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SHARP BOUNDS ON THE AUGMENTED ZAGREB INDEX OF GRAPH OPERATIONS

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Abstract. Let G be a finite and simple graph with edge set $E(G)$. The *augmented Zagreb index* of G is

$$AZI(G) = \sum_{uv \in E(G)} \left(\frac{d_G(u) d_G(v)}{d_G(u) + d_G(v) - 2} \right)^3,$$

where $d_G(u)$ denotes the degree of a vertex u in G. In this paper, we give some bounds of this index for join, corona, cartesian and composition product of graphs by general sum-connectivity index and general Randić index and compute the sharp amount of that for the regular graphs.

1. Introduction

Let G be a finite and simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the order and the size of the graph G, respectively. For a vertex $v \in V(G)$, the *open neighborhood* of v, denoted by $N_G(v) = N(v)$, is the set $\{u \in V(G) \mid uv \in E(G)\}$. The *degree* of $v \in V(G)$, denoted by $d_G(v)$, is defined by $d_G(v) = |N_G(v)|$. The maximum (resp. minimum) degree of vertices of G is denoted by Δ_G (resp. δ_G). We use Bondy and Murty [10] for terminology and notation not defined here.

Several authors defined and studied more vertex degree-based graph invariants such as [16]. One of them is *augmented Zagreb index* of G that is proposed in 2010 by Furtula et al. [15] as

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$$AZI(G) = \sum_{uv \in E(G)} \left(\frac{d_G(u)d_G(v)}{d_G(u) + d_G(v) - 2} \right)^3,$$

where $d_G(u)$ denotes the degree of a vertex u in G. The researchers give a good bounds for it by using different graph parameters, investigate the impact of removing and adding the edge for graph on the augmented Zagreb index. For details see [1,18,24,27].

In 2009, Zu and Trijnastić [28] defined the *sum-connectivity index* as

$$\chi(G) = \sum_{uv \in E(G)} d_G(u) + d_G(v)$$

and one year later, they in [29] introduced the *general sum-connectivity index* as

$$\chi_\lambda(G) = \sum_{uv \in E(G)} \left(d_G(u) + d_G(v)\right)^\lambda, \quad \text{for } \lambda \in \mathbb{R}.$$

There are good results on general sum-connectivity index such as [22, 23]. In 1975, the chemist Milan Randić [21] introduced a topological index $R(G)$ under the name *branching index*. The branching index was renamed the *molecular connectivity index* and is often referred to as the *Randić index* and later named *second Zagreb index*. In 1998, Bollobas and Erdos [9] proposed the generealization state of it named *general Randić index*, $R_\lambda(G)$, as

$$R_\lambda(G) = \sum_{uv \in E(G)} \left(d_G(u)d_G(v)\right)^\lambda, \quad \text{for } \lambda \in \mathbb{R}$$

later that is named *second general Zagreb index*.

The relation between several indices and operations of graphs were very studied. (see [2–8, 11–14, 17, 19, 20, 25, 26]). In this paper, we calculate bounds of the augmented Zagreb index by two other indices, the general sum-connectivity index and the Randić index for join, corona, cartesian and composition product of graphs and compute the sharp amount of that for the regular graphs.

2. The Join of Graphs

The join $G + H$ of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is the graph union $G \cup H$ together with all the edges joining $V(G)$ and $V(H)$. Obviously, $|V(G + H)| = |V(G)| + |V(H)|$ and $|E(G + H)| = |E(G)| + |E(H)| + |V(G)||V(H)|$.

Theorem 2.1. *Let G be a graph of order n_1 and of size m_1 and let H be a graph of order n_2 and of size m_2. Then*

$$\begin{aligned} AZI(G+H) \leq & \frac{(\Delta_G - 1)^3 AZI(G)}{(\Delta_G + n_2 - 1)^3} + \frac{n_2^3 \chi_3(G) + (3n_2^2\Delta_G^2 + 3n_2^4)\chi_2(G) + 3n_2^5 \chi_1(G)}{8(\delta_G + n_2 - 1)^3} \\ & + \frac{(6n_2\Delta_G + 3n_2^2)R_2(G) + (12n_2^3\Delta_G + 3n_2^4)R_1(G) + m_1 n_2^6}{8(\delta_G + n_2 - 1)^3} \end{aligned}$$

$$
\begin{aligned}
&+\frac{(\Delta_H-1)^3 AZI(H)}{(\Delta_H+n_1-1)^3}\\
&+\frac{n_1^3\chi_3(H)+(3n_1^2\Delta_H^2+3n_1^4)\chi_2(H)+3n_1^5\chi_1(H)}{8(\delta_H+n_1-1)^3}\\
&+\frac{(6n_1\Delta_H+3n_1^2)R_2(H)+(12n_1^3\Delta_H+3n_1^4)R_1(H)+m_2n_1^6}{8(\delta_H+n_1-1)^3}\\
&+n_1n_2\left(\frac{(\Delta_G+n_2)(\Delta_H+n_1)}{\delta_G+\delta_H+n_1+n_2-2}\right)^3,
\end{aligned}
$$

with equality if and only if G and H are regular graphs.

Proof. By definition,

$$AZI(G+H)=\sum_{uv\in E(G+H)}\left(\frac{d_{G+H}(u)d_{G+H}(v)}{d_{G+H}(u)+d_{G+H}(v)-2}\right)^3.$$

We partition the edges of $G+H$ in to three subset E_1, E_2 and E_3, as follows:

$$
\begin{aligned}
E_1 =&\{e=uv \mid u,v\in V(G)\},\\
E_2 =&\{e=uv \mid u,v\in V(H)\},\\
E_3 =&\{e=uv \mid u\in V(G), v\in V(H)\}.
\end{aligned}
$$

Let $e=uv\in E_1$. Then $d_{G+H}(u)=d_G(u)+n_2$ and $d_{G+H}(v)=d_G(v)+n_2$. Hence

$$
\begin{aligned}
((d_G(u)+n_2)(d_G(v)+n_2))^3 =&(d_G(u)d_G(v))^2[3n_2(d_G(u)+d_G(v))+3n_2^2]\\
&+(d_G(u)d_G(v))^3+d_G(u)d_G(v)\\
&\times[6n_2^3(d_G(u)+d_G(v))+3n_2^4]\\
&+n_2^3(d_G(u)+d_G(v))^3+3n_2^5(d_G(u)+d_G(v))\\
&+(d_G(u)+d_G(v))^2[3n_2^2d_G(u)d_G(v)+3n_2^4]+n_2^6
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\frac{d_{G+H}(u)d_{G+H}(v)}{d_{G+H}(u)+d_{G+H}(v)-2}\right)^3\\
=&\left(1-\frac{2n_2}{d_G(u)+d_G(v)+2n_2-2}\right)^3\left(\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2}\right)^3\\
&+\frac{n_2^3(d_G(u)+d_G(v))^3+[3n_2^2(d_G(u)d_G(v))^2+3n_2^4](d_G(u)+d_G(v))^2}{(d_G(u)+d_G(v)+2n_2-2)^3}\\
&+\frac{3n_2^5(d_G(u)+d_G(v))+[3n_2(d_G(u)+d_G(v))+3n_2^2](d_G(u)d_G(v))^2}{(d_G(u)+d_G(v)+2n_2-2)^3}\\
&+\frac{[6n_2^3(d_G(u)+d_G(v))+3n_2^4]d_G(u)d_G(v)+n_2^6}{(d_G(u)+d_G(v)+2n_2-2)^3}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left(\frac{\Delta_G - 1}{\Delta_G + n_2 - 1}\right)^3 \left(\frac{d_G(u)d_G(v)}{d_G(u) + d_G(v) - 2}\right)^3 \\
&\quad + \frac{n_2^3(d_G(u) + d_G(v))^3 + (3n_2^2\Delta_G^2 + 3n_2^4)(d_G(u) + d_G(v))^2}{8(\delta_G + n_2 - 1)^3} \\
&\quad + \frac{3n_2^5(d_G(u) + d_G(v)) + (6n_2\Delta_G + 3n_2^2)(d_G(u)d_G(v))^2}{8(\delta_G + n_2 - 1)^3} \\
&\quad + \frac{(12n_2^3\Delta_G + 3n_2^4)d_G(u)d_G(v) + n_2^6}{8(\delta_G + n_2 - 1)^3}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sum_{uv \in E_1} \left(\frac{d_{G+H}(u)d_{G+H}(v)}{d_{G+H}(u) + d_{G+H}(v) - 2}\right)^3 &\leq \left(\frac{\Delta_G - 1}{\Delta_G + n_2 - 1}\right)^3 AZI(G) + \frac{m_1 n_2^6}{8(\delta_G + n_2 - 1)^3} \\
&\quad + \frac{n_2^3\chi_3(G) + (3n_2^2\Delta_G^2 + 3n_2^4)\chi_2(G) + n_2^5\chi_1(G)}{8(\delta_G + n_2 - 1)^3} \\
&\quad + \frac{(6n_2\Delta_G + 3n_2^2)R_2(G) + (12n_2^3\Delta_G + 3n_2^4)R_1(G)}{8(\delta_G + n_2 - 1)^3}.
\end{aligned} \tag{2.1}
$$

Obviously, equality holds if and only if $\Delta_G = \delta_G$. Similarly

$$
\begin{aligned}
\sum_{uv \in E_2} \left(\frac{d_{G+H}(u)d_{G+H}(v)}{d_{G+H}(u) + d_{G+H}(v) - 2}\right)^3 &\leq \left(\frac{\Delta_H - 1}{\Delta_H + n_1 - 1}\right)^3 AZI(H) + \frac{m_2 n_1^6}{8(\delta_H + n_1 - 1)^3} \\
&\quad + \frac{n_1^3\chi_3(H) + (3n_1^2\Delta_H^2 + 3n_1^4)\chi_2(H) + 3n_1^5\chi_1(H)}{8(\delta_H + n_1 - 1)^3} \\
&\quad + \frac{(6n_1\Delta_H + 3n_1^2)R_2(H) + (12n_1^3\Delta_H + 3n_1^4)R_1(H)}{8(\delta_H + n_1 - 1)^3}.
\end{aligned} \tag{2.2}
$$

Equality holds if and only if $\Delta_H = \delta_H$. Let $e = uv \in E_3$ such that $u \in V(G)$ and $v \in V(H)$. Then $d_{G+H}(u) = d_G(u) + n_2$ and $d_{G+H}(v) = d_H(v) + n_1$. Hence for every edge $e = uv \in E_3$,

$$
\begin{aligned}
\left(\frac{d_{G+H}(u)d_{G+H}(v)}{d_{G+H}(u) + d_{G+H}(v)}\right)^3 &= \left(\frac{(d_G(u) + n_2)(d_H(v) + n_1)}{d_G(u) + d_H(v) + n_1 + n_2 - 2}\right)^3 \\
&\leq \left(\frac{(\Delta_G + n_2)(\Delta_H + n_1)}{\delta_G + \delta_H + n_1 + n_2 - 2}\right)^3.
\end{aligned}
$$

Therefore,

$$
\sum_{uv \in E_3} \left(\frac{d_{G+H}(u)d_{G+H}(v)}{d_{G+H}(u) + d_{G+H}(v)}\right)^3 \leq n_1 n_2 \left(\frac{(\Delta_G + n_2)(\Delta_H + n_1)}{\delta_G + \delta_H + n_1 + n_2 - 2}\right)^3, \tag{2.3}
$$

with equality if and only if $\Delta_G = \delta_G$ and $\Delta_H = \delta_H$. By Equations (2.1), (2.2) and (2.3), we have:

$$\begin{aligned}
AZI(G+H) \leq & \frac{(\Delta_G-1)^3 AZI(G)}{(\Delta_G+n_2-1)^3} + \frac{n_2^3\chi_3(G) + (3n_2^2\Delta_G^2 + 3n_2^4)\chi_2(G) + 3n_2^5\chi_1(G)}{8(\delta_G+n_2-1)^3} \\
& + \frac{(6n_2\Delta_G + 3n_2^2)R_2(G) + (12n_2^3\Delta_G + 3n_2^4)R_1(G) + m_1n_2^6}{8(\delta_G+n_2-1)^3} \\
& + \frac{(\Delta_H-1)^3 AZI(H)}{(\Delta_H+n_1-1)^3} + \frac{n_1^3\chi_3(H) + (3n_1^2\Delta_H^2 + 3n_1^4)\chi_2(H) + 3n_1^5\chi_1(H)}{8(\delta_H+n_1-1)^3} \\
& + \frac{(6n_1\Delta_H + 3n_1^2)R_2(H) + (12n_1^3\Delta_H + 3n_1^4)R_1(H) + m_2n_1^6}{8(\delta_H+n_1-1)^3} \\
& + n_1n_2\left(\frac{(\Delta_G+n_2)(\Delta_H+n_1)}{\delta_G+\delta_H+n_1+n_2-2}\right)^3.
\end{aligned}$$

Equality holds if and only if G and H are regular graphs. □

Theorem 2.2. *Let G be a graph of order n_1 and of size m_1 and let H be a graph of order n_2 and of size m_2. Then*

$$\begin{aligned}
AZI(G+H) \geq & \frac{(\delta_G-1)^3 AZI(G)}{(\delta_G+n_2-1)^3} + \frac{n_2^3\chi_3(G) + (3n_2^2\delta_G^2 + 3n_2^4)\chi_2(G) + 3n_2^5\chi_1(G)}{8(\Delta_G+n_2-1)^3} \\
& + \frac{(6n_2\delta_G + 3n_2^2)R_2(G) + (12n_2^3\delta_G + 3n_2^4)R_1(G) + m_1n_2^6}{8(\Delta_G+n_2-1)^3} \\
& + \frac{(\delta_H-1)^3 AZI(H)}{(\delta_H+n_1-1)^3} + \frac{n_1^3\chi_3(H) + (3n_1^2\delta_H^2 + 3n_1^4)\chi_2(H) + 3n_1^5\chi_1(H)}{8(\Delta_H+n_1-1)^3} \\
& + \frac{(6n_1\delta_H + 3n_1^2)R_2(H) + (12n_1^3\delta_H + 3n_1^4)R_1(H) + m_2n_1^6}{8(\Delta_H+n_1-1)^3} \\
& + n_1n_2\left(\frac{(\delta_G+n_2)(\delta_H+n_1)}{\Delta_G+\Delta_H+n_1+n_2-2}\right)^3,
\end{aligned}$$

with equality if and only if G and H are regular graphs.

Proof. Using an argument similar to that described in proof of Theorem 2.1, we obtained the result. □

Corollary 2.1. *Let G be a k-regular graph of order n_1 and let H be a r-regular graph of order n_2. Then*

$$AZI(G+H) = \frac{k(k+n_2)^6}{16(k+n_2-1)^3} + \frac{r(r+n_1)^6}{16(r+n_1-1)^3} + \frac{n_1n_2(k+n_2)^3(r+n_1)^3}{(k+r+n_1+n_2-2)^3}.$$

3. The Corona Product of Graphs

The corona product $G \circ H$ of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is as the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the i-th vertex of G to every vertex in i-th copy of H. Obviously, $|V(G \circ H)| = |V(G)| + |V(G)||V(H)|$ and $|E(G \circ H)| = |E(G)| + |V(G)||E(H)| + |V(G)||V(H)|$.

Theorem 3.1. *Let G be a graph of order n_1 and of size m_1 and let H be a graph of order n_2 and of size m_2. Then*

$$\begin{aligned} AZI(G \circ H) \leq & \frac{(\Delta_G - 1)^3 AZI(G)}{(\Delta_G + n_2 - 1)^3} + \frac{n_2^3 \chi_3(G) + (3n_2^2\Delta_G^2 + 3n_2^4)\chi_2(G) + 3n_2^5\chi_1(G)}{8(\delta_G + n_2 - 1)^3} \\ & + \frac{(6n_2\Delta_G + 3n_2^2)R_2(G) + (12n_2^3\Delta_G + 3n_2^4)R_1(G) + m_1 n_2^6}{8(\delta_G + n_2 - 1)^3} \\ & + \frac{(\Delta_H - 1)^3 AZI(H)}{\Delta_H^3} + \frac{\chi_3(H) + (3\Delta_H^2 + 3)\chi_2(H) + 3\chi_1(H)}{8\delta_H^3} \\ & + \frac{(6\Delta_H + 3)R_2(H) + (12\Delta_H + 3)R_1(H) + m_2}{8\delta_H^3} \\ & + n_1 n_2 \left(\frac{(\Delta_G + n_2)(\Delta_H + 1)}{\delta_G + \delta_H + n_2 - 1} \right)^3, \end{aligned}$$

with equality if and only if G and H are regular graphs.

Proof. We partition the edges of G in to three subset E_1, E_2 and E_3 such that $E_1 = \{e = uv \mid u, v \in V(G)\}$, $E_2 = \{e = uv \mid u, v \in V(H)\}$ and $E_3 = \{e = uv \mid u \in V(G), v \in V(H)\}$.

If $e = uv \in E_1$, then $d_{G\circ H}(u) = d_G(u) + n_2$ and $d_{G\circ H}(v) = d_G(v) + n_2$ and if $e = uv \in E_2$, then $d_{G\circ H}(u) = d_H(u) + 1$ and $d_{G\circ H}(v) = d_H(v) + 1$. By used of proof of Theorem 2.1, we have,

$$\begin{aligned} \sum_{uv \in E_1} \left(\frac{d_{G\circ H}(u) d_{G\circ H}(v)}{d_{G\circ H}(u) + d_{G\circ H}(v) - 2} \right)^3 \leq & \frac{(\Delta_G - 1)^3 AZI(G)}{(\Delta_G + n_2 - 1)^3} \\ & + \frac{n_2^3 \chi_3(G) + (3n_2^2\Delta_G^2 + 3n_2^4)\chi_2(G) + n_2^5\chi_1(G)}{8(\delta_G + n_2 - 1)^3} \\ & + \frac{(6n_2\Delta_G + 3n_2^2)R_2(G) + (12n_2^3\Delta_G + 3n_2^4)R_1(G)}{8(\delta_G + n_2 - 1)^3} \\ & + \frac{m_1 n_2^6}{8(\delta_G + n_2 - 1)^3}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} \sum_{uv \in E_2} \left(\frac{d_{G\circ H}(u) d_{G\circ H}(v)}{d_{G\circ H}(u) + d_{G\circ H}(v) - 2} \right)^3 \leq & \frac{(\Delta_H - 1)^3 AZI(H)}{\Delta_H^3} \\ & + \frac{\chi_3(H) + (3\Delta_H^2 + 3)\chi_2(H) + 3\chi_1(H)}{8\delta_H^3} \\ & + \frac{(6\Delta_{Hh} + 3)R_2(H) + (12\Delta_H + 3)R_1(H) + m_2}{8\delta_H^3}. \end{aligned} \tag{3.2}$$

Obviously, equalities hold if and only if $\Delta_G = \delta_G$ and $\Delta_H = \delta_H$.

Let $e = uv \in E_3$ such that $u \in V(G)$ and $v \in V(H)$. Then $d_{G\circ H}(u) = d_G(u) + n_2$ and $d_{G\circ H}(v) = d_H(v) + 1$. Hence for every edge $e = uv \in E_3$,

$$\begin{aligned}\left(\frac{d_{G\circ H}(u)d_{G\circ H}(v)}{d_{G\circ H}(u) + d_{G\circ H}(v) - 2}\right)^3 &= \left(\frac{(d_G(u) + n_2)(d_H(v) + 1)}{d_G(u) + d_H(v) + n_2 + 1 - 2}\right)^3 \\ &\leq \left(\frac{(\Delta_G + n_2)(\Delta_H + 1)}{\delta_G + \delta_H + n_2 - 1}\right)^3.\end{aligned}$$

Therefore,

$$\sum_{uv\in E_3}\left(\frac{d_{G\circ H}(u)d_{G\circ H}(v)}{d_{G\circ H}(u) + d_{G\circ H}(v) - 2}\right)^3 \leq \frac{n_1n_2(\Delta_G + n_2)^3(\Delta_H + 1)^3}{(\delta_G + \delta_H + n_2 - 1)^3}, \tag{3.3}$$

with equality if and only if $\Delta_G = \delta_G$ and $\Delta_H = \delta_H$. By Equations (3.1), (3.2) and (3.3), we have:

$$\begin{aligned}AZI(G\circ H) \leq &\frac{(\Delta_G - 1)^3 AZI(G)}{(\Delta_G + n_2 - 1)^3} + \frac{n_2^3\chi_3(G) + (3n_2^2\Delta_G^2 + 3n_2^4)\chi_2(G) + 3n_2^5\chi_1(G)}{8(\delta_G + n_2 - 1)^3} \\ &+ \frac{(6n_2\Delta_G + 3n_2^2)R_2(G) + (12n_2^3\Delta_G + 3n_2^4)R_1(G) + m_1n_2^6}{8(\delta_G + n_2 - 1)^3} \\ &+ \frac{(\Delta_H - 1)^3 AZI(H)}{\Delta_H^3} + \frac{\chi_3(H) + (3\Delta_H^2 + 3)\chi_2(H) + 3\chi_1(H)}{8\delta_H^3} \\ &+ \frac{(6\Delta_H + 3)R_2(H) + (12\Delta_H + 3)R_1(H) + m_2}{8\delta_H^3} \\ &+ n_1n_2\left(\frac{(\Delta_G + n_2)(\Delta_H + 1)}{\delta_G + \delta_H + n_2 - 1}\right)^3.\end{aligned}$$

Equality holds if and only if G and H are regular graphs. □

Theorem 3.2. *Let G be a graph of order n_1 and of size m_1 and let H be a graph of order n_2 and of size m_2. Then*

$$\begin{aligned}AZI(G\circ H) \geq &\frac{(\delta_G - 1)^3 AZI(G)}{(\delta_G + n_2 - 1)^3} + \frac{n_2^3\chi_3(G) + (3n_2^2\delta_G^2 + 3n_2^4)\chi_2(G) + 3n_2^5\chi_1(G)}{8(\Delta_G + n_2 - 1)^3} \\ &+ \frac{(6n_2\delta_G + 3n_2^2)R_2(G) + (12n_2^3\delta_G + 3n_2^4)R_1(G) + m_1n_2^6}{8(\Delta_G + n_2 - 1)^3} \\ &+ \frac{(\delta_H - 1)^3 AZI(H)}{\delta_H^3} + \frac{\chi_3(H) + (3\delta_H^2 + 3)\chi_2(H) + 3\chi_1(H)}{8\Delta_H^3} \\ &+ \frac{(6\delta_H + 3)R_2(H) + (12\delta_H + 3)R_1(H) + m_2}{8\Delta_H^3} \\ &+ \frac{n_1n_2(\delta_G + n_2)^3(\delta_H + 1)^3}{(\Delta_G + \Delta_H + n_2 - 1)^3},\end{aligned}$$

with equality if and only if G and H are regular graphs.

Proof. The proof of the result is similar to this given in Theorem 3.1. $\square$

Corollary 3.1. *Let G be a k-regular graph of order n_1 and let H be a r-regular graph of order n_2. Then*

$$AZI(G \circ H) = \frac{k(k+n_2)^6}{16(k+n_2-1)^3} + \frac{r(r+1)^6}{16r^3} + \frac{n_1 n_2 (k+n_2)^3 (r+1)^3}{(k+r+n_2-1)^3}.$$

4. The Cartesian Product of Graphs

The Cartesian product $G \times H$ of graphs G and H has the vertex set $V(G \times H) = V(G) \times V(H)$ and $(u,x)(v,y)$ is an edge of $G \times H$ if $uv \in E(G)$ and $x = y$, or $u = v$ and $xy \in E(H)$. Obviously, $|V(G \times H)| = |V(G)||V(H)|$ and $|E(G \times H)| = |E(G)||V(H)| + |V(G)||E(H)|$.

Theorem 4.1. *Let G be a graph of order n_1 and of size m_1 and let H be a graph of order n_2 and of size m_2. Then*

$$\begin{aligned}
AZI(G \times H) \leq & \frac{n_2(\Delta_G + \Delta_H - \delta_H - 1)^3 AZI(G) + n_1(\Delta_G + \Delta_H - \delta_G - 1)^3 AZI(H)}{(\Delta_G + \Delta_H - 1)^3} \\
& + \frac{n_2 \Delta_H^3 \chi_3(G) + n_2(3\Delta_H^2 \Delta_G^2 + 3\Delta_H^4)\chi_2(G) + 3n_2 \Delta_H^5 \chi_1(G) + \Delta_G^6 m_2}{8(\delta_G + \delta_H - 1)^3} \\
& + \frac{n_1 \Delta_G^3 \chi_3(H) + n_1(3\Delta_H^2 \Delta_G^2 + 3\Delta_G^4)\chi_2(H) + 3n_1 \Delta_G^5 \chi_1(H) + \Delta_H^6 m_1}{8(\delta_G + \delta_H - 1)^3} \\
& + \frac{n_2(6\Delta_H \Delta_G + 3\Delta_H^2) R_2(G) + n_2(12\Delta_H^3 \Delta_G + 3\Delta_H^4) R_1(G)}{8(\delta_G + \delta_H - 1)^3} \\
& + \frac{n_1(6\Delta_H \Delta_G + 3\Delta_G^2) R_2(H) + n_1(12\Delta_G^3 \Delta_H + 3\Delta_G^4) R_1(H)}{8(\delta_G + \delta_H - 1)^3},
\end{aligned}$$

with equality if and only if G and H are regular graphs.

Proof. By definition,

$$AZI(G \times H) = \sum_{(u,x)(v,y) \in E(G \times H)} \left(\frac{d_{G\times H}(u,x) d_{G\times H}(v,y)}{d_{G\times H}(u,x) + d_{G\times H}(v,y) - 2} \right)^3.$$

We partition the edges of $G \times H$ in to two subset E_1 and E_2, as follows:

$$\begin{aligned}
E_1 =& \{e = (u,x)(v,y) \mid uv \in E(G), x = y\}, \\
E_2 =& \{e = (u,x)(v,y) \mid xy \in E(H), u = v\}.
\end{aligned}$$

Let $e = (u,x)(v,x) \in E_1$. Then $d_{G\times H}(u,x) = d_G(u) + d_H(x)$ and $d_{G\times H}(v,x) = d_G(v) + d_H(x)$. By used of proof of Theorem 2.1, we have

$$\left(\frac{d_{G\times H}(u,x) d_{G\times H}(v,x)}{d_{G\times H}(u,x) + d_{G\times H}(v,x) - 2} \right)^3 \leq \frac{(\Delta_G + \Delta_H - \delta_H - 1)^3}{(\Delta_G + \Delta_H - 1)^3} \left(\frac{d_G(u) d_G(v)}{d_G(u) + d_G(v) - 2} \right)^3$$

$$+\frac{\Delta_H^3(d_G(u)+d_G(v))^3)(d_G(u)+d_G(v))^2}{8(\delta_G+\delta_H-1)^3}$$
$$+\frac{(3\Delta_H^2\Delta_G^2+3\Delta_H^4)(d_G(u)+d_G(v))^2}{8(\delta_G+\delta_H-1)^3}$$
$$+\frac{3\Delta_H^5(d_G(u)+d_G(v))}{8(\delta_G+\delta_H-1)^3}$$
$$+\frac{(6\Delta_H\Delta_G+3\Delta_H^2)(d_G(u)d_G(v))^2}{8(\delta_G+\delta_H-1)^3}$$
$$+\frac{(12\Delta_H^3\Delta_G+3\Delta_H^4)d_G(u)d_G(v)+\Delta_H^6}{8(\delta_G+\delta_H-1)^3}.$$

Therefore,

$$\begin{aligned}
&\sum_{(u,x)(v,x)\in E_1}\left(\frac{d_{G\times H}(u,x)d_{G\times H}(v,x)}{d_{G\times H}(u,x)+d_{G\times H}(v,x)-2}\right)^3\\
\leq&\frac{n_2(\Delta_G+\Delta_H-\delta_H-1)^3AZI(G)}{(\Delta_G+\Delta_H-1)^3}+\frac{n_2\Delta_H^3\chi_3(G)+3n_2\Delta_H^5\chi_1(G)}{8(\delta_G+\delta_H-1)^3}\\
&+\frac{n_2(3\Delta_H^2\Delta_G^2+3\Delta_H^4)\chi_2(G)}{8(\delta_G+\delta_H-1)^3}+\frac{n_2(6\Delta_H\Delta_G+3\Delta_H^2)R_2(G)}{8(\delta_G+\delta_H-1)^3}\\
&+\frac{n_2(12\Delta_H^3\Delta_G+3\Delta_H^4)R_1(G)+\Delta_H^6n_2m_1}{8(\delta_G+\delta_H-1)^3}
\end{aligned}\tag{4.1}$$

Obviously, equality holds if and only if $\Delta_G=\delta_G$ and $\Delta_H=\delta_H$. Similarly,

$$\begin{aligned}
\sum_{(u,x)(u,y)\in E_2}\left(\frac{d_{G\times H}(u,x)d_{G\times H}(u,y)}{d_{G\times H}(u,x)+d_{G\times H}(u,y)-2}\right)^3\leq&\frac{n_1(\Delta_G+\Delta_H-\delta_G-1)^3AZI(H)}{(\Delta_G+\Delta_H-1)^3}\\
&+\frac{n_1\Delta_G^3\chi_3(H)+3n_1\Delta_G^5\chi_1(H)}{8(\delta_G+\delta_H-1)^3}\\
&+\frac{n_1(3\Delta_H^2\Delta_G^2+3\Delta_G^4)\chi_2(H)}{8(\delta_G+\delta_H-1)^3}\\
&+\frac{n_1(6\Delta_H\Delta_G+3\Delta_G^2)R_2(H)}{8(\delta_G+\delta_H-1)^3}\\
&+\frac{n_1(12\Delta_G^3\Delta_H+3\Delta_G^4)R_1(H)+\Delta_G^6n_1m_2}{8(\delta_G+\delta_H-1)^3}.
\end{aligned}\tag{4.2}$$

Equality holds if and only if $\Delta_G=\delta_G$ and $\Delta_H=\delta_H$. By Equations (4.1) and (4.2), we have:

$$\begin{aligned}
AZI(G\times H)\leq&\frac{n_2(\Delta_G+\Delta_H-\delta_H-1)^3AZI(G)+n_1(\Delta_G+\Delta_H-\delta_G-1)^3AZI(H)}{(\Delta_G+\Delta_H-1)^3}\\
&+\frac{n_2\Delta_H^3\chi_3(G)+n_2(3\Delta_H^2\Delta_G^2+3\Delta_H^4)\chi_2(G)+3n_2\Delta_H^5\chi_1(G)+\Delta_G^6m_2}{8(\delta_G+\delta_H-1)^3}
\end{aligned}$$

$$+\frac{n_1\Delta_G^3\chi_3(H)+n_1(3\Delta_H^2\Delta_G^2+3\Delta_G^4)\chi_2(H)+3n_1\Delta_G^5\chi_1(H)+\Delta_H^6 m_1}{8(\delta_G+\delta_H-1)^3}$$
$$+\frac{n_2(6\Delta_H\Delta_G+3\Delta_H^2)R_2(G)+n_2(12\Delta_H^3\Delta_G+3\Delta_H^4)R_1(G)}{8(\delta_G+\delta_H-1)^3}$$
$$+\frac{n_1(6\Delta_H\Delta_G+3\Delta_G^2)R_2(H)+n_1(12\Delta_G^3\Delta_H+3\Delta_G^4)R_1(H)}{8(\delta_G+\delta_H-1)^3},$$

with equality if and only if G and H are regular graphs. □

Theorem 4.2. *Let G be a graph of order n_1 and of size m_1 and let H be a graph of order n_2 and of size m_2. Then*

$$AZI(G\times H)\geq\frac{n_2(\delta_G+\delta_H-\Delta_H-1)^3AZI(G)+n_1(\delta_G+\delta_H-\Delta_G-1)^3AZI(H)}{(\delta_G+\delta_H-1)^3}$$
$$+\frac{n_2\delta_H^3\chi_3(G)+n_2(3\delta_H^2\delta_G^2+3\delta_H^4)\chi_2(G)+3n_2\delta_H^5\chi_1(G)}{8(\Delta_G+\Delta_H-1)^3}$$
$$+\frac{n_2(6\delta_H\delta_G+3\delta_H^2)R_2(G)+n_2(12\delta_H^3\delta_G+3\delta_H^4)R_1(G)+\delta_H^6m_1}{8(\Delta_G+\Delta_H-1)^3}$$
$$+\frac{n_1\delta_G^3\chi_3(H)+n_1(3\delta_H^2\delta_G^2+3\delta_G^4)\chi_2(H)+3n_1\delta_G^5\chi_1(H)}{8(\Delta_G+\Delta_H-1)^3}$$
$$+\frac{n_1(6\delta_H\delta_G+3\delta_G^2)R_2(H)+n_1(12\delta_G^3\delta_H+3\delta_G^4)R_1(H)+\delta_G^6m_2}{8(\Delta_G+\Delta_H-1)^3},$$

with equality if and only if G and H are regular graphs.

Proof. Using an argument similar to that described in proof of Theorem 4.1, we obtained the result. □

Corollary 4.1. *Let G be a k-regular graph of order n_1 and let H be a r-regular graph of order n_2. Then $AZI(G\times H)=\frac{n_1n_2(k+r)^7}{16(k+r-1)^3}$.*

5. The Composition Product of Graphs

The composition $G[H]$ of graphs G and H has the vertex set $V(G[H])=V(G)\times V(H)$ and $(u,x)(v,y)$ is an edge of $G[H]$ if ($uv\in E(G)$) or ($xy\in E(H)$ and $u=v$). Obviously, $|V(G[H])|=|V(G)||V(H)|$ and $|E(G[H])|=|E(G)||V(H)|^2+|E(H)||V(G)|$.

Theorem 5.1. *Let G be a graph of order n_1 and of size m_1 and let H be a graph of order n_2 and of size m_2. Then*

$$AZI(G[H])$$
$$\leq\frac{n_2^5(n_2\Delta_G+\Delta_H-\delta_H-n_2)^3AZI(G)+n_1(\Delta_H+n_2\Delta_G-n_2\delta_G-1)^3AZI(H)}{(n_2\Delta_G+\Delta_H-1)^3}$$

$$+\frac{n_2^5\Delta_H^3\chi_3(G)+n_2^2(3n_2^4\Delta_H^2\Delta_G^2+3n_2^2\Delta_H^4)\chi_2(G)+3n_2^3\Delta_H^5\chi_1(G)}{8(n_2\delta_G+\delta_H-1)^3}$$
$$+\frac{n_1n_2^3\Delta_G^3\chi_3(H)+n_1(3n_2^2\Delta_H^2\Delta_G^2+3n_2^4\Delta_G^4)\chi_2(H)+3n_1n_2^5\Delta_G^5\chi_1(H)}{8(n_2\delta_G+\delta_H-1)^3}$$
$$+\frac{n_2^2(6n_2^5\Delta_H\Delta_G+3n_2^4\Delta_H^2)R_2(G)+n_2^2(12n_2^3\Delta_H^3\Delta_G+3n_2^2\Delta_H^4)R_1(G)+n_2^2m_1\Delta_H^6}{8(n_2\delta_G+\delta_H-1)^3}$$
$$+\frac{n_1(6n_2\Delta_H\Delta_G+3n_2^2\Delta_G^2)R_2(H)+n_1(12n_2^3\Delta_G^3\Delta_H+3n_2^4\Delta_G^4)R_1(H)+n_1m_2n_2^6\Delta_G^6}{8(n_2\delta_G+\delta_H-1)^3},$$

with equality if and only if G and H are regular graphs.

Proof. We partition the edges of $G[H]$ in to two subset E_1 and E_2, as follows:

$$E_1=\{e=(u,x)(v,y)\mid uv\in E(G)\},$$
$$E_2=\{e=(u,x)(v,y)\mid xy\in E(H), u=v\}.$$

Let $e=(u,x)(v,y)\in E_1$. Then $d_{G[H]}(u,x)=n_2d_G(u)+d_H(x)$ and $d_{G[H]}(v,y)=n_2d_G(v)+d_H(y)$. By used of proof of Theorem 2.1, we have,

$$\begin{aligned}\left(\frac{d_{G[H]}(u,x)d_{G[H]}(v,y)}{d_{G[H]}(u,x)+d_{G[H]}(v,y)-2}\right)^3 \leq & \frac{n_2^5(n_2\Delta_G+\Delta_H-\delta_H-n_2)^3}{(n_2\Delta_G+\Delta_H-1)^3}\left(\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2}\right)^3\\
&+\frac{n_2^3\Delta_H^3(d_G(u)+d_G(v))^3+3n_2\Delta_H^5(d_G(u)+d_G(v))}{8(n_2\delta_G+\delta_H-1)^3}\\
&+\frac{(3n_2^4\Delta_H^2\Delta_G^2+3n_2^2\Delta_H^4)(d_G(u)+d_G(v))^2}{8(n_2\delta_G+\delta_H-1)^3}\\
&+\frac{(6n_2^5\Delta_H\Delta_G+3n_2^4\Delta_H^2)(d_G(u)d_G(v))^2}{8(n_2\delta_G+\delta_H-1)^3}\\
&+\frac{(12n_2^3\Delta_H^3\Delta_G+3n_2^2\Delta_H^4)d_G(u)d_G(v)+\Delta_H^6}{8(n_2\delta_G+\delta_H-1)^3}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{(u,x)(v,y)\in E_1}\left(\frac{d_{G[H]}(u,x)d_{G[H]}(v,x)}{d_{G[H]}(u,x)+d_{G[H]}(v,x)-2}\right)^3 \leq & \frac{n_2^5(n_2\Delta_G+\Delta_H-\delta_H-n_2)^3AZI(G)}{(n_2\Delta_G+\Delta_H-1)^3}\\
&+\frac{n_2^2(6n_2^5\Delta_H\Delta_G+3n_2^4\Delta_H^2)R_2(G)}{8(n_2\delta_G+\delta_H-1)^3}\\
&+\frac{n_2^2(3n_2^4\Delta_H^2\Delta_G^2+3n_2^2\Delta_H^4)\chi_2(G)}{8(n_2\delta_G+\delta_H-1)^3}\\
&+\frac{n_2^5\Delta_H^3\chi_3(G)}{8(n_2\delta_G+\delta_H-1)^3}+\frac{3n_2^3\Delta_H^5\chi_1(G)}{8(n_2\delta_G+\delta_H-1)^3}\\
&+\frac{n_2^2(12n_2^3\Delta_H^3\Delta_G+3n_2^2\Delta_H^4)R_1(G)}{8(n_2\delta_G+\delta_H-1)^3}\end{aligned}$$

$$+\frac{n_2^2 m_1 \Delta_H^6}{8(n_2\delta_G+\delta_H-1)^3}. \tag{5.1}$$

Obviously, equality holds if and only if $\Delta_G=\delta_G$ and $\Delta_H=\delta_H$. Similarly,

$$\begin{aligned}\sum_{(u,x)(u,y)\in E_2}\left(\frac{d_{G[H]}(u,x)d_{G[H]}(u,y)}{d_{G[H]}(u,x)+d_{G[H]}(u,y)-2}\right)^3 \leq & \frac{n_1(\Delta_H+n_2\Delta_G-n_2\delta_G-1)^3 AZI(H)}{(n_2\Delta_G+\Delta_H-1)^3}\\ &+\frac{n_1(6n_2\Delta_H\Delta_G+3n_2^2\Delta_G^2)R_2(H)}{8(n_2\delta_G+\delta_H-1)^3}\\ &+\frac{n_1(3n_2^2\Delta_H^2\Delta_G^2+3n_2^4\Delta_G^4)\chi_2(H)}{8(n_2\delta_G+\delta_H-1)^3}\\ &+\frac{n_1n_2^3\Delta_G^3\chi_3(H)}{8(\delta_G+\delta_H-1)^3}+\frac{3n_1n_2^5\Delta_G^5\chi_1(H)}{8(n_2\delta_G+\delta_H-1)^3}\\ &+\frac{n_1(12n_2^3\Delta_G^3\Delta_H+3n_2^4\Delta_G^4)R_1(H)}{8(n_2\delta_G+\delta_H-1)^3}\\ &+\frac{n_1m_2n_2^6\Delta_G^6}{8(n_2\delta_G+\delta_H-1)^3}.\end{aligned} \tag{5.2}$$

Equality holds if and only if $\Delta_G=\delta_G$ and $\Delta_H=\delta_H$. By Equations (5.1) and (5.2), we have:

$$\begin{aligned}&AZI(G[H])\\ \leq&\frac{n_2^5(n_2\Delta_G+\Delta_H-\delta_H-n_2)^3AZI(G)+n_1(\Delta_H+n_2\Delta_G-n_2\delta_G-1)^3AZI(H)}{(n_2\Delta_G+\Delta_H-1)^3}\\ &+\frac{n_2^5\Delta_H^3\chi_3(G)+n_2^2(3n_2^4\Delta_H^2\Delta_G^2+3n_2^2\Delta_H^4)\chi_2(G)+3n_2^3\Delta_H^5\chi_1(G)}{8(n_2\delta_G+\delta_H-1)^3}\\ &+\frac{n_1n_2^3\Delta_G^3\chi_3(H)+n_1(3n_2^2\Delta_H^2\Delta_G^2+3n_2^4\Delta_G^4)\chi_2(H)+3n_1n_2^5\Delta_G^5\chi_1(H)}{8(n_2\delta_G+\delta_H-1)^3}\\ &+\frac{n_2^2(6n_2^5\Delta_H\Delta_G+3n_2^4\Delta_H^2)R_2(G)+n_2^2(12n_2^3\Delta_H^3\Delta_G+3n_2^2\Delta_H^4)R_1(G)+n_2^2m_1\Delta_H^6}{8(n_2\delta_G+\delta_H-1)^3}\\ &+\frac{n_1(6n_2\Delta_H\Delta_G+3n_2^2\Delta_G^2)R_2(H)+n_1(12n_2^3\Delta_G^3\Delta_H+3n_2^4\Delta_G^4)R_1(H)+n_1m_2n_2^6\Delta_G^6}{8(n_2\delta_G+\delta_H-1)^3},\end{aligned}$$

with equality if and only if G and H are regular graphs. $\square$

Theorem 5.2. *Let G be a graph of order n_1 and of size m_1 and let H be a graph of order n_2 and of size m_2. Then*

$$\begin{aligned}&AZI(G[H])\\ \geq&\frac{n_2^5(n_2\delta_G+\delta_H-\Delta_H-n_2)^3AZI(G)+n_1(\delta_H+n_2\delta_G-n_2\Delta_G-1)^3AZI(H)}{(n_2\delta_G+\delta_H-1)^3}\end{aligned}$$

$$+\frac{n_2^5\delta_H^3\chi_3(G)+n_2^2(3n_2^4\delta_H^2\delta_G^2+3n_2^2\delta_H^4)\chi_2(G)+3n_2^3\delta_H^5\chi_1(G)}{8(n_2\Delta_G+\Delta_H-1)^3}$$
$$+\frac{n_1n_2^3\delta_G^3\chi_3(H)+n_1(3n_2^2\delta_H^2\delta_G^2+3n_2^4\delta_G^4)\chi_2(H)+3n_1n_2^5\delta_G^5\chi_1(H)}{8(n_2\Delta_G+\Delta_H-1)^3}$$
$$+\frac{n_2^2(6n_2^5\delta_H\delta_G+3n_2^4\delta_H^2)R_2(G)+n_2^2(12n_2^3\delta_H^3\delta_G+3n_2^2\delta_H^4)R_1(G)+n_2^2m_1\delta_H^6}{8(n_2\Delta_G+\Delta_H-1)^3}$$
$$+\frac{n_1(6n_2\delta_H\delta_G+3n_2^2\delta_G^2)R_2(H)+n_1(12n_2^3\delta_G^3\delta_H+3n_2^4\delta_G^4)R_1(H)+n_1m_2n_2^6\delta_G^6}{8(n_2\Delta_G+\Delta_H-1)^3},$$

with equality if and only if G and H are regular graphs.

Proof. The proof of the result is similar to this given in Theorem 5.1. □

Corollary 5.1. *Let G be a k-regular graph of order n_1 and let H be a r-regular graph of order n_2. Then $AZI(G[H])=\frac{n_1n_2(n_2k+r)^7}{16(n_2k+r-1)^3}$.*

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