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SOME REMARKS ON VARIOUS SCHUR CONVEXITY

FARZANEH GORJIZADEH[1] AND NOHA EFTEKHARI[1]

Abstract. The aim of this work is to investigate the Schur convexity, Schur geometrically convexity, Schur harmonically convexity and Schur power convexity of some special functions. Some sufficient conditions are obtained to guarantee the above-mentioned properties satisfy. We attain some special inequalities. Also, we obtain some applications of main results.

1. introduction

Throughout this work, we denote $\mathbb{R}_+^n = \{(x_1, \ldots, x_n) : \ x_i > 0, \ i = 1, 2, \ldots, n\}$. For the convenience of the readers, we recall the relevant material.

Definition 1.1 ([5]). Let $n \geq 2$ and $x, y \in \mathbb{R}^n$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. We say that x is majorized by y and denoted by $x \prec y$, if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}, \quad \text{for } 1 \leq k \leq n-1,$$
$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}$ are rearrangements of x and y in decreasing order.

Let $E \subseteq \mathbb{R}^n$ be a set with nonempty interior. We say $\varphi : E \to \mathbb{R}$ is Schur convex if $x \prec y$ implies $\varphi(x) \leq \varphi(y)$ and φ is said to be Schur concave if $-\varphi$ is Schur convex.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called a symmetric function, if $f(Px) = f(x)$ for any $x \in \mathbb{R}^n$ and any $n \times n$ permutation matrix P. A set $E \subseteq \mathbb{R}^n$ is called symmetric, if

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$x \in E$ implies $xP \in E$ for any $n \times n$ permutation matrix P. Also, a set $E \subseteq \mathbb{R}^n$ is called a convex set if for any $x, y \in E$ and $\lambda \in [0,1]$, we have $\lambda x + (1-\lambda)y \in E$.

In this work, we need the following three lemmas.

Lemma 1.1 ([5]). *Let $E \subseteq \mathbb{R}^n$ be a symmetric convex set with nonempty interior and $\varphi : E \to \mathbb{R}$ is a continuous symmetric function on E. If φ is differentiable on* $\operatorname{int} E$, *then φ is Schur convex (Schur concave) on E if and only if*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \geq 0 \quad \text{or } (\leq 0)$$

holds for all $x = (x_1, \ldots, x_n) \in \operatorname{int} E$.

Lemma 1.2 ([2,7]). *Let $E \subset \mathbb{R}_+^n$ be a symmetric geometrically convex set with a nonempty interior and $\varphi : E \to \mathbb{R}_+$ be continuous on E and differentiable on* $\operatorname{int} E$. *Then φ is Schur geometrically convex (Schur geometrically concave) if and only if φ is symmetric on E and*

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial \varphi}{\partial x_1} - x_2\frac{\partial \varphi}{\partial x_2}\right) \geq 0 \quad \text{or } (\leq 0) \tag{1.1}$$

holds for all $x = (x_1, \ldots, x_n) \in \operatorname{int} E$, where E is a geometrically convex set, if for any $x, y \in E$ and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta = 1$, we have $x^\alpha y^\beta \in E$.

Since for any $x_1, x_2 \in \mathbb{R}$, we have

$$(x_1 - x_2)(\log x_1 - \log x_2) \geq 0,$$

we can reduce (1.1) *to the following inequality*

$$(x_1 - x_2)\left(x_1\frac{\partial \varphi}{\partial x_1} - x_2\frac{\partial \varphi}{\partial x_2}\right) \geq 0 \quad \text{or } (\leq 0). \tag{1.2}$$

Lemma 1.3. ([6, Lemma 2.2]). *Let $E \subseteq \mathbb{R}_+^n$ be a symmetric harmonic convex set with nonempty interior and $\varphi : E \to \mathbb{R}_+$ be a continuous symmetric function on E. If φ is differentiable on* $\operatorname{int} E$, *then φ is Schur harmonic convex (Schur harmonic concave) on E if and only if*

$$(x_1 - x_2)\left(x_1^2\frac{\partial \varphi}{\partial x_1} - x_2^2\frac{\partial \varphi}{\partial x_2}\right) \geq 0 \quad \text{or } (\leq 0)$$

holds for all $x = (x_1, \ldots, x_n) \in \operatorname{int} E$, where E is a harmonic convex set, if for any $x, y \in E$, we have $\frac{2xy}{x+y} \in E$.

In 1923, the Schur convexity was discovered by I. Schur. It has many interested applications of symmetric functions in Hadamard's inequality, analytic inequalities, stochastic ordering and some other branches of graphs and matrices, see for example [1,3,4].

We organize this paper as follow. We establish the integral mean of fg is Schur convex, Schur geometrical convex, Schur harmonic convex, and Schur power convex on $[0,\infty)\times[0,\infty)$, for convex, continuous and similarly ordered functions f and g. In Section 3, we obtain some applications of results in Section 2.

2. Main Results

In this section, we obtain some results for special functions to be Schur convex (Schur concave), Schur geometrically convex, Schur harmonically convex, and Schur power convex.

We say that $f,g:\mathbb{R}\to\mathbb{R}$ are similarly ordered function if for all $x,y\in\mathbb{R}$, we have

$$(f(x)-f(y))(g(x)-g(y))\geq 0,$$

if the above inequality reversed, we say that f and g have oppositely ordered.

Lemma 2.1. *Let $f,g:\mathbb{R}\to[0,\infty)$ be convex, continuous and similarly ordered functions. Then for $x,y\in\mathbb{R}$, we have*

$$\frac{1}{y-x}\int_x^y f(t)g(t)dt\leq\frac{f(x)g(x)+f(y)g(y)}{2}.$$

Proof. Since f and g have similarly ordered, for any $x,y\in\mathbb{R}$ we have

$$(f(x)-f(y))(g(x)-g(y))\geq 0.$$

It follows that

$$f(x)g(y)+f(y)g(x)\leq f(x)g(x)+f(y)g(y). \tag{2.1}$$

On the other hand, f and g are convex functions, so for $x,y\in\mathbb{R}$ and $t\in[0,1]$, we have

$$\begin{aligned}f(tx+(1-t)y)&\leq tf(x)+(1-t)f(y),\\ g(tx+(1-t)y)&\leq tg(x)+(1-t)g(y).\end{aligned}$$

By multiplying both sides of the latter inequalities together and integrating on $[0,1]$, we get

$$\begin{aligned}&\int_0^1 f(tx+(1-t)y)g(tx+(1-t)y)dt\\ &\leq\int_0^1[t^2f(x)g(x)+t(1-t)[f(x)g(y)+g(x)f(y)]+(1-t)^2f(y)g(y)]dt,\end{aligned}$$

with change of variable $u=tx+(1-t)y=t(x-y)+y$, it follows

$$\begin{aligned}\frac{1}{y-x}\int_x^y f(u)g(u)du&\leq\frac{f(x)g(x)+f(y)g(y)}{3}+\frac{f(x)g(y)+f(y)g(x)}{6}\\ &\leq\frac{f(x)g(x)+f(y)g(y)}{2}.\end{aligned}$$

Now, (2.1) follows from the last inequality. □

Theorem 2.1. *Let $f, g : \mathbb{R} \to [0, \infty)$ be convex, continuous and similarly ordered functions. Then*

$$F(x,y) = \begin{cases} \dfrac{1}{y-x}\displaystyle\int_x^y f(t)g(t)dt, & x \neq y, \\ f(x)g(x), & x = y, \end{cases}$$

is Schur convex on $\mathbb{R}^2$.

Proof. By Lemma 2.1, we have

$$\begin{aligned} \left(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\right)(y-x) = & \left[-\frac{1}{(y-x)^2}\int_x^y f(t)g(t)dt + \frac{f(y)g(y)}{y-x}\right. \\ & \left. -\frac{1}{(y-x)^2}\int_x^y f(t)g(t)dt + \frac{f(x)g(x)}{y-x}\right](y-x) \\ = & f(x)g(x) + f(y)g(y) - \frac{2}{y-x}\int_x^y f(t)g(t)dt \geq 0. \end{aligned}$$

Now Lemma 1.1 implies that F is Schur convex. □

Corollary 2.1. *Let $\alpha \geq 1$. Then*

$$F(x,y) = \begin{cases} \dfrac{1}{y-x}\displaystyle\int_x^y t^\alpha e^t dt, & x \neq y, \\ x^\alpha e^x, & x = y, \end{cases}$$

is Schur convex on $[0, \infty) \times [0, \infty)$.

Proof. Suppose that $f, g : [0, \infty) \to [0, \infty)$ are defined by $f(t) = t^\alpha$ and $g(t) = e^t$. Since $\alpha \geq 1$, the function f is increasing and convex, according to Theorem 2.1, F is Schur convex. □

The next two corollaries are results of Theorem 2.1.

Corollary 2.2. *Let $f : \mathbb{R} \to [0, \infty)$ be increasing, continuous and convex function. Then*

$$F(x,y) = \begin{cases} \dfrac{1}{y-x}\displaystyle\int_x^y e^t f(t)dt, & x \neq y, \\ e^x f(x), & x = y, \end{cases}$$

is Schur convex on $\mathbb{R}^2$.

Corollary 2.3. *Let $f : [0, \infty) \to [0, \infty)$ be increasing, continuous and convex function and $\alpha \geq 1$. Then*

$$F(x,y) = \begin{cases} \dfrac{1}{y-x}\displaystyle\int_x^y t^\alpha f(t)dt, & x \neq y, \\ x^\alpha f(x), & x = y, \end{cases}$$

is Schur convex on $[0, \infty) \times [0, \infty)$.

Similar to Lemma 2.1, we have the following lemma for concave and oppositely ordered functions.

Lemma 2.2. *Let $f, g : \mathbb{R} \to [0,\infty)$ be concave, continuous and oppositely ordered functions. Then for $x, y \in \mathbb{R}$ we have*

$$\frac{1}{y-x}\int_x^y f(t)g(t)dt \geq \frac{f(x)g(x)+f(y)g(y)}{2}.$$

Theorem 2.2. *Let $f, g : \mathbb{R} \to [0,\infty)$ be concave, continuous and oppositely ordered functions. Then*

$$F(x,y) = \begin{cases} \dfrac{1}{y-x}\displaystyle\int_x^y f(t)g(t)dt, & x \neq y, \\ f(x)g(x), & x = y, \end{cases}$$

is Schur concave on $\mathbb{R}^2$.

Proof. The result follows by similar arguments to the proof of Theorem 2.1 and using Lemma 2.2. ☐

Theorem 2.2 implies next two corollaries.

Corollary 2.4. *Let $f : [0,\infty) \to [0,\infty)$ be decreasing and concave function and $0 < \alpha < 1$. Then*

$$F(x,y) = \begin{cases} \dfrac{1}{y-x}\displaystyle\int_x^y t^\alpha f(t)dt, & x \neq y, \\ x^\alpha f(x), & x = y, \end{cases}$$

is Schur concave on $[0,\infty) \times [0,\infty)$.

Corollary 2.5. *The function*

$$F(x,y) = \begin{cases} \dfrac{1}{y-x}\displaystyle\int_x^y \operatorname{sech} t \ln t dt, & x \neq y, \\ \operatorname{sech} x \ln x, & x = y, \end{cases}$$

is Schur concave on $[0,\infty) \times [0,\infty)$.

By Lemmas 1.1, 1.2 and 1.3, we have the following theorem.

Theorem 2.3. *Let f and g be two real continuous functions defined on $\mathbb{R}$, then*

$$F(x,y) = \begin{cases} \dfrac{1}{y-x}\displaystyle\int_x^y f(t)g(t)dt, & x \neq y, \\ f(x)g(x), & x = y, \end{cases}$$

is Schur convex (concave) on $[0,\infty) \times [0,\infty)$ if and only if

$$F(x,y) \leq (\geq) \frac{f(x)g(x)+f(y)g(y)}{2}, \tag{2.2}$$

is Schur geometrically convex (*concave*) *on* $[0,\infty)\times[0,\infty)$ *if and only if*

$$F(x,y) \le (\ge) \frac{xf(x)g(x)+yf(y)g(y)}{x+y}, \tag{2.3}$$

and is Schur harmonically convex (*concave*) *on* $\mathbb{R}^2_+$ *if and only if*

$$F(x,y) \le (\ge) \frac{x^2f(x)g(x)+y^2f(y)g(y)}{x^2+y^2}. \tag{2.4}$$

Proof. From Lemma 1.1 it follows that F is Schur convex (concave) on $[0,\infty)\times[0,\infty)$ if and only if

$$(y-x)\left(\frac{\partial F}{\partial y}-\frac{\partial F}{\partial x}\right) \ge 0\,(\le 0).$$

On the other hand, as in the proof of Theorem 2.1, we have

$$(y-x)\left(\frac{\partial F}{\partial y}-\frac{\partial F}{\partial x}\right) = f(x)g(x)+f(y)g(y)-\frac{2}{y-x}\int_x^y f(t)g(t)dt.$$

This implies (2.2).

From Lemma 1.2 it follows that F is Schur geometrically convex (concave) on $[0,\infty)\times[0,\infty)$ if and only if

$$(y-x)\left(y\frac{\partial F}{\partial y}-x\frac{\partial F}{\partial x}\right) \ge 0\,(\le 0).$$

But

$$\begin{aligned}(y-x)\left(y\frac{\partial F}{\partial y}-x\frac{\partial F}{\partial x}\right) &= (y-x)\left[-\frac{y}{(y-x)^2}\int_x^y f(t)g(t)dt+\frac{yf(y)g(y)}{y-x}\right.\\ &\quad\left.-\frac{x}{(y-x)^2}\int_x^y f(t)g(t)dt+\frac{xf(x)g(x)}{y-x}\right]\\ &= xf(x)g(x)+yf(y)g(y)-\frac{x+y}{y-x}\int_x^y f(t)g(t)dt,\end{aligned}$$

hence (2.3) follows.

From Lemma 1.3 it follows that F is Schur harmonically convex (concave) on $\mathbb{R}^2_+$ if and only if

$$(y-x)\left(y^2\frac{\partial F}{\partial y}-x^2\frac{\partial F}{\partial x}\right) \ge 0\,(\le 0).$$

On the other hand, we have

$$\begin{aligned}(y-x)\left(y^2\frac{\partial F}{\partial y}-x^2\frac{\partial F}{\partial x}\right) &= (y-x)\left[-\frac{y^2}{(y-x)^2}\int_x^y f(t)g(t)dt+\frac{y^2f(y)g(y)}{y-x}\right.\\ &\quad\left.-\frac{x^2}{(y-x)^2}\int_x^y f(t)g(t)dt+\frac{x^2f(x)g(x)}{y-x}\right]\\ &= x^2f(x)g(x)+y^2f(y)g(y)-\frac{x^2+y^2}{y-x}\int_x^y f(t)g(t)dt,\end{aligned}$$

Therefore, (2.4) holds. □

In [8, Definition 2.3], we put $f(x) = x^\alpha$, then the following definition follows.

Definition 2.1. Let α be a positive real number and $E \subseteq \mathbb{R}_+^n$ be such that $x \in E$ implies $x^{\frac{1}{\alpha}} = (x_1^{\frac{1}{\alpha}}, \dots, x_n^{\frac{1}{\alpha}}) \in E$. A real-valued function $F : E \to \mathbb{R}$ is said to be Schur power convex if

$$F(x_1, \dots, x_n) \le F(y_1, \dots, y_n),$$

holds for each pair of n-tuples $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ in E such that

$$(x_1^\alpha, \dots, x_n^\alpha) \prec (y_1^\alpha, \dots, y_n^\alpha),$$

and F is Schur power concave if $-F$ is Schur power convex.

Remark 2.1. Let $E \subseteq \mathbb{R}_+^n$ and α be a positive real number. Then $F : E \to (0, \infty)$ is Schur power convex on E if and only if $F(x^{\frac{1}{\alpha}})$ is Schur convex function.

Lemma 2.3. *Let $E \in \mathbb{R}_+^n$ be a symmetric convex set with nonempty interior and $F : E \to \mathbb{R}$ be a continuous symmetric function on E. If F is differentiable on* $\operatorname{int} E$, *then F is Schur power convex* (*Schur power concave*) *on E if and only if*

$$(x_1^\alpha - x_2^\alpha)\left(x_1^{1-\alpha}\frac{\partial F}{\partial x_1} - x_2^{1-\alpha}\frac{\partial F}{\partial x_2}\right) \ge 0\,(\le 0),$$

for all $x = (x_1, \dots, x_n) \in \operatorname{int} E$ and $\alpha \in \mathbb{R}_+$.

Proof. The result follows by using Definition 2.1 and Remark 2.1 and Lemma 1.1. □

Theorem 2.4. *Let $\alpha \in \mathbb{R}_+$. Let f and g be two real continuous functions defined on* $\mathbb{R}$, *then*

$$F(x, y) = \begin{cases} \dfrac{1}{y - x}\displaystyle\int_x^y f(t)g(t)dt, & x \ne y, \\ f(x)g(x), & x = y, \end{cases}$$

is Schur power convex (*concave*) *on* $[0, \infty) \times [0, \infty)$ *if and only if*

$$F(x, y) \le (\ge) \frac{x^{1-\alpha} f(x)g(x) + y^{1-\alpha} f(y)g(y)}{x^{1-\alpha} + y^{1-\alpha}}.$$

Proof. Let $x, y \in [0, \infty)$ and $x \ne y$. According to Lemma 2.3, $F(x, y)$ is Schur power convex (concave) if and only if

$$(y^\alpha - x^\alpha)\left(y^{1-\alpha}\frac{\partial F}{\partial y} - x^{1-\alpha}\frac{\partial F}{\partial x}\right) \ge 0\,(\le 0).$$

But we have

$$(y^\alpha - x^\alpha)\left(y^{1-\alpha}\frac{\partial F}{\partial y} - x^{1-\alpha}\frac{\partial F}{\partial x}\right)$$

$$
\begin{aligned}
=&(y^\alpha - x^\alpha)\left[-\frac{y^{1-\alpha}}{(y-x)^2}\int_x^y f(t)g(t)dt + \frac{y^{1-\alpha}f(y)g(y)}{y-x}\right.\\
&\left.-\frac{x^{1-\alpha}}{(y-x)^2}\int_x^y f(t)g(t)dt + \frac{x^{1-\alpha}f(x)g(x)}{y-x}\right]\\
=&\frac{x^{1-\alpha}f(x)g(x)+y^{1-\alpha}f(y)g(y)}{y-x} - \frac{x^{1-\alpha}+y^{1-\alpha}}{(y-x)^2}\int_x^y f(t)g(t)dt.
\end{aligned}
$$

As F is symmetric, that is $F(x,y)=F(y,x)$, we get the conclusion. $\square$

Corollary 2.6. *Let $\alpha,\beta\in(0,\infty)$ and f be a real continuous function defined on $\mathbb{R}$, then*

$$
F(x,y)=\begin{cases} \dfrac{1}{y-x}\displaystyle\int_x^y t^\beta f(t)dt, & x\neq y,\\ x^\beta f(x), & x=y, \end{cases}
$$

is Schur power convex on $[0,\infty)\times[0,\infty)$ if and only if

$$
F(x,y)\leq \frac{x^{1-\alpha+\beta}f(x)+y^{1-\alpha+\beta}f(y)}{x^{1-\alpha}+y^{1-\alpha}}.
$$

Proof. In Theorem 2.4, put $g(x)=x^\beta$. $\square$

Theorem 2.5. *Let $f,g:[0,\infty)\to[0,\infty)$ be convex (concave), continuous and similarly (oppositely) ordered functions on $[0,\infty)$. Then*

$$
F(x,y)=\begin{cases} \dfrac{1}{y-x}\displaystyle\int_x^y f(t)g(t)dt, & x\neq y,\\ f(x)g(x), & x=y, \end{cases}
$$

(i) *is Schur geometrically convex (concave) on $[0,\infty)\times[0,\infty)$;*
(ii) *is Schur harmonically convex (concave) on $[0,\infty)\times[0,\infty)$;*
(iii) *is Schur power convex (concave) on $[0,\infty)\times[0,\infty)$, if $0<\alpha<1$.*

Proof. (i) As f and g have similarly (oppositely) ordered and nonnegative on $[0,\infty)$, then for all $x,y\in[0,\infty)$, we have

$$
(y-x)(f(y)g(y)-f(x)g(x))\geq 0\,(\leq 0). \tag{2.5}
$$

This implies that

$$
xf(y)g(y)+yf(x)g(x)\leq(\geq)xf(x)g(x)+yf(y)g(y),
$$

and it follows that

$$
\frac{f(y)g(y)+f(x)g(x)}{2}\leq(\geq)\frac{xf(x)g(x)+yf(y)g(y)}{x+y}. \tag{2.6}
$$

Now, from (2.6) and Lemma 2.1 (Lemma 2.2) together with Theorem 2.3 it follows that $F(x,y)$ is Schur geometrically convex (concave) on $[0,\infty)\times[0,\infty)$.

(ii) Since f and g have similarly (oppositely) ordered and nonnegative on $[0,\infty)$, then for all $x,y\in[0,\infty)$, we have (2.5). It follows that

$$(y^2-x^2)(f(y)g(y)-f(x)g(x))\geq 0\,(\leq 0).$$

This implies that

$$x^2f(y)g(y)+y^2f(x)g(x)\leq(\geq)x^2f(x)g(x)+y^2f(y)g(y),$$

and it follows that

$$\frac{f(y)g(y)+f(x)g(x)}{2}\leq(\geq)\frac{x^2f(x)g(x)+y^2f(y)g(y)}{x^2+y^2}. \tag{2.7}$$

From (2.7) and Lemma 2.1 (Lemma 2.2) together with Theorem 2.3 it follows that $F(x,y)$ is Schur harmonically convex (concave) on $[0,\infty)\times[0,\infty)$.

(iii) Since f and g have similarly (oppositely) ordered and nonnegative on $[0,\infty)$ and $0<\alpha<1$, then for all $x,y\in[0,\infty)$ we have

$$(y^{1-\alpha}-x^{1-\alpha})(f(y)g(y)-f(x)g(x))\geq 0\,(\leq 0).$$

It follows that

$$x^{1-\alpha}f(y)g(y)+y^{1-\alpha}f(x)g(x)\leq(\geq)x^{1-\alpha}f(x)g(x)+y^{1-\alpha}f(y)g(y).$$

This yields

$$\frac{f(y)g(y)+f(x)g(x)}{2}\leq(\geq)\frac{x^{1-\alpha}f(x)g(x)+y^{1-\alpha}f(y)g(y)}{x^{1-\alpha}+y^{1-\alpha}}. \tag{2.8}$$

Now, from the inequality (2.8) and Lemma 2.1 (Lemma 2.2) together with Theorem 2.3 it follows that $F(x,y)$ is Schur power convex (concave) on $[0,\infty)\times[0,\infty)$. □

Corollary 2.7. *Let $\alpha,\beta\in(1,2)$. Then*

$$F(x,y)=\begin{cases}\dfrac{1}{y-x}\displaystyle\int_x^y t^{\alpha-1}(1-t)^{\beta-1}dt, & x\neq y,\\ x^{\alpha-1}(1-x)^{\beta-1}, & x=y,\end{cases}$$

is Schur concave, geometrically Schur concave and harmonically Schur concave on $[0,1]\times[0,1]$. Also, for all $x,y\in[0,1]$ such that $x\neq y$ the following inequalities hold

$$\frac{1}{y-x}\int_x^y t^{\alpha-1}(1-t)^{\beta-1}dt\geq\frac{x^{\alpha-1}(1-x)^{\beta-1}+y^{\alpha-1}(1-y)^{\beta-1}}{2},$$

$$\frac{1}{y-x}\int_x^y t^{\alpha-1}(1-t)^{\beta-1}dt\geq\frac{x^{\alpha}(1-x)^{\beta-1}+y^{\alpha}(1-y)^{\beta-1}}{x+y},$$

$$\frac{1}{y-x}\int_x^y t^{\alpha-1}(1-t)^{\beta-1}dt\geq\frac{x^{\alpha+1}(1-x)^{\beta-1}+y^{\alpha+1}(1-y)^{\beta-1}}{x^2+y^2}.$$

Proof. In Theorems 2.2, 2.5, we put $f(x) = x^{\alpha-1}$ and $g(x) = (1-x)^{\beta-1}$. Since $\alpha, \beta \in (1,2)$ on $[0,1]$ the function f is increasing and concave and g is decreasing and concave. It follows that on $[0,1]$ the functions f and g are concave, continuous and oppositely ordered. Now, Theorem 2.3 implies the results. □

Theorem 2.6. *Let α be a positive real number and $f : (0,\infty) \to (0,\infty)$ be a log-concave function. Then $t^\alpha f(t)$ is log-concave and the following inequality holds*

$$\frac{1}{y-x}\int_x^y t^\alpha f(t)dt \geq \frac{x^\alpha f(x) - y^\alpha f(y)}{\ln(x^\alpha f(x)) - \ln(y^\alpha f(y))}.$$

Proof. For $\alpha > 0$, function t^α is log-concave. Since $\ln t$ is concave and $\alpha > 0$, we have

$$\lambda\alpha(\ln x) + (1-\lambda)\alpha \ln y \leq \alpha \ln(\lambda x + (1-\lambda)y),$$

so

$$\lambda(\ln x^\alpha) + (1-\lambda)\ln y^\alpha \leq \ln(\lambda x + (1-\lambda)y)^\alpha.$$

Thus, t^α is log-concave. Put $g(x) = x^\alpha f(x)$, then for $t \in [0,1]$, we have

$$\begin{aligned}
g(tx + (1-t)y) &= (tx + (1-t)y)^\alpha f(tx + (1-t)y) \\
&\geq (x^\alpha)^t (y^\alpha)^{1-t} (f(x))^t (f(y))^{1-t} \\
&= (x^\alpha f(x))^t (y^\alpha f(y))^{1-t} \\
&= (g(x))^t (g(y))^{1-t} \\
&= \left(\frac{x^\alpha f(x)}{y^\alpha f(y)}\right)^t (y^\alpha f(y)),
\end{aligned}$$

that is, $g(x) = x^\alpha f(x)$ is log-concave. By integrating both sides of the above inequality on $[0,1]$ and change of variable $u = tx + (1-t)y$, getting $w = \dfrac{x^\alpha f(x)}{y^\alpha f(y)}$, then we have

$$\int_0^1 (tx + (1-t)y)^\alpha f(tx + (1-t)y)dt \geq y^\alpha f(y) \int_0^1 \left(\frac{x^\alpha f(x)}{y^\alpha f(y)}\right)^t dt,$$

$$\begin{aligned}
\frac{1}{y-x}\int_x^y u^\alpha f(u)du &\geq y^\alpha f(y) \int_0^1 w^t dt \\
&= \frac{y^\alpha f(y)}{\ln\left(\frac{x^\alpha f(x)}{y^\alpha f(y)}\right)} \left(\frac{x^\alpha f(x)}{y^\alpha f(y)} - 1\right) \\
&= \frac{x^\alpha f(x) - y^\alpha f(y)}{\ln(x^\alpha f(x)) - \ln(y^\alpha f(y))}.
\end{aligned}$$

□

Lemma 2.4. *Let I be an interval in $\mathbb{R}$ and $f, g : I \to [0, \infty)$ be continuous functions. Then for $x \in I^n \subseteq \mathbb{R}^n$ the function*

$$F(x) = \sum_{i=1}^{n} \int_0^{x_i} f(t)g(t)dt$$

is Schur convex if and only if f and g are similarly ordered functions (is Schur concave if and only if f and g are oppositely ordered functions).

Proof. Clearly F is symmetric. According to Lemma 1.1, F is Schur convex if and only if for $x_1, x_2 \in I$, we have

$$(x_1 - x_2)\left(\frac{\partial F}{\partial x_1} - \frac{\partial F}{\partial x_2}\right) = (x_1 - x_2)\left(f(x_1)g(x_1) - f(x_2)g(x_2)\right) \geq 0,$$

if and only if f and g are similarly ordered functions. □

Lemma 2.5. *Let I be an interval in $\mathbb{R}$ and $f : I \to (0, \infty)$ be differentiable on* $\operatorname{int} I$. *Then for $x \in I^n \subseteq \mathbb{R}^n$ the function*

$$F(x) = \prod_{i=1}^{n} f(x_i)$$

is Schur convex if and only if $\frac{f'}{f}$ is increasing on I (is Schur concave if and only if $\frac{f'}{f}$ is decreasing on I).

Proof. Clearly F is symmetric. According to Lemma 1.1, F is Schur convex if and only if for $x_1, x_2 \in I$, we have

$$\begin{aligned}(x_1 - x_2)\left(\frac{\partial F}{\partial x_1} - \frac{\partial F}{\partial x_2}\right) &= (x_1 - x_2)\left(f'(x_1)\prod_{i=2}^{n} f(x_i) - f'(x_2)\prod_{i=1, i\neq 2}^{n} f(x_i)\right) \\ &= (x_1 - x_2)\prod_{i=3}^{n} f(x_i)(f'(x_1)f(x_2) - f(x_1)f'(x_2)) \geq 0,\end{aligned}$$

if and only if $\frac{f'}{f}$ is increasing on I. □

Remark 2.2. As in the literature, the infinite decreasing sequence $x = (x_n)$ majorized by the infinite decreasing sequence $y = (y_n)$ and denoted by $x \prec y$, if there exists an infinite doubly stochastic square matrix $P = (p_{ij})$ (i.e., $p_{ij} \geq 0$ for all $i, j \in \mathbb{N}$, and all rows sum and all columns sum are equal one) such that $x = y.P$. If (α_n) be a sequence in the interval $[0, 1]$, we take $x_1 = \alpha_1 y_1 + (1 - \alpha_1)y_2$, $x_2 = (1 - \alpha_1)y_1 + \alpha_1 y_2$, and $x_3 = \alpha_2 y_3 + (1 - \alpha_2)y_4$, $x_4 = (1 - \alpha_2)y_3 + \alpha_2 y_4, \dots$, where $y = (y_n)$ is an infinite decreasing real sequence. If we put

$$P=\begin{bmatrix} \alpha_1 & 1-\alpha_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1-\alpha_1 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \alpha_2 & 1-\alpha_2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1-\alpha_2 & \alpha_2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \ddots & \ddots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 & \cdots \end{bmatrix},$$

then $x = yP$ and $x \prec y$.

Example 2.1. In Lemma 2.5, set $f(x) = \sin x$ and $I = (0, \pi)$. The function $f'(x) = \cos x$ and $\frac{f'(x)}{f(x)} = \cot x$ is decreasing on I. So, $F(x) = \prod_{i=1}^{n} \sin x_i$ is Schur concave. Let $x = (x_n)$ and $y = (y_n)$ be two decreasing sequence in $I = (0, \pi)$, such that $x \prec y$ as in Remark 2.2. Since F is Schur concave, we have $F(x) \geq F(y)$ and so

$$\begin{aligned} &\sin(\alpha_1 y_1 + (1-\alpha_1)y_2)\sin((1-\alpha_1)y_1 + \alpha_1 y_2)\sin(\alpha_2 y_3 + (1-\alpha_2)y_4) \\ &\times \sin((1-\alpha_2)y_3 + \alpha_2 y_4)\cdots \geq \prod_{i=1}^{\infty} \sin y_i. \end{aligned}$$

In the special case, $\alpha_i = \frac{1}{2}$ for all $i \in \mathbb{N}$, we have

$$\sin\left(\frac{y_1+y_2}{2}\right)\sin\left(\frac{y_3+y_4}{2}\right)\cdots \geq \left(\prod_{i=1}^{\infty}\sin y_i\right)^{\frac{1}{2}}.$$

Example 2.2. In Lemma 2.5, put $f(x) = \cos x$ and $I = (0, \frac{\pi}{2})$. The function $f'(x) = -\sin x$ and $\frac{f'(x)}{f(x)} = -\tan x$ is decreasing on I. So $F(x) = \prod_{i=1}^{n} \cos x_i$ is Schur concave. Let $x = (x_n)$ and $y = (y_n)$ be two decreasing sequence in $I = (0, \frac{\pi}{2})$, such that $x \prec y$ as in Remark 2.2. Since F is Schur concave, we have $F(x) \geq F(y)$ and so

$$\begin{aligned} &\cos(\alpha_1 y_1 + (1-\alpha_1)y_2)\cos((1-\alpha_1)y_1 + \alpha_1 y_2)\cos(\alpha_2 y_3 + (1-\alpha_2)y_4) \\ &\times \cos((1-\alpha_2)y_3 + \alpha_2 y_4)\cdots \geq \prod_{i=1}^{\infty} \cos y_i. \end{aligned}$$

In the special case, $\alpha_i = \frac{1}{2}$ for all $i \in \mathbb{N}$, we have

$$\cos\left(\frac{y_1+y_2}{2}\right)\cos\left(\frac{y_3+y_4}{2}\right)\cdots \geq \left(\prod_{i=1}^{\infty}\cos y_i\right)^{\frac{1}{2}}.$$

As in [9], let $I = (0, l)$ and $L_n = \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum\limits_{i=1}^{n} x_i = ml\right\}$ for some $0 < m < n$, $D_n = I^n \cap L_n$ and $\Omega = (y, \ldots, y)$, where $y = \frac{1}{n}\sum\limits_{i=1}^{n} x_i = \frac{ml}{n}$.

Lemma 2.6. ([9, Lemma 2.1]). *If $f : I^n \to \mathbb{R}$ is a Schur-convex function, then $f(\Omega)$ is a global minimum in D_n. If f is strictly Schur-convex on I^n, then $f(\Omega)$ is the unique global mimimum in D_n.*

Remark 2.3. In Example 2.2 and Lemma 2.6, put $l = \frac{\pi}{2}$ and $x_i \in (0, \frac{\pi}{2})$, for $i = 1, 2, \ldots, n$, and $\sum\limits_{i=1}^{n} x_i = \pi$. Then $\Omega = (\frac{\pi}{n}, \ldots, \frac{\pi}{n})$ and we have $F(x) \leq F(\Omega)$, that is

$$\prod_{i=1}^{n} \cos x_i \leq \left(\cos \frac{\pi}{n}\right)^n.$$

Similarly in Example 2.1, for $l = \frac{\pi}{2}$, we have

$$\prod_{i=1}^{n} \sin x_i \leq \left(\sin \frac{\pi}{n}\right)^n.$$

Lemma 2.7. *Let I be an interval in $\mathbb{R}$ and $f : I \to (0, \infty)$ be continuous, then for each $x \in I^n \subset \mathbb{R}^n$, the function*

$$F(x) = \prod_{i=1}^{n} \int_0^{x_i} f(t)dt,$$

is Schur convex if and only if $\dfrac{\int_0^x f(t)dt}{f(x)}$ is decreasing on I (is Schur concave if and only if $\frac{\int_0^x f(t)dt}{f(x)}$ is increasing on I).

Proof. Clearly F is symmetric. According to Lemma 1.1, F is Schur convex if and only if for $x_1, x_2 \in I$, we have

$$\begin{aligned}
(x_1 - x_2)\left(\frac{\partial F}{\partial x_1} - \frac{\partial F}{\partial x_2}\right) =& (x_1 - x_2)\left(f(x_1)\prod_{i=2}^{n}\int_0^{x_i} f(t)dt - f(x_2)\prod_{i=1, i\neq 2}^{n}\int_0^{x_i} f(t)dt\right)\\
=&(x_1 - x_2)\prod_{i=3}^{n}\int_0^{x_i} f(t)dt\\
&\times\left(f(x_1)\int_0^{x_2} f(t)dt - f(x_2)\int_0^{x_1} f(t)dt\right)\\
\geq& 0,
\end{aligned}$$

if and only if $\frac{\int_0^x f(t)dt}{f(x)}$ is decreasing on I. □

3. Applications

In this section, we obtain some inequalities, which are the applications of the results in Section 2.

The next two examples are the applications of Lemma 2.1 and Theorems 2.1, 2.3 and 2.5.

Example 3.1. Let $\alpha \geq 1$ and $E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)}$ be the Mittage-Leffler function. Let

$$F(x, y) = \begin{cases} \dfrac{1}{y - x}\displaystyle\int_x^y t^\alpha E_\alpha(t^\alpha)dt, & x \neq y,\\ x^\alpha E_\alpha(x^\alpha), & x = y.\end{cases}$$

Since t^α and $E_\alpha(t)$ are convex, continuous and similarly ordered on $[0,\infty)$, then Lemma 2.1 and Theorems 2.1, 2.3 and 2.5 imply that F is Schur convex, Schur geometrically convex and Schur harmonically convex on $[0,\infty)\times[0,\infty)$ and for $x,y\in[0,\infty)$, the following inequalities hold

$$\begin{aligned}\frac{1}{y-x}\int_x^y t^\alpha E_\alpha(t^\alpha)dt \leq &\frac{x^\alpha E_\alpha(x^\alpha)+y^\alpha E_\alpha(y^\alpha)}{2},\\ \frac{1}{y-x}\int_x^y t^\alpha E_\alpha(t^\alpha)dt \leq &\frac{x^{\alpha+1} E_\alpha(x^\alpha)+y^{\alpha+1} E_\alpha(y^\alpha)}{x+y},\\ \frac{1}{y-x}\int_x^y t^\alpha E_\alpha(t^\alpha)dt \leq &\frac{x^{\alpha+2} E_\alpha(x^\alpha)+y^{\alpha+2} E_\alpha(y^\alpha)}{x^2+y^2}.\end{aligned}$$

Example 3.2. Let $\alpha>0$ and

$$F(x,y)=\begin{cases}\dfrac{1}{y-x}\displaystyle\int_x^y \Gamma(t)E_\alpha(t^\alpha)dt, & x\neq y,\\ \Gamma(x)E_\alpha(x^\alpha), & x=y.\end{cases}$$

Since $\Gamma(t)$ and $E_\alpha(t)$ are convex, continuous and similarly ordered on $\left[\frac{3}{2},\infty\right)$, then Lemma 2.1 and Theorems 2.1, 2.3 and 2.5 imply that F is Schur convex, Schur geometrically convex and Schur harmonically convex on $\left[\frac{3}{2},\infty\right)\times\left[\frac{3}{2},\infty\right)$ and for $x,y\in\left[\frac{3}{2},\infty\right)$, the following inequalities hold

$$\begin{aligned}\frac{1}{y-x}\int_x^y \Gamma(t)E_\alpha(t^\alpha)dt \leq &\frac{\Gamma(x)E_\alpha(x^\alpha)+\Gamma(y)E_\alpha(y^\alpha)}{2},\\ \frac{1}{y-x}\int_x^y \Gamma(t)E_\alpha(t^\alpha)dt \leq &\frac{x\Gamma(x)E_\alpha(x^\alpha)+y\Gamma(y)E_\alpha(y^\alpha)}{x+y},\\ \frac{1}{y-x}\int_x^y \Gamma(t)E_\alpha(t^\alpha)dt \leq &\frac{x^2\Gamma(x)E_\alpha(x^\alpha)+y^2\Gamma(y)E_\alpha(y^\alpha)}{x^2+y^2}.\end{aligned}$$

Remark 3.1. For $x,y\in[0,\infty)$, the following majorizations hold

$$(1+x,1+y)\prec(1+x+y,1), \tag{3.1}$$

$$\left(\frac{1}{H_2(x,y)},\frac{1}{H_2(x,y)}\right)\prec\left(\frac{1}{x},\frac{1}{y}\right), \tag{3.2}$$

$$\left(\frac{x+y}{2},\frac{x+y}{2}\right)\prec(x,y), \tag{3.3}$$

where $H_2(x,y)=\frac{2}{\frac{1}{x}+\frac{1}{y}}$.

Example 3.3. Let $f,g:[0,\infty)\to[0,\infty)$ be convex, continuous and similarly ordered functions. Then for $x,y\in[0,\infty)$ with $x\neq y$, (3.1), (3.2) and (3.3) and Theorem 2.1 imply the following inequalities

$$\frac{1}{y-x}\int_{1+x}^{1+y} f(t)g(t)dt \leq \frac{-1}{x+y}\int_{1+x+y}^{1} f(t)g(t)dt,$$

$$f\left(\frac{1}{H_2(x,y)}\right)g\left(\frac{1}{H_2(x,y)}\right) \leq \frac{1}{\frac{1}{y}-\frac{1}{x}}\int_{\frac{1}{x}}^{\frac{1}{y}} f(t)g(t)dt,$$
$$f\left(\frac{x+y}{2}\right)g\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_{x}^{y} f(t)g(t)dt.$$

For increasing, continuous and convex function $f:[0,\infty)\to[0,\infty)$ and $\alpha\geq 1$, Remark 3.1 and Corollaries 2.1, 2.3 imply the following inequalities

$$\frac{1}{y-x}\int_{1+x}^{1+y} t^{\alpha}e^{t}dt \leq \frac{-1}{y+x}\int_{1+x+y}^{1} t^{\alpha}e^{t}dt,$$
$$\left(\frac{1}{H_2(x,y)}\right)^{\alpha} e^{\frac{1}{H_2(x,y)}} \leq \frac{1}{\frac{1}{y}-\frac{1}{x}}\int_{\frac{1}{x}}^{\frac{1}{y}} t^{\alpha}e^{t}dt,$$
$$\left(\frac{x+y}{2}\right)^{\alpha} e^{\frac{x+y}{2}} \leq \frac{1}{y-x}\int_{x}^{y} t^{\alpha}e^{t}dt,$$
$$\frac{1}{y-x}\int_{1+x}^{1+y} t^{\alpha}f(t)dt \leq \frac{-1}{x+y}\int_{1+x+y}^{1} t^{\alpha}f(t)dt,$$
$$\left(\frac{1}{H_2(x,y)}\right)^{\alpha} f\left(\frac{1}{H_2(x,y)}\right) \leq \frac{1}{\frac{1}{y}-\frac{1}{x}}\int_{\frac{1}{x}}^{\frac{1}{y}} t^{\alpha}f(t)dt,$$
$$\left(\frac{x+y}{2}\right)^{\alpha} f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_{x}^{y} t^{\alpha}f(t)dt.$$

For increasing, continuous and convex function $f:[0,\infty)\to[0,\infty)$, Remark 3.1 and Corollary 2.2 imply the following inequalities

$$\frac{1}{y-x}\int_{1+x}^{1+y} e^{t}f(t)dt \leq \frac{-1}{x+y}\int_{1+x+y}^{1} e^{t}f(t)dt,$$
$$e^{\frac{1}{H_2(x,y)}} f\left(\frac{1}{H_2(x,y)}\right) \leq \frac{1}{\frac{1}{y}-\frac{1}{x}}\int_{\frac{1}{x}}^{\frac{1}{y}} e^{t}f(t)dt,$$
$$e^{\frac{x+y}{2}} f\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x}\int_{x}^{y} e^{t}f(t)dt.$$

Remark 3.1 and Corollary 2.5 imply the following inequalities

$$\frac{1}{y-x}\int_{1+x}^{1+y} \operatorname{sech} t \ln t dt \geq \frac{-1}{x+y}\int_{1+x+y}^{1} \operatorname{sech} t \ln t dt,$$
$$\operatorname{sech}\left(\frac{1}{H_2(x,y)}\right)\ln\left(\frac{1}{H_2(x,y)}\right) \geq \frac{1}{\frac{1}{y}-\frac{1}{x}}\int_{\frac{1}{x}}^{\frac{1}{y}} \operatorname{sech} t \ln t dt,$$
$$\operatorname{sech}\left(\frac{x+y}{2}\right)\ln\left(\frac{x+y}{2}\right) \geq \frac{1}{y-x}\int_{x}^{y} \operatorname{sech} t \ln t dt.$$

Remark 3.1 and Corollary 2.7 imply the following inequalities, for $\alpha,\beta\in(1,2)$,

$$\frac{1}{y-x}\int_{1+x}^{1+y} t^{\alpha-1}(1-t)^{\beta-1}dt \geq \frac{-1}{x+y}\int_{1+x+y}^{1} t^{\alpha-1}(1-t)^{\beta-1}dt,$$

$$\left(\frac{1}{H_2(x,y)}\right)^{\alpha-1}\left(1-\frac{1}{H_2(x,y)}\right)^{\beta-1} \geq \frac{1}{\frac{1}{y}-\frac{1}{x}}\int_{\frac{1}{x}}^{\frac{1}{y}} t^{\alpha-1}(1-t)^{\beta-1}dt,$$

$$\left(\frac{x+y}{2}\right)^{\alpha-1}\left(1-\frac{x+y}{2}\right)^{\beta-1} \geq \frac{1}{y-x}\int_{x}^{y} t^{\alpha-1}(1-t)^{\beta-1}dt.$$

Remark 3.1 and Example 3.1 imply the following inequalities, for $\alpha \geq 1$ and the Mittage-Leffler function $E_\alpha(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+1)}$

$$\frac{1}{y-x}\int_{1+x}^{1+y} t^\alpha E_\alpha(t^\alpha)dt \leq \frac{-1}{x+y}\int_{1+x+y}^{1} t^\alpha E_\alpha\left(t^\alpha\right)dt,$$

$$\left(\frac{1}{H_2(x,y)}\right)^{\alpha} E_\alpha\left(\left(\frac{1}{H_2(x,y)}\right)^{\alpha}\right) \leq \frac{1}{\frac{1}{y}-\frac{1}{x}}\int_{\frac{1}{x}}^{\frac{1}{y}} t^\alpha E_\alpha(t^\alpha)dt,$$

$$\left(\frac{x+y}{2}\right)^{\alpha} E_\alpha\left(\left(\frac{x+y}{2}\right)^{\alpha}\right) \leq \frac{1}{y-x}\int_{x}^{y} t^\alpha E_\alpha(t^\alpha)dt.$$

Remark 3.2. Let $\alpha \geq 1$ and $E_\alpha(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+1)}$ be the Mittage-Leffler function on $(0,\infty)$. In Lemma 2.4, set $f(t) = E_\alpha(t)$ and $g(t) = 1$. Then the function

$$\begin{aligned} F(x) =& \sum_{i=1}^{n}\sum_{k=0}^{\infty}\int_0^{x_i}\frac{t^k}{\Gamma(\alpha k+1)}dt = \sum_{i=1}^{n}\sum_{k=0}^{\infty}\frac{x_i^{k+1}}{(k+1)\Gamma(\alpha k+1)} \\ =& \sum_{k=0}^{\infty}\sum_{i=1}^{n}\frac{x_i^{k+1}}{(k+1)\Gamma(\alpha k+1)} = \sum_{k=0}^{\infty}\frac{\frac{\sum_{i=1}^{n}x_i^{k+1}}{k+1}}{\Gamma(\alpha k+1)} \end{aligned}$$

is Schur convex on $\mathbb{R}_+^n$.

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[1]Department of Pure Mathematics,
Faculty of Mathematical Sciences,
Shahrekord University,
P.O. Box 115, Shahrekord, 88186-34141, Iran
Email address: gorjizadeh@stu.sku.ac.ir, farzaneanaliz@gmail.com
Email address: eftekhari-n@sci.sku.ac.ir, eftekharinoha@yahoo.com