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CONVERGENCE AND DIFFERENCE ESTIMATES BETWEEN MASTROIANNI AND GUPTA OPERATORS

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This paper is dedicated to Prof. Dr. Gradimir V. Milovanović

Abstract. Gupta operators are a modified form of Srivastava-Gupta operators and we are concerned about investigating the difference of operators and we estimate the difference of Mastroianni operators with Gupta operators in terms of modulus of continuity of first order. We also study the weighted approximation of functions and obtain the rate of convergence with the help of the moduli of continuity as well as Peetre's K-functional of Gupta operators.

1. Introduction and preliminaries

Acu-Rasa [3], Aral et al. [4] and Gupta [17] studied some fascinating results for the difference of operators in general sense. Several results on this topic are compiled in the recent book of Gupta et al. [19]. We extend here the study for some important operators. The Mastroianni operators [23] are mentioned below:

$$\mathcal{M}_{n,c}(f;x)=\sum_{i=0}^{\infty} v_{n,i}(x,c)\mathcal{F}_{n,i}(f), \tag{1.1}$$

where

$$v_{n,i}(x,c)=\frac{(-x)^i}{i!}\tau_{n,c}^{(i)}(x),\quad \mathcal{F}_{n,i}(f)=f\left(\frac{i}{n}\right),$$

with individual cases, which are mentioned below.

(*i*) If $\tau_{n,0}(x)=exp(-nx)$, then $v_{n,i}(x,0)=exp(-nx)\frac{(nx)^i}{i!}$ and the operators $\mathcal{M}_{n,0}$ becomes Szász operators.

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(*ii*) If $c \in \mathbb{N}$ and $\tau_{n,c}(x) = \frac{1}{(1+cx)^{n/c}}$, then we have $v_{n,i}(x,c) = \frac{(n/c)_i}{i!} \cdot \frac{(cx)^i}{(1+cx)^{\frac{n}{c}+i}}$ and we obtain classical Baskakov operators.

(*iii*) If $\tau_{n,-1}(x) = (1-x)^n$, then $v_{n,i}(x,-1) = \binom{n}{i} x^i (1-x)^{n-i}$ and the operators (1.1) reduce to Bernstein polynomials,

where $\mathcal{F}_{n,i} : \mathcal{S} \to \mathbb{R}$ is a functional (linear and positive) defined on $\mathcal{S}$ and $\mathcal{S} \subset C[0,\infty)$. Case (*iii*) has not been considered here, we will continue with this case in our next upcoming paper.

Srivastava-Gupta operator (see [10, 29]) reproduce only constant functions, recently Gupta in [16] studied few examples of the genuine operators (operators preserving linear functions), we consider here following operators

$$\mathcal{G}_{n;c}(f;x) = \sum_{i=0}^{\infty} v_{n,i}(x,c) \mathcal{H}_{n,i}(f), \tag{1.2}$$

where $v_{n,i}(x,c)$ is defined in (1.1) and

$$\mathcal{H}_{n,i}(f) = (n+c) \int_0^\infty v_{n+2c,i-1}(t,c) f(t) dt, \quad 1 \le i < \infty, \quad \mathcal{H}_{n,0}(f) = f(0).$$

Remark 1.1. For operators (1.1), we have $\mathcal{F}_{n,i}(f) = f\left(\frac{i}{n}\right)$ such that

$$\mathcal{F}_{n,i}(e_0) = 1 \quad \text{and} \quad b^{\mathcal{F}_{n,i}} := \mathcal{F}_{n,i}(e_1).$$

If we denote $T_r^{\mathcal{F}_{n,i}} = \mathcal{F}_{n,i}(e_1 - b^{\mathcal{F}_{n,i}} e_0)^r$, $r \in \mathbb{N}$, then by simple computation, we have

$$T_r^{\mathcal{F}_{n,i}} = \mathcal{F}_{n,i}(e_1 - b^{\mathcal{F}_{n,i}} e_0)^r = 0, \quad r = 2, 4.$$

2. Preliminaries

Remark 2.1. For the Gupta type operators (1.2), by simple computation, we have

$$\mathcal{H}_{n,i}(e_r) = \frac{(i+r-1)!}{(i-1)!} \cdot \frac{\Gamma\left(\frac{n}{c} - r + 1\right)}{c^r \Gamma\left(\frac{n}{c} + 1\right)},$$

where $\mathcal{H}_{n,i}(e_0) = 1$, $b^{\mathcal{H}_{n,i}} := \mathcal{H}_{n,i}(e_1) = \frac{i}{n}$. If we denote $T_r^{\mathcal{H}_{n,i}} = \mathcal{H}_{n,i}(e_1 - b^{\mathcal{H}_{n,i}} e_0)^r$, $r \in \mathbb{N}$, then after simple computation, we have

$$T_2^{\mathcal{H}_{n,i}} := \mathcal{H}_{n,i}(e_1 - b^{\mathcal{H}_{n,i}} e_0)^2 = \frac{ci^2 + ni}{n^2(n-c)}$$

and

$$\begin{aligned} T_4^{\mathcal{H}_{n,i}} :=& \mathcal{H}_{n,i}(e_1 - b^{\mathcal{H}_{n,i}} e_0)^4 \\ =& \mathcal{H}_{n,i}(e_4, x) - 4\mathcal{H}_{n,i}(e_3, x)\left(\frac{i}{n}\right) + 6\mathcal{H}_{n,i}(e_2, x)\left(\frac{i}{n}\right)^2 \\ & - 4\mathcal{H}_{n,i}(e_1, x)\left(\frac{i}{n}\right)^3 + \mathcal{H}_{n,i}(e_0, x)\left(\frac{i}{n}\right)^4 \end{aligned}$$

$$= \frac{(i+3)(i+2)(i+1)i}{n(n-c)(n-2c)(n-3c)} - 4\frac{(i+2)(i+1)i^2}{n^2(n-c)(n-2c)} + 6\frac{(i+1)i^3}{n^3(n-c)} - \frac{3i^4}{n^4}.$$

Lemma 2.1. *Few moments of Mastroianni operators are given by*

$$\begin{aligned}
\mathcal{M}_n(e_0;x) =&1,\\
\mathcal{M}_n(e_1;x) =&x,\\
\mathcal{M}_n(e_2;x) =&\frac{x}{n}[x(n+c)+1],\\
\mathcal{M}_n(e_3;x) =&\frac{x}{n^2}[x^2(n+c)(n+2c)+3x(n+c)+1],\\
\mathcal{M}_n(e_4;x) =&\frac{x}{n^3}[x^3(n+c)(n+2c)(n+3c)+6x^2(n+c)(n+2c)+7x(n+c)+1],\\
\mathcal{M}_n(e_5;x) =&\frac{x}{n^4}[x^4(n+c)(n+2c)(n+3c)(n+4c)+10x^3(n+c)(n+2c)(n+3c)\\
&+25x^2(n+c)(n+2c)+15x(n+c)+1],\\
\mathcal{M}_n(e_6;x) =&\frac{x}{n^5}[x^5(n+c)(n+2c)(n+3c)(n+4c)(n+5c)+15x^4(n+c)(n+2c)\\
&\times(n+3c)(n+4c)+65x^3(n+c)(n+2c)(n+3c)+90x^2(n+c)(n+2c)\\
&+31x(n+c)+1].
\end{aligned}$$

Lemma 2.2. *Let $f(t)=e_i,\ i=0,1,2,3,4,$ and c is the element of the set $\{0,1,2\}$, then we have*

$$\begin{aligned}
\mathcal{G}_{n,c}(e_0;x) =&1,\\
\mathcal{G}_{n,c}(e_1;x) =&x,\\
\mathcal{G}_{n,c}(e_2;x) =&\frac{(n+c)}{(n-c)}x^2+\frac{2}{(n-c)}x, \quad n>c,\\
\mathcal{G}_{n,c}(e_3;x) =&\frac{(n+c)(n+2c)}{(n-c)(n-2c)}x^3+\frac{6(n+c)}{(n-c)(n-2c)}x^2+\frac{6}{(n-c)(n-2c)}x, \quad n>2c,\\
\mathcal{G}_{n,c}(e_4;x) =&\frac{(n+c)(n+2c)(n+3c)}{(n-c)(n-2c)(n-3c)}x^4+\frac{12(n+c)(n+2c)}{(n-c)(n-2c)(n-3c)}x^3\\
&+\frac{36(n+c)}{(n-c)(n-2c)(n-3c)}x^2+\frac{24}{(n-c)(n-2c)(n-3c)}x, \quad n>3c.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathcal{G}_{n,c}((e_1-x);x) =&0,\\
\mathcal{G}_{n,c}\left((e_1-x)^2;x\right) =&\frac{2x(1+cx)}{n-c}, \quad n>c,\\
\mathcal{G}_{n,c}\left((e_1-x)^4;x\right) =&\frac{12c^2(n+7c)}{(n-c)(n-2c)(n-3c)}x^4+\frac{24c^2(13n+c)}{(n-c)(n-2c)(n-3c)}x^3\\
&+\frac{12c^2(n+9c)}{(n-c)(n-2c)(n-3c)}x^2
\end{aligned}$$

$$+\frac{24}{(n-c)(n-2c)(n-3c)}x, \quad n>3c.$$

Very recently, Pratap and Deo [28] considered genuine Gupta-Srivastava operators and studied fundamental properties, the rate of convergence, Voronovskaya type estimates, convergence estimates and weighted approximation. In the year 2018, Garg et al. [13] studied the weighted approximation properties for Stancu generalized Baskakov operators. In the same year, Acu et al. [2] also studied the order of approximation for Srivastava-Gupta operators via Peetre's K-functional and weighted approximation properties and some numerical considerations regarding the approximation properties, were considered. Several researchers studied approximation operators and its variants, and they were given some impressive results like asymptotic formula, Voronovskaya-type formula, rate of convergence and bounded variation (see [1, 2, 4–9, 11, 12, 14, 18, 24–27]).

The purpose of this paper to study the approximation properties of Gupta operators and the approximation of difference of operators and find an estimate for the difference of Mastroianni operators with Gupta operators in terms of modulus of continuity of first order. In the third section, we give the rate of convergence with the help of the moduli of continuity and the Peetre's K-functional and the last section of this paper the weighted approximation of functions are studied.

3. Difference of Operators

Let $C_B[0,\infty)$ be the class of bounded continuous functions defined on the interval $[0,\infty)$ equipped with the norm $||\cdot|| = \sup_{x\in[0,\infty)} |f(x)| < \infty$.

Theorem 3.1 (Theorem A)**.** ([15, 17]). *Let $f^{(s)} \in C_B[0,\infty)$, s is a member of set $\{0,1,2\}$ and x belongs to $[0,\infty)$, then for all natural numbers n, we get*

$$|(\mathcal{G}_{n,c}-\mathcal{M}_{n,c})(f,x)| \leq ||f''||\alpha(x)+\omega(f'',\delta_1)(1+\alpha(x))+2\omega(f,\delta_2(x)),$$

where

$$\alpha(x)=\frac{1}{2}\sum_{i=0}^{\infty}v_{n,i}(x,c)(T_2^{\mathcal{F}_{n,i}}+T_2^{\mathcal{H}_{n,i}}),$$

and

$$\delta_1^2=\frac{1}{2}\sum_{i=0}^{\infty}v_{n,i}(x,c)(T_4^{\mathcal{F}_{n,i}}+T_4^{\mathcal{H}_{n,i}}), \quad \delta_2^2=\sum_{i=0}^{\infty}v_{n,i}(x,c)(b^{\mathcal{F}_{n,i}}-b^{\mathcal{H}_{n,i}})^2.$$

We give the quantitative estimate for difference of Mastroianni and Gupta type operators as an application of Theorem A.

Theorem 3.2. *Let $f^{(j)} \in C_B[0,\infty)$, j is a member of set $\{0,1,2\}$ and x belongs to $[0,\infty)$, then for all natural numbers n, we get*

$$|(\mathcal{G}_{n,c}-\mathcal{M}_{n,c})(f;x)| \leq ||f''||\beta(x)+\omega(f'',\delta_1)(1+\beta(x)),$$

where

$$\beta(x) = \frac{cx[x(n+c)+1]}{2n(n-c)} + \frac{nx}{2n(n-c)}$$

and

$$\delta_1^2 = \frac{1}{2n^4(n-c)(n-2c)(n-3c)}\left[\left\{3c^2\,(n+c)\,(n+2c)\,(n+3c)\,(n+6c)\right\}x^4\right.$$
$$+\,6c\,(n+c)\,(n+2c)\,\{3c\,(n+6c)+2n\,(n+2c)\}\,x^3$$
$$+\,(n+c)\left\{21c^2\,(n+6c)+36nc\,(n+2c)+n^2\,(3n+c)\right\}x^2$$
$$\left.+\left\{3c^2\,(n+6c)+12nc\,(n+2c)+n^2\,(3n+c)+6n^3\right\}x\right].$$

Proof. First using Remark 1.1, Remark 2.1 and applying Lemma 2.1, we get

$$\begin{aligned}\beta(x) =& \frac{1}{2}\sum_{i=0}^{\infty} v_{n,i}(x,c)(T_2^{\mathcal{F}_{n,i}} + T_2^{\mathcal{H}_{n,i}})\\ =& \frac{1}{2}\sum_{i=0}^{\infty} v_{n,i}(x,c)\frac{ci^2+ni}{n^2(n-c)}\\ =& \frac{c}{2(n-c)}\mathcal{M}_n(e_2,x) + \frac{n}{2n(n-c)}\mathcal{M}_n(e_1,x)\\ =& \frac{cx[x(n+c)+1]}{2n(n-c)} + \frac{nx}{2n(n-c)}.\end{aligned}$$

Next, by Remark 1.1 and Remark 2.1, we get

$$\begin{aligned}\delta_1^2 =& \frac{1}{2}\sum_{i=0}^{\infty} v_{n,i}(x,c)(T_4^{\mathcal{F}_{n,i}} + T_4^{\mathcal{H}_{n,i}})\\ =& \frac{1}{2}\sum_{i=0}^{\infty} v_{n,i}(x,c)T_4^{\mathcal{H}_{n,i}}\\ =& \frac{1}{2}\sum_{i=0}^{\infty} v_{n,i}(x,c)\left[\frac{(i+3)(i+2)(i+1)i}{n(n-c)(n-2c)(n-3c)} - 4\frac{(i+2)(i+1)i^2}{n^2(n-c)(n-2c)}\right.\\ &\left. +\,6\frac{(i+1)i^3}{n^3(n-c)} - \frac{3i^4}{n^4}\right]\\ =& \frac{1}{2}\sum_{i=0}^{\infty}\frac{v_{n,i}(x,c)}{n^4(n-c)(n-2c)(n-3c)}\left[\left(i^4+6i^3+11i^2+6i\right)n^3\right.\\ &-4\left(i^4+3i^3+2i^2\right)n^2(n-3c) + 6(i^4+i^3)n(n-2c)(n-3c)\\ &\left.-3i^4(n-c)(n-2c)(n-3c)\right]\\ =& \frac{1}{2}\sum_{i=0}^{\infty}\frac{v_{n,i}(x,c)}{n^4(n-c)(n-2c)(n-3c)}\left[i^4\left\{n^3-4n^2(n-3c)+6n(n-2c)(n-3c)\right.\right.\\ &\left.-3(n-c)(n-2c)(n-3c)\right\} + i^3\left\{6n^3-12n^2\,(n-3c)+6n\,(n-2c)\,(n-3c)\right\}\end{aligned}$$

$$
\begin{aligned}
&+ i^2 \left\{11n^3 - 8n^2(n-3c)\right\} + 6in^3\Big] \\
=&\frac{1}{2}\sum_{i=0}^{\infty} \frac{v_{n,i}(x,c)}{n^4(n-c)(n-2c)(n-3c)} \\
&\times \left[3i^4c^2(n+6c) + 12i^3nc(n+2c) + i^2n^2(3n+c) + 6in^3\right] \\
=&\frac{1}{2n^4(n-c)(n-2c)(n-3c)} \Big[3n^4c^2(n+6c)\mathcal{M}_n(e_4,x) + 12n^4c(n+2c)\mathcal{M}_n(e_3,x) \\
&+ n^4(3n+c)\mathcal{M}_n(e_2,x) + 6n^4\mathcal{M}_n(e_1,x)\Big] \\
=&\frac{3xc^2(n+6c)\left\{x^3(n+c)(n+2c)(n+3c) + 6x^2(n+c)(n+2c) + 7x(n+c) + 1\right\}}{2n^4(n-c)(n-2c)(n-3c)} \\
&+ \frac{6nc(n+2c)x\left\{x^2(n+c)(n+2c) + 3x(n+c) + 1\right\}}{n^4(n-c)(n-2c)(n-3c)} \\
&+ \frac{n^2(3n+c)x\left\{x(n+c)+1\right\}}{2n^4(n-c)(n-2c)(n-3c)} + \frac{3n^3x}{n^4(n-c)(n-2c)(n-3c)} \\
=&\frac{1}{2n^4(n-c)(n-2c)(n-3c)} \Big[\left\{3c^2(n+c)(n+2c)(n+3c)(n+6c)\right\}x^4 \\
&+ 6c(n+c)(n+2c)\left\{3c(n+6c) + 2n(n+2c)\right\}x^3 \\
&+ (n+c)\left\{21c^2(n+6c) + 36nc(n+2c) + n^2(3n+c)\right\}x^2 \\
&+ \left\{3c^2(n+6c) + 12nc(n+2c) + n^2(3n+c) + 6n^3\right\}x\Big]
\end{aligned}
$$

and

$$\delta_2^2 = \sum_{i=0}^{\infty} v_{n,i}(x,c)\left(b^{\mathcal{F}_{n,i}} - b^{\mathcal{H}_{n,i}}\right)^2 = 0. \qquad \square$$

4. Weighted Approximation

The usual first order of modulus of continuity of f on bounded interval $[0,b]$ is defined as:

$$\omega_b(f;\delta) = \sup_{0<|t-x|\le\delta}\ \sup_{t,x\in[0,b]} |f(t)-f(x)|.$$

Let

$$B_2[0,\infty) := \left\{f : [0,\infty) \to \mathbb{R} : |f(x)| \le M_f\left(1+x^2\right)\right\},$$

where M_f is a constant dependant on f, with the norm

$$\|f\|_2 = \sup_{x\ge 0} \frac{|f(x)|}{1+x^2}.$$

Let

$$C_2[0,\infty) = C[0,\infty) \cap B_2[0,\infty).$$

In [20], Ispir acquainted the weighted modulus of continuity $\Omega(f;\delta)$ as:

$$\Omega(f;\delta) = \sup_{0\le|k|<\delta, x\ge 0} \frac{|f(x+k)-f(x)|}{(1+k^2)(1+x^2)}, \quad f \in C_2[0,\infty). \tag{4.1}$$

Let

$$C_2'[0,\infty) = \left\{ f \in C_2[0,\infty) : \lim_{t\to\infty} \frac{|f(x)|}{1+t^2} < \infty \right\}.$$

From [20,21], if $f \in C_2'[0,\infty)$, then $\lim_{\delta\to 0} \Omega(f,\delta) = 0$ and

$$\Omega(f;p\delta) \le 2(1+p)\left(1+\delta^2\right)\Omega(f;\delta), \quad p>0. \tag{4.2}$$

From (4.1) and (4.2) and for $f \in C_2'[0,\infty)$, we have

$$\begin{aligned}|f(t)-f(x)| &\le \left(1+(t-x)^2\right)\left(1+x^2\right)\Omega(f;|t-x|)\\ &\le 2\left(1+\frac{|t-x|}{\delta}\right)\left(1+\delta^2\right)\Omega(f;\delta)\left(1+(t-x)^2\right)\left(1+x^2\right).\end{aligned}$$

Now we give rate of approximation of unbounded functions in theorem of first order of modulus of continuity.

Theorem 4.1. *Let $f \in C_2[0,\infty)$, then we get*

$$|\mathcal{G}_{n,c}(f,x)-f(x)| \le 4M_f\left(1+b^2\right)\delta_n^2(x) + 2\omega_{b+1}(f,\delta),$$

where $\delta = \delta_n(x) = \sqrt{\mathcal{G}_{n,c}\left((t-x)^2, x\right)}$.

Proof. For $x \in [0,b]$ and $t \ge 0$, we have

$$|f(t)-f(x)| \le 4M_f\left(1+b^2\right)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{b+1}(f,\delta), \quad \delta>0.$$

Applying operator $\mathcal{G}_{n,c}$ and using Cauchy-Schwarz inequality, we have

$$\begin{aligned}|\mathcal{G}_{n,c}(f;x)-f(x)| \le& 4M_f\left(1+b^2\right)\mathcal{G}_{n,c}\left((t-x)^2,x\right)\\ &+\left(1+\frac{\mathcal{G}_{n,c}(|t-x|,x)}{\delta}\right)\omega_{b+1}(f,\delta)\\ \le& 4M_f\left(1+b^2\right)\mathcal{G}_{n,c}\left((t-x)^2,x\right)\\ &+\left(1+\frac{1}{\delta}\sqrt{\mathcal{G}_{n,c}\left((t-x)^2,x\right)}\right)\omega_{b+1}(f,\delta).\end{aligned}$$

After choosing $\delta = \sqrt{\mathcal{G}_{n,c}\left((t-x)^2,x\right)}$, we obtain the required result. □

Theorem 4.2. *Let $f \in C_2'[0,\infty)$, then we have*

$$\lim_{n\to\infty} \|\mathcal{G}_{n,c}(f)-f\|_2 = 0.$$

Proof. From [22], it is sufficient to verify the following by well-known Bohman-Korovkin theorem as:

$$\lim_{n\to\infty}\left\|\mathcal{G}_{n,c}\left(t^i;x\right)-x^i\right\|_2=0,\quad i=0,1,2.$$

From Lemma 2, the result is true for $i=0,1$. Again using Lemma 2, we get

$$\left\|\mathcal{G}_{n,c}\left(t^2;x\right)-x^2\right\|_2=\sup_{x\geq 0}\left|\frac{(n+c)}{(n-c)}x^2+\frac{2}{(n-c)}x-x^2\right|.$$

Finally, we have

$$\lim_{n\to\infty}\left\|\mathcal{G}_{n,c}\left(t^2;x\right)-x^2\right\|_2=0.$$

Thus, we get the desired result. □

Theorem 4.3. *Let $g\in C_2'[0,\infty)$ and $\eta>0$, we have*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{\left|\mathcal{G}_{n,c}(g;x)-g(x)\right|}{(1+x^2)^{1+\eta}}=0,\quad x_0\in(0,\infty].$$

Proof. Let $x_0>0$ be any arbitrary fixed value and $x_0\in(0,\infty]$ then, we have

$$\begin{aligned}\sup_{x\in[0,\infty)}\frac{\left|\mathcal{G}_{n,c}(g;x)-g(x)\right|}{(1+x^2)^{1+\eta}}&\leq\sup_{x\leq x_0}\frac{\left|\mathcal{G}_{n,c}(g;x)-g(x)\right|}{(1+x^2)^{1+\eta}}+\sup_{x>x_0}\frac{\left|\mathcal{G}_{n,c}(g;x)-g(x)\right|}{(1+x^2)^{1+\eta}}\\&\leq\left|\mathcal{G}_{n,c}(g)-g\right|_{C[0,x_0]}+\|g\|_2\sup_{x>x_0}\frac{\left|\mathcal{G}_{n,c}(1+t^2;x)\right|}{(1+x^2)^{1+\eta}}\\&\quad+\sup_{x>x_0}\frac{|g(x)|}{(1+x_0^2)^{1+\eta}}.\end{aligned}$$

From Theorem 4.2, the first term of the above inequality tends to zero.

Since $|g(x)|\leq\|g\|_2(1+x^2)$, we have

$$\sup_{x>x_0}\frac{|g(x)|}{(1+x^2)^{1+\eta}}\leq\frac{\|g\|_2}{(1+x_0^2)^{\eta}}.$$

Let $\varepsilon>0$ be arbitrary and if we choose x_0 very big then

$$\frac{\|g\|_2}{(1+x_0^2)^{\eta}}<\frac{\varepsilon}{2}.\tag{4.3}$$

Since $\lim_{n\to\infty}\sup_{x>x_0}\frac{\mathcal{G}_{n,c}\left(1+t^2;x\right)}{1+x^2}=1$, we have

$$\sup_{x>x_0}\frac{\mathcal{G}_{n,c}(1+t^2;x)}{1+x^2}\leq\frac{(1+x_0^2)^{\eta}}{\|g\|_2}\cdot\frac{\varepsilon}{2}+1\quad\text{as}\quad n\to\infty.$$

Therefore,

$$\|g\|_2\sup_{x>x_0}\frac{\mathcal{G}_{n,c}(1+t^2;x)}{(1+x^2)^{1+\eta}}\leq\frac{\|g\|_2}{(1+x_0{}^2)^{\eta}}\sup_{x>x_0}\frac{\mathcal{G}_{n,c}(1+t^2;x)}{(1+x^2)}\leq\frac{\varepsilon}{2}+\frac{\|g\|_2}{(1+x^2)^{\eta}}.$$

From Theorem 4.1, and for sufficient large n, we have

$$\left\|\mathcal{G}_{n,c}(g)-g\right\|_{C[0,x_0]}<\varepsilon.\tag{4.4}$$

Estimates from (4.3) to (4.4), the theorem is proved. □

Theorem 4.4. *Let $f \in C_2' [0, \infty)$. For sufficient large n, we have*

$$\sup_{x\in[0,\infty)} \frac{|\mathcal{G}_{n,c}(f;x) - f(x)|}{(1+x^2)^{5/2}} \leq \hat{C}\Omega\left(f; n^{-1/2}\right),$$

where $\hat{C} > 0$ is constant.

Proof. For x is a point of interval $\in [0, \infty)$ and δ is a positive number and by using definition of the weighted modulus of continuity and Lemma 2.2, we obtain

$$\begin{aligned} |f(t) - f(x)| \leq& \left(1 + (x + |t-x|)^2\right) \Omega(f; |t-x|) \\ \leq& 2\left(1+x^2\right)\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta}\right)\Omega(f;\delta). \end{aligned}$$

Applying operator $\mathcal{G}_{n,c}$ both sides, we get

$$\begin{aligned} |\mathcal{G}_{n,c}(f;x) - f(x)| \leq& 2\left(1+x^2\right)\Omega(f;\delta)\left\{1 + \mathcal{G}_{n,c}\left((t-x)^2; x\right)\right. \\ &\left. + \mathcal{G}_{n,c}\left(\left(1+(t-x)^2\right)\frac{|t-x|}{\delta}; x\right)\right\}. \end{aligned}$$

Applying Cauchy-Schwarz inequality, Lemma 2.2 and choosing $\delta = \frac{1}{\sqrt{n}}$, we obtain the required result. □

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