 1 \) and for \( |z| \geq k \)

\[
T(z) = \beta n(|\alpha| - k)\gamma H(z) + G(z) + (1 + k^n)z(\gamma D_\alpha H(z) + D_\alpha G(z)) \neq 0,
\]
or, equivalently, for \( |z| \geq k \)

\[
T(z) = \gamma (1 + k^n)zD_\alpha H(z) + n\beta(|\alpha| - k)H(z) + (1 + k^n)zD_\alpha G(z) + n\beta(|\alpha| - k)G(z) \neq 0.
\]

Using (ii) of Lemma 3.2 we have for \( |\gamma| < 1 \) and for \( |z| \geq k \)

\[
|zD_\alpha q(z) + n\beta \left( \frac{|\alpha| - k}{1 + k^n} \right) q(z)| - n|\lambda| |z + \beta \left( \frac{|\alpha| - k}{1 + k^n} \right)| M
\]

\[
|zD_\alpha P(z) + n\beta \left( \frac{|\alpha| - k}{1 + k^n} \right) P(z) + n\lambda z^n \left( \alpha + \beta \left( \frac{|\alpha| - k}{1 + k^n} \right) \right) M|.
\]

By Corollary 2.2, it is possible to choose the argument of \( \lambda \) such that

\[
|zD_\alpha P(z) + n\beta \left( \frac{|\alpha| - k}{1 + k^n} \right) P(z)| = n \frac{|z|^n}{k^n} \alpha + \beta \left( \frac{|\alpha| - k}{1 + k^n} \right) M.
\]

Using (3.3) in (3.2) and letting \( |\lambda| \to 1 \) we get for \( |\alpha| > k \) and \( |\beta| < 1 \)

\[
|zD_\alpha q(z) + n\beta \left( \frac{|\alpha| - k}{1 + k^n} \right) q(z)| - n |z + \beta \left( \frac{|\alpha| - k}{1 + k^n} \right)| M
\]

\[
\leq n \frac{|z|^n}{k^n} \alpha + \beta \left( \frac{|\alpha| - k}{1 + k^n} \right) M - |zD_\alpha P(z) + n\beta \left( \frac{|\alpha| - k}{1 + k^n} \right) P(z)|.
\]

That is

\[
|zD_\alpha P(z) + n\beta \left( \frac{|\alpha| - k}{1 + k^n} \right) P(z)| + |zD_\alpha q(z) + n\beta \left( \frac{|\alpha| - k}{1 + k^n} \right) q(z)|
\]

\[
\leq n \left\{ \frac{|z|^n}{k^n} \alpha + \beta \left( \frac{|\alpha| - k}{1 + k^n} \right) + |z + \beta \left( \frac{|\alpha| - k}{1 + k^n} \right)| \right\} M.
\]
Using Lemma 3.3 in (3.4) we get
\[ |zD_\alpha P(z) + n\beta \left( \frac{1}{1+k^n} \right) P(z)| + |zD_\alpha q(z) + n\beta \left( \frac{1}{1+k^n} \right) q(z)| \leq n \left\{ |z|^n \right\} \alpha + \beta \left( \frac{1}{1+k^n} \right) + k^n \left| z + \beta \left( \frac{1}{1+k^n} \right) \right| \max_{|z|=1} |P(z)|.

That proves Lemma 3.5 completely. □

4. Proof of Theorems

Proof of Theorem 2.1. By Rouche’s Theorem, the polynomial \( \lambda P(z) - Q(z) \) has all its zeros in \( |z| \leq k, k \geq 1 \) for \( |\lambda| < 1 \). Therefore, for \( r > 1 \), all the zeros of \( \lambda P(rz) - Q(rz) \) lie in \( |z| \leq \frac{r}{k} < k \). By applying Lemma 3.4 to the polynomial \( \lambda P(rz) - Q(rz) \), we have for \( |z| = 1 \)
\[ n(|\alpha| - k)|\lambda P(rz) - Q(rz)| \leq (1+k^n)|z(\lambda D_\alpha P(rz) - D_\alpha Q(rz))|. \]
As in the proof of Lemma 3.1, we have for \( |\beta| < 1 \) and for \( |z| \geq k \)
\[ (1+k^n)z\{\lambda D_\alpha P(rz) - D_\alpha Q(rz)\} + n\beta(|\alpha| - k)\{\lambda P(rz) - Q(rz)\} \neq 0. \]
This implies for \( |z| \geq k \)
\[ (1+k^n)|zD_\alpha Q(z)| \geq n(|\alpha| - k)|Q(z)|. \]

This gives for every \( \beta \) with \( |\beta| \leq 1 \)
\[ (1+k^n)|zD_\alpha Q(z)| - n|\beta|(|\alpha| - k)|Q(z)| \geq 0. \]
Therefore, it is possible to choose the argument of \( \beta \) in the right hand side of Theorem 2.1 such that
\[ (1+k^n)|zD_\alpha Q(z) - n|\beta|(|\alpha| - k)|Q(z)| = |zD_\alpha Q(z)| - n|\beta| \left( \frac{|\alpha| - k}{1+k^n} \right) |Q(z)|. \]
Using (4.2) in Theorem 2.1 and letting \( |\beta| \to 1 \), we get the desired result. □

Proof of Theorem 2.3. Let \( P(z) \) be a polynomial of degree \( n \) which does not vanish in \( |z| \leq k, k \geq 1 \). If \( q(z) = \left( \frac{z}{k} \right)^n P \left( \frac{k^2}{z^2} \right) \), then \( q(z) \) has all its zeros in \( |z| \leq k, k \geq 1 \)
and \( |P(z)| = |q(z)| \) for \( |z| = k \). Hence, by Theorem 2.1, we have for all \( \alpha, \beta \) satisfying \( |\alpha| \geq k, \ |\beta| \leq 1 \)

\[
(4.3) 
\begin{align*}
|zD_\alpha P(z) + n\beta \left( \frac{|\alpha| - k}{1 + k^n} \right) P(z)| &\leq |zD_\alpha q(z) + n\beta \left( \frac{|\alpha| - k}{1 + k^n} \right) q(z)|, \quad \text{for } |z| \geq k.
\end{align*}
\]

Let \( m = \min_{|z|=k} |P(z)| \). If \( P(z) \) has a zero in \( |z| = k \), then \( m = 0 \) and result follows by combining Lemma 3.5 with (4.3). Therefore, we suppose that all the zeros of \( P(z) \) lie in \( |z| > k \) and so \( m > 0 \). We have \( |\gamma m| < |P(z)| \) on \( |z| = k \) for any \( \gamma \) with \( |\gamma| < 1 \). By Rouche’s Theorem the polynomial \( F(z) = P(z) + \gamma m \) has no zeros in \( |z| < k \).

Therefore, the polynomial \( G(z) = \left( \frac{z}{k} \right)^n F \left( \frac{k^n}{z} \right) = q(z) - \gamma m z^n \) will have all its zeros in \( |z| \leq k \). Also \( |F(z)| = |G(z)| \) for \( |z| = k \). On applying Theorem 2.1, we get for any \( \beta, \alpha \) with \( |\beta| \leq 1, |\alpha| \geq k \)

\[
|zD_\alpha F(z) + n\beta \left( \frac{|\alpha| - k}{1 + k^n} \right) F(z)| \leq |zD_\alpha G(z) + n\beta \left( \frac{|\alpha| - k}{1 + k^n} \right) G(z)|, \quad \text{for } |z| \geq k.
\]

Equivalently,

\[
(4.4) 
\begin{align*}
|zD_\alpha P(z) + n\beta \left( \frac{|\alpha| - k}{1 + k^n} \right) P(z)| - n|\gamma| |z + \beta \left( \frac{|\alpha| - k}{1 + k^n} \right) m| &\leq |zD_\alpha q(z) + n\beta \left( \frac{|\alpha| - k}{1 + k^n} \right) q(z) - n\gamma \frac{z^n}{k^n} \left( \alpha + \left( \frac{|\alpha| - k}{1 + k^n} \right) m \right)|.
\end{align*}
\]

Since \( q(z) \) has all its zeros in \( |z| \leq k \) and \( \min_{|z|=k} |p(z)| = \min_{|z|=k} |q(z)| = m \), therefore, by Corollary 2.3, we have

\[
(4.5) 
\begin{align*}
|zD_\alpha q(z) + n\beta \left( \frac{|\alpha| - k}{1 + k^n} \right) q(z)| &\geq n \left| \frac{z^n}{k^n} \right| + \beta \left( \frac{|\alpha| - k}{1 + k^n} \right) m.
\end{align*}
\]

Therefore, we can write (4.4) in view of (4.5) as

\[
(4.6) 
\begin{align*}
|zD_\alpha P(z) + n\beta \left( \frac{|\alpha| - k}{1 + k^n} \right) P(z)| - n |z + \beta \left( \frac{|\alpha| - k}{1 + k^n} \right) m| &\leq |zD_\alpha q(z) + n\beta \left( \frac{|\alpha| - k}{1 + k^n} \right) q(z) - n \left| \frac{z^n}{k^n} \right| + \beta \left( \frac{|\alpha| - k}{1 + k^n} \right) m|.
\end{align*}
\]

Letting \( |\gamma| \to 1 \), we get from inequality (4.6)

\[
(4.7) 
\begin{align*}
|zD_\alpha P(z) + n\beta \left( \frac{|\alpha| - k}{1 + k^n} \right) P(z)| - |zD_\alpha q(z) + n\beta \left( \frac{|\alpha| - k}{1 + k^n} \right) q(z)| &\leq - n \left| \frac{z^n}{k^n} \right| + \beta \left( \frac{|\alpha| - k}{1 + k^n} \right) - |z + \beta \left( \frac{|\alpha| - k}{1 + k^n} \right) m|.
\end{align*}
\]
Now, by Lemma 3.5, we have
\[
\left|zD_\alpha P(z) + n\beta \left(\frac{\vert \alpha \vert - k}{1 + k^n}\right) P(z)\right| + \left|zD_\alpha q(z) + \beta \left(\frac{\vert \alpha \vert - k}{1 + k^n}\right) q(z)\right| \\
\leq n \left\{ \left|z\right|^n \alpha + \beta \left(\frac{\vert \alpha \vert - k}{1 + k^n}\right) \left|z + \beta \left(\frac{\vert \alpha \vert - k}{1 + k^n}\right)\right| \right\} \max_{\left|z\right|=1} |P(z)|. 
\]
Inequalities (4.7) and (4.8) together lead to
\[
\left|zD_\alpha P(z) + n\beta \left(\frac{\vert \alpha \vert - k}{1 + k^n}\right) P(z)\right| \\
\leq \frac{n}{2} \left\{ \left|z\right|^n \alpha + \beta \left(\frac{\vert \alpha \vert - k}{1 + k^n}\right) \left|z + \beta \left(\frac{\vert \alpha \vert - k}{1 + k^n}\right)\right| \right\} \max_{\left|z\right|=1} |P(z)| \\
- \frac{n}{2} \left\{ \left|z\right|^n \alpha + \beta \left(\frac{\vert \alpha \vert - k}{1 + k^n}\right) \left|z + \beta \left(\frac{\vert \alpha \vert - k}{1 + k^n}\right)\right| \right\} m.
\]
That proves Theorem 2.3 completely.

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References


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