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ON FUZZY METRIZABLE Q-SUBALGEBRAS

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Abstract. This paper applies the concepts of (Kramosil and Michalek) KM-fuzzy metric spaces, Q-algebras and lattice subsets and introduces the notion of KM-fuzzy metric Q-algebras. In this study the the time parameter plays the main role in the construct of KM-fuzzy metric Q-algebras. We investigate some properties of KM-fuzzy metric Q-algebras and try to solve a mathematics model depended to the time parameter.

1. Introduction

Today's logic algebras are applied in real world problems and BCK-algebras and BCI-algebras as branches of logic algebra are introduced by Imai and Iseki [6,7]. In recent years, BCK-algebras, a specific subclass within the broader category of BCI-algebras, have garnered significant attention from researchers. Subsequently, other scholars have explored different extensions of logical algebras, leading to structures such as BCC-algebras and BCH-algebras. Neggers and collaborators introduced the concept of Q-algebras as a generalization of BCI-algebras, extending various theorems previously established in the context of BCI-algebras [13]. Recognizing that abstract algebra alone lacks direct real-world applications, mathematicians have sought to integrate these logical frameworks with other mathematical concepts. For instance, crisp sets have been generalized to fuzzy sets [15], which has opened new avenues for applications across diverse scientific domains.

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Fuzzy set theory, conceived by Zadeh to address uncertainty [15], represents a generalization of classical set theory. It plays a pivotal role in modeling and controlling uncertain systems across nature, society, and industry. Additionally, it has proven invaluable in describing complex phenomena that classical set theory struggles to characterize effectively. Following Zadeh's seminal research, extensive efforts have been devoted to formulating fuzzy analogues of classical theories. Among these advancements, fuzzy topology has emerged as a key area of focus. As a foundational subset of fuzzy theory, fuzzy topology has witnessed active research in recent years due to its extensive practical applications.

One pressing challenge in fuzzy topology involves establishing an appropriate definition for a fuzzy metric space. Efforts to address this issue have been undertaken by numerous researchers from various perspectives. Several methodologies have been proposed to define a metric space within a fuzzy framework ([2, 3, 5, 8]), each serving as a generalization of traditional metric spaces. Among these approaches, the concept of KM-fuzzy metric spaces has been explored extensively, utilizing the notion of t-norms as a foundational element. In 2015, I. Mardones-Perez and colleagues investigated the extent to which certain topological and uniform properties are upheld in KM-fuzzy pseudometric spaces. They also explored connections between KM-fuzzy pseudometric spaces and specific fuzzy structures that naturally arise, referred to as fuzzifying structures [11]. The notion of metrizable structures is one of the main tools in the Mathematics, so we see some researches in the recent years such as a survey of generalized metrizable properties in topological groups and weakly topological groups [14], continuity of homomorphisms on complete metrizable topological algebras [4], metrizable quotients of free topological groups [9], and algebraic constructions of groupoid for metric spaces [10].

Our main motivation for this paper is to construct a KM-fuzzy Q-algebra based of KM-fuzzy pseudometric spaces and triangular norms. It is defined the notation of metrizable Q-algebra in this study. Indeed, we consider a Q-algebra and define a map related to its binary operation and convert it to a metric space. The main aim of this relation is an extension of a Q-algebra to a metric Q-algebra. It is tried to apply the binary operations to generate of a distance map(metric) and apply the distance map to generate a binary relation and so generate of Q-algebras from metric spaces and generate of metric spaces from Q-algebras. The class of Q-algebras necessarily are not well-ordered, and one of our motivation of this study is to construct of well-ordered Q-algebras and to this aim, applied the notation of homomorphisms. We introduced the concept of metrizable Q-algebra and showed that the any countable Q-algebra is a metrizable Q-algebra. Since the fuzzy metric space is a generalization of metric space, we motivated that extend the class of metrizable Q-algebra to the class of fuzzy metrizable Q-algebra and apply the notation of KM-fuzzy Q-algebra for this motivation. This study try to investigate the properties of fuzzy metrizable Q-algebra and to extend the larger class as product of fuzzy metrizable Q-algebras. Indeed, we consider an arbitrary Q-algebra and construct a real Q-algebra (subset of real numbers) isomorphic to it and convert it to a metric space. This makes to introduce the notion of (fuzzy) metrizable Q-algebras and investigate their properties. The triangular norms(such as nilpotent triangular norm) are important tools in this study and help to become simpler complex computations on the fuzzy metrizable Q-algebras. In final, we establish the notions of KM-fuzzy Q-algebra by the level subsets and present the extension of KM-fuzzy Q-algebras.

2. Preliminaries

In this section, we gather essential concepts relevant to our work.

Definition 2.1 ([13])**.** A Q-algebra is a nonempty set $\mathtt{X}$ equipped with a constant element $0_{\mathtt{X}}$ and a binary operation $*$ that adhere to the following axioms:

(i) $x * x = 0_{\mathtt{X}}$,
(ii) $x * 0_{\mathtt{X}} = x$,
(iii) $(x * y) * z = (x * z) * y$ for all $x, y, z \in \mathtt{X}$.

In $(\mathtt{X}, *)$, we can define a binary relation $\leq_{\mathtt{X}}$ by $x \leq_{\mathtt{X}} y$ if and only if $x * y = 0_{\mathtt{X}}$.

Theorem 2.1 ([13])**.** *Let* $(\mathtt{X}, *, 0)$ *be a* Q*-algebra and* $x, y, z \in \mathtt{X}$*. Then,*

(i) $(x * (x * y)) * y = 0$ *for all* $x, y \in \mathtt{X}$*,*
(ii) *if* $(x * y) * (x * z) = z * y$*, for all* $x, y, z \in \mathtt{X}$*, then* $(\mathtt{X}, *, 0)$ *is a BCI-algebra,*
(iii) *if* $x \leq_{\mathtt{X}} 0$*, for all* $\in \mathtt{X}$*, then* $\mathtt{X}$ *contains only* 0*,*
(iv) *if* $x \leq_{\mathtt{X}} y$*, then* $x * (x * (x * y)) = 0$*,*

Theorem 2.2 ([13])**.** *Any quadratic* Q*-algebra on finite fields is isomorphic to any other similar algebra on finite fields.*

Theorem 2.3 ([13])**.** *Any quadratic* Q*-algebra on finite fields generates the abelian groups.*

Definition 2.2 ([1])**.** A fuzzy subset $A = \{(x, \xi(x)) \mid x \in \mathtt{X}\}$ of a Q-algebra $(\mathtt{X}, *, 0_{\mathtt{X}})$ is a fuzzy sub-Q-algebra of $\mathtt{X}$, if for any $x, y \in \mathtt{X}$, $\xi(x * y) \geq \xi(x) \wedge \xi(y)$ and in equality, it is called strong.

Definition 2.3 ([12])**.** A binary operation $\mathtt{T} : \mathtt{I}^2 \to \mathtt{I}$ is a t-norm if it for all $x, y, z, w \in \mathtt{I}$ satisfies the following ($\mathtt{I} = [0, 1]$):

(i) $\mathtt{T}(1, x) = x$;
(ii) $\mathtt{T}(x, y) = \mathtt{T}(y, x)$;
(iii) $\mathtt{T}(\mathtt{T}(x, y), z) = \mathtt{T}(x, \mathtt{T}(y, z))$;
(iii) if $w \leq x$ and $y \leq z$, then $\mathtt{T}(w, y) \leq \mathtt{T}(x, z)$.

Definition 2.4 ([11])**.** A triplet $(\mathtt{X}, \mathtt{L}, \mathtt{T})$ is called a KM-fuzzy metric space, if $\mathtt{X}$ is an arbitrary non-empty set, $\mathtt{T}$ is a left-continuous t-norm and $\mathtt{L} : \mathtt{X}^2 \times \mathbb{R}^{\geq 0} \to \mathtt{I}$ is a fuzzy set, such that for each $x, y, z, \in \mathtt{X}$ and $t, s \geq 0$, we have:

(i) $\mathtt{L}(x, y, 0) = 0$;
(ii) $\mathtt{L}(x, x, t) = 1$, for all $t > 0$;

(*iii*) $\mathtt{L}(x, y, t) = \mathtt{L}(y, x, t)$:
(*iv*) $\mathtt{T}(\mathtt{L}(x, y, t), \mathtt{L}(y, z, s)) \leq \mathtt{L}(x, z, t + s)$;
(*vi*) $\mathtt{L}(x, y, -) : \mathbb{R}^{\geq 0} \to \mathtt{I}$ is a left-continuous map;
(*vii*) $\mathtt{L}(x, y, t) \to 1$, when $t \to +\infty$;
(*viii*) $\mathtt{L}(x, y, t) = 1$, for all $t > 0$, implies that $x = y$.

If $(\mathtt{X}, \mathtt{L}, T)$ satisfies in conditions (*i*)-(*vii*), then it is called *KM*-fuzzy pseudometric space and $\mathtt{L}$ is called a *KM*-fuzzy pseudometric.

3. Derived Metric Space from Q-algebras

In this section, we focus on constructing non-commutative Q-algebras while expanding the class of Q-algebras. Additionally, we aim to build well-ordered Q-algebras derived from countable sets. Furthermore, we introduce the concept of metrizable Q-algebras and establish a connection between Q-algebras and metric spaces.

Let $(\mathtt{X}, *_{\mathtt{X}}, 0_{\mathtt{X}})$ and $(\mathtt{X}', *'_{\mathtt{X}}, 0'_{\mathtt{X}})$ be two Q-algebras and $f : \mathtt{X} \to \mathtt{X}'$ be a map. Then f is called a *homomorphism*, if for any $x, y \in \mathtt{X}, f(x *_X y) = f(x) *'_{\mathtt{X}} f(y)$ and is called an *isomorphism*, if f is a bijection . It is clear that $f(0_{\mathtt{X}}) = f(x *_X x) = f(x) *'_{\mathtt{X}} f(x) = 0_{\mathtt{X}'}$ and $f(x *_X 0_{\mathtt{X}}) = f(x) *'_{\mathtt{X}} f(0_{\mathtt{X}}) = f(x) *'_{\mathtt{X}} 0_{\mathtt{X}'} = f(x)$.

From now on, we consider $\{0, 1, 2, 3, \ldots, k-1\} = \mathbb{N}^*_{k-1}$ and try construct well ordered Q-algebras by rearranging from the other same-cardinal Q-algebras.

Theorem 3.1. *Let* $(\mathtt{X}, *_{\mathtt{X}}, 0_{\mathtt{X}})$ *be a countable Q-algebra. Then,* $(\mathtt{X}, *_{\mathtt{X}}, 0_{\mathtt{X}})$ *is isomorphic to a well ordered Q-algebra.*

Proof. Let $(\mathtt{X}, *, 0_{\mathtt{X}})$ be a any arbitrary Q-algebra and $|\mathtt{X}| = k$, which $k \in \mathbb{N}$. Then, define $\mathtt{X} \overset{\psi}{\sim} \mathbb{N}^*_{k-1}$ ($\psi : \mathtt{X} \to \mathbb{N}^*_{k-1}$ is a bijection) that for any $1 \leq i, j \leq k-1$, there is $x, y \in \mathtt{X}$ such that $i = \psi(x)$, $j = \psi(y)$ and $0 = \psi(0_{\mathtt{X}})$. Now, define a binary operation $*' : \mathbb{N}^*_{k-1} \times \mathbb{N}^*_{k-1} \to \mathbb{N}^*_{k-1}$, by $i *' j = \psi(x * y)$ and $0 = \psi(0_{\mathtt{X}})$. Since ψ is a bijection, based the definition of ψ, the binary operation $*'$ is well-defined. Assume that $i, j, k \in \{0, 1, 2, \ldots, k-2, k-1\}$. There exist unique $x, y, z \in \mathtt{X}$ such that $i *' i = \psi(x * x) = \psi(0_{\mathtt{X}}) = 0$, $i * 0 = \psi(x * 0_{\mathtt{X}}) = \psi(x) = i$ and

$$\begin{aligned}(i *' j) *' k =& \psi(x * y) *' k = \psi(x * y) *' \psi(z) = \psi((x * y) * z) = \psi((x * z) * y) \\ =& \psi(x * z) *' \psi(y) = (i *' k) *' j.\end{aligned}$$

Hence, $(\mathbb{N}^*_{k-1}, *', 0)$ is a Q-algebra. Now, assume that $(\mathtt{X}, *, 0_{\mathtt{X}})$ is an arbitrary Q-algebra which $\mathtt{X}$ is infinite set. Since $\mathtt{X}$ is countable, we can see that $\mathtt{X} \overset{\psi}{\sim} \mathbb{N}^* = \{0\} \cup \mathbb{N}$($\psi : \mathtt{X} \to \mathbb{N}^*$ is a bijection) that for any $i, j \in \mathbb{N}^*$, there is $x, y \in \mathtt{X}$ such that $i = \psi(x)$, $j = \psi(y)$ and $0 = \psi(0_{\mathtt{X}})$. In a similar way, we see that $(\mathbb{N}^*, *')$ is a Q-algebra, which $*' : \mathbb{N}^* \times \mathbb{N}^* \to \mathbb{N}^*$, by $i *' j = \psi(x * y)$ and $0 = \psi(0_{\mathtt{X}})$. ☐

Based the Theorem 3.1, we have the following example.

Example 3.1. Let $\mathtt{X} = \{0_{\mathtt{X}}, a, b, c\}$. Then, $(\mathtt{X}, *)$ is a Q-algebra as Table 1, which $\mathtt{X} \overset{\psi}{\sim} \mathbb{N}_3^*, \psi = \{(0_{\mathtt{X}}, 0), (a, 1), (b, 2), (c, 3)\}$ and so, from Theorem 3.1, $(\mathbb{N}_3^*, *)$ is a Q-algebra in Table 2.

$*$	$0_{\mathtt{X}}$	a	b	c
$0_{\mathtt{X}}$	$0_{\mathtt{X}}$	a	b	c
a	a	$0_{\mathtt{X}}$	c	b
b	b	c	$0_{\mathtt{X}}$	a
c	c	b	a	$0_{\mathtt{X}}$

TABLE 1. $(\mathtt{X}, *, 0_{\mathtt{X}})$

$*'$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

TABLE 2. $(\mathbb{N}_3^*, *', 0)$

A Q-algebra $(\mathtt{X}, *)$ is called commutative, if for any $x, y \in \mathtt{X}$, $x * y = y * x$. In what follow, we try construct the non-commutative Q-algebras.

Proposition 3.1. *Any non-empty set can be a non-commutative Q-algebra.*

Proof. Let $\mathtt{X}$ be a non-empty set and $0_{\mathtt{X}} \in \mathtt{X}$. Define a binary operation " $*$ " on $\mathtt{X}$ by

$$x * y = \begin{cases} 0_{\mathtt{X}}, & \text{if } x = y, \\ x, & \text{if } x \neq y. \end{cases}$$

Computations demonstrate that $(\mathtt{X}, *)$ is a non-commutative Q-algebra. □

In what follow, we want to construct a metric space $(\mathtt{X}, \mathtt{D}_*)$ from a Q-algebra $(\mathtt{X}, *)$, where the map $\mathtt{D}_* : \mathtt{X} \times \mathtt{X} \to \mathbb{R}^{\geq 0}$ is related to the binary operation $*$.

Let $(\mathtt{X}, *)$ be a Q-algebra. We say that $(\mathtt{X}, *)$ is a diagonal Q-algebra, if for any $x, y \in \mathtt{X}, x * y = 0_{\mathtt{X}}$, implies that $x = y$.

Example 3.2. Let $(\mathtt{X}, *)$ be a commutative diagonal Q-algebra. Define $\mathtt{D}_* : \mathtt{X} \times \mathtt{X} \to [0, +\infty)$ by

$$\mathtt{D}_*(x, y) = \begin{cases} 0, & \text{if } x * y = 0_{\mathtt{X}}, \\ 1, & \text{if } x * y \neq 0_{\mathtt{X}}. \end{cases}$$

Since $(\mathtt{X}, *)$ is a commutative diagonal Q-algebra, $x * y = 0_{\mathtt{X}}$ implies that $x = y$ and for any $x \in \mathtt{X}$, $\mathtt{D}_*(x, x) = 0$. For any distinct $x, y \in \mathtt{X}$,

$$\begin{aligned} &\mathtt{D}_*(x, y) = 1 = \mathtt{D}_*(y, x), \quad \text{for any } x \neq y \neq z \in \mathtt{X}, \\ &\mathtt{D}_*(x, z) = 1 \leq 2 = 1 + 1 = \mathtt{D}_*(x, y) + \mathtt{D}_*(y, z), \quad \text{for any } x \neq z, x = y \in \mathtt{X}, \\ &\mathtt{D}_*(x, z) = 1 \leq 1 = 0 + 1 = \mathtt{D}_*(x, y) + \mathtt{D}_*(y, z), \quad \text{for any } x = z, y \in \mathtt{X}, \\ &\mathtt{D}_*(x, z) = 0 \leq \mathtt{D}_*(x, y) + \mathtt{D}_*(y, z). \end{aligned}$$

Hence, $(\mathtt{X}, *, \mathtt{D}_*)$ is a metric Q-algebra.

In what follow, we consider two Q-algebras and convert them to metric spaces.

Example 3.3. (*i*) Consider the Q-algebra $(\mathtt{X},*)$, which $\mathtt{X}=\mathbb{N}_4^*=\{0,1,2,3,4\}$ and $*:\mathtt{X}\times\mathtt{X}\to\mathtt{X}$ is defined by

$$x*y=\begin{cases}0, & \text{if } x=y,\\ \max\{x,y\}, & \text{if } x\neq y.\end{cases}$$

Now, define $\mathtt{D}_*:\mathtt{X}\times\mathtt{X}\to[0,+\infty)$, by $\mathtt{D}_*(x,y)=x*y$, where $x,y\in\mathtt{X}$. Since $(\mathtt{X},*)$ is a Q-algebra, for any $x\in\mathtt{X}$, $\mathtt{D}_*(x,x)=x*x=0$, for any $x\neq y\in\mathtt{X}$,

$$\begin{aligned}\mathtt{D}_*(x,y)=&x*y=\max\{x,y\}=\max\{y,x\}=y*x=\mathtt{D}_*(y,x),\quad \text{for all } x,y,z\in\mathtt{X},\\ \mathtt{D}_*(x,z)=&x*z=\max\{x,z\}\leq\max\{x,y\}+\max\{y,z\}\\ =&(x*y)+(y*z)=\mathtt{D}_*(x,y)+\mathtt{D}_*(y,z).\end{aligned}$$

Hence, $(\mathtt{X},*,\mathtt{D}_*)$ is a metric space.

(*ii*) Consider the Q-algebra $(\mathtt{X},*)$, which $\mathtt{X}=\mathbb{N}^*$ and $*:\mathtt{X}\times\mathtt{X}\to\mathtt{X}$ is defined by

$$x*y=\begin{cases}0, & \text{if } x=y,\\ x+y, & \text{if } x\neq y.\end{cases}$$

Define $\mathtt{D}_*:\mathtt{X}\times\mathtt{X}\to[0,+\infty)$, by $\mathtt{D}_*(x,y)=x*y$, where $x,y\in\mathtt{X}$. Since $(\mathtt{X},*)$ is a Q-algebra, for any $x\in\mathtt{X}$, $\mathtt{D}_*(x,x)=x*x=0$, and for any $x\neq y\in\mathtt{X}$,

$$\begin{aligned}\mathtt{D}_*(x,y)=&x*y=\max\{x,y\}=x+y=y+x=y*x=\mathtt{D}_*(y,x),\quad \text{for all } x,y,z\in\mathtt{X},\\ \mathtt{D}_*(x,z)=&x*z=x+z\leq(x+y)+(y+z)=(x*y)+(y*z)=\mathtt{D}_*(x,y)+\mathtt{D}_*(y,z).\end{aligned}$$

Hence, $(\mathtt{X},*,\mathtt{D}_*)$ is a metric space.

Let $(\mathtt{X},*_{\mathtt{X}},0_{\mathtt{X}})$ be a Q-algebra. We say that $(\mathtt{X},*_{\mathtt{X}},0_{\mathtt{X}})$ is a metrizable Q-algebra, if there is a map $\mathtt{D}_*:\mathtt{X}'\times\mathtt{X}'\to[0,+\infty)$ such that $\mathtt{X}'$ is an isomorphic Q-algebra to Q-algebra $\mathtt{X}$ and $(\mathtt{X}',*,\mathtt{D}_*)$ is a metric space.

Theorem 3.2. *Let* $(\mathtt{X},*)$ *be an arbitrary countable* Q*-algebra. Then,* $(\mathtt{X},*)$ *is a metrizable* Q*-algebra.*

Proof. Let $(\mathtt{X},*)$ be a countable Q-algebra. Then, $\mathtt{X}\overset{\psi}{\sim}\mathbb{N}^*$ or there is $k\in\mathbb{N}$ such that $\mathtt{X}\overset{\psi}{\sim}\mathbb{N}_k^*$. Now, based Theorem 3.1, define a map $\mathtt{D}_{max}:\mathtt{X}\times\mathtt{X}\to\mathbb{R}^{\geq 0}$, by

$$\mathtt{D}_{max}(x,y)=\begin{cases}0, & \text{if } x=y,\\ \max\{i,j\}, & \text{if } x\neq y,\end{cases}$$

such that $i=\psi(x)$, $j=\psi(y)$ and $0=\psi(0_{\mathtt{X}})$. Since $(\mathtt{X},*)$ is a Q-algebra, for any $x\in\mathtt{X}$, $\mathtt{D}_{max}(x,x)=x*x=0$, for any $x\neq y,z\in\mathtt{X}$, there exist $i,j,k\in\mathbb{N}$, such that $i=\psi(x)$, $j=\psi(y)$, $k=\psi(z)$ and

$$\begin{aligned}\mathtt{D}_{max}(x,y)=&\max\{i,j\}=\max\{j,i\}=\mathtt{D}_{max}(y,x),\quad \text{for any } x,y,z\in\mathtt{X},\\ \mathtt{D}_{max}(x,z)=&\max\{i,k\}\leq\max\{i,j\}+\max\{j,k\}=\mathtt{D}_{max}(x,y)+\mathtt{D}_{max}(y,z).\end{aligned}$$

Hence, $(\mathtt{X},*,\mathtt{D}_{max})$ is a metric space. □

Example 3.4. Consider the Q-algebra $(\mathtt{X}, *)$ in Table 1. Define a map $\psi : \mathtt{X} \to \mathbb{N}_3^*$ by $\psi = \{(0_{\mathtt{X}}, 0), (a, 1), (b, 2), (c, 3)\}$. Clearly, ψ is an isomorphism and so $(\{0_{\mathtt{X}}, a, b, c\}, *)$ isomorphic to Q-algebra $(\mathbb{N}_3^*, *', 0)$. Now, from Theorem 3.2, we define a map $\mathtt{D}_{max} : \mathbb{N}_3^* \times \mathbb{N}_3^* \to \mathbb{R}^{\geq 0}$, by

$$\mathtt{D}_{max}(x, y) = \begin{cases} 0, & \text{if } x = y, \\ \max\{i, j\}, & \text{if } x \neq y. \end{cases}$$

Hence, $(\mathtt{X}, *)$ is a metrizable Q-algebra.

$\mathtt{D}$	$0_{\mathtt{X}}$	a	b	c
$0_{\mathtt{X}}$	0	1	2	3
a	1	0	2	3
b	2	2	0	3
c	3	3	3	0

TABLE 3. Metric Q-algebra $(\mathtt{X}, *, \mathtt{D})$

In similar to Theorem 3.1 and Theorem 3.2, we have the following corollary.

Corollary 3.1. *Let* $(\mathtt{X}, *)$ *be a* Q*-algebra and* $\mathtt{X}$ *be an uncountable set. Then,*

(*i*) $(\mathtt{X}, *_{\mathtt{X}}, 0_{\mathtt{X}})$ *is isomorphic to a well ordered* Q*-algebra;*
(*ii*) $(\mathtt{X}, *)$ *is a metrizable* Q*-algebra.*

Example 3.5. Consider the Q-algebra $(\mathbb{R}, *)$, which $x * y = x - y$. Now define, $\mathtt{D}_*(x, y) = |x * y|$. It is easy to see that $(\mathtt{X}, \mathtt{D}_*)$ is a metric space.

Let $(\mathtt{X}, *_{\mathtt{X}})$ and $(\mathtt{Y}, *_{\mathtt{Y}})$ be Q-algebras. For any $(x, y), (x', y') \in \mathtt{X} \times \mathtt{Y}$, consider $(x, y) *_{\mathtt{X}\times\mathtt{Y}} (x', y') = (x *_X x', y *_Y y')$. Hence, we have the following theorem.

Theorem 3.3. *Let* $(\mathtt{X}, *_{\mathtt{X}}, 0_{\mathtt{X}})$ *and* $(\mathtt{Y}, *_{\mathtt{Y}}, 0_{\mathtt{Y}})$ *be* Q*-algebras. Then,* $(\mathtt{X} \times \mathtt{Y}, *_{\mathtt{X}\times\mathtt{Y}})$ *is a* Q*-algebra.*

4. KM-FUZZY METRIC Q-ALGEBRA

In this section, we present a novel concept referred to as KM-fuzzy metric Q-algebras and examine some of their fundamental properties. It is demonstrated that the class of KM-fuzzy metric Q-algebras serves as a generalization of Q-algebras, implying that any Q-algebra can be extended into a metrizable Q-algebra. Furthermore, the process of extending a standard metric space to a fuzzy metric space is explored, thereby facilitating the extension of metrizable Q-algebras into fuzzy metrizable Q-algebras.

Definition 4.1. Let $(\mathtt{X}, \mathtt{L}, \mathtt{T})$ be a KM-fuzzy metric space and $(\mathtt{X}, *)$ be a Q-algebra. Then, $(\xi, \mathtt{L}, \mathtt{T})$ is called a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$ (a strong KM-fuzzy metric Q-subalgebra of $\mathtt{X}$), if $A = \{(x, \xi(x)) \mid x \in \mathtt{X}\}$ is a fuzzy subset of $\mathtt{X}$, and there is a time t (for $t = 0$, we call starting time) such that for all $x, y \in \mathtt{X}$, we have

$\mathtt{T}(\xi(x*y),\mathtt{T}(\xi(x),\xi(y))) \leq \mathtt{L}(x,y,t)$. We call $(\xi,\mathtt{L},\mathtt{T})$ is a strong KM-fuzzy metric Q-subalgebra of $\mathtt{X}$, if for all $x,y\in\mathtt{X}$, $\mathtt{T}(\xi(x*y),\mathtt{T}(\xi(x),\xi(y))) = \mathtt{L}(x,y,t)$.

Example 4.1. Let $\mathtt{X}=\mathbb{N}_7\cup\{0_{\mathtt{X}}\}$. Then $(\mathtt{X},*,0_{\mathtt{X}})$ is a Q-algebra based the Table 4. Define a fuzzy subset ξ of $\mathtt{X}$ as shown in Table 5 and KM-fuzzy metric

$$\mathtt{L}(x,y,t)=\begin{cases}0, & \text{if } t=0,\\ \frac{\min\{x,y\}+t}{\max\{x,y\}+t}, & \text{if } t>0.\end{cases}$$

If $(\xi,\mathtt{L},\mathtt{T}_{\min})$ is a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$, then $t\geq 293$, since $x*x=0_{\mathtt{X}}$ and $\mathtt{L}(x,x,t)=1$, we understood

$$\bigvee_{x,y\in\mathtt{X}}\Big(\bigwedge_{x,y\in\mathtt{X}}(\xi(x*y),\xi(x),\xi(y))\Big)\leq\bigwedge_{x,y\in\mathtt{X}}\frac{\min\{x,y\}+t}{\max\{x,y\}+t}$$

implies $\frac{98}{100}\leq\frac{1+t}{7+t}$, i.e., $t\geq 293$.

Table 4. Q-algebra $(\mathtt{X},*,0_{\mathtt{X}})$

$*$	$0_{\mathtt{X}}$	1	2	3	4	5	6	7
$0_{\mathtt{X}}$	$0_{\mathtt{X}}$	1	2	3	4	5	6	7
1	1	$0_{\mathtt{X}}$	3	2	5	4	7	6
2	2	3	$0_{\mathtt{X}}$	1	6	7	4	5
3	3	2	1	$0_{\mathtt{X}}$	7	6	5	4
4	4	5	6	7	$0_{\mathtt{X}}$	1	2	3
5	5	4	7	6	1	$0_{\mathtt{X}}$	3	2
6	6	7	4	5	2	3	$0_{\mathtt{X}}$	1
7	7	6	5	4	3	2	1	$0_{\mathtt{X}}$

Let $(\mathtt{X},*_{\mathtt{X}},0_{\mathtt{X}})$ be a Q-algebra. We say that $(\mathtt{X},*_{\mathtt{X}},0_{\mathtt{X}})$ is a fuzzy metrizable, if it has a KM-fuzzy metric Q-subalgebra. For any $x,y\in\mathtt{I}$, $\mathtt{T}_{\min}(x,y)=\min\{x,y\}$, is called Gödel t-norm or Minimum t-norm and for any fuzzy subset ξ of $\mathtt{X},\xi$ is called almost everywhere constant, if there is $\emptyset\neq\mathtt{Y}\subseteq\mathtt{X}$, that for any $y\in\mathtt{Y}$, $\xi(y)=c$, which $c\in(0,1]$.

Theorem 4.1. *Let* $(\mathtt{X},*_{\mathtt{X}},0_{\mathtt{X}})$ *be a countable Q-algebra. Then,* $(\mathtt{X},*_{\mathtt{X}},0_{\mathtt{X}})$ *is a fuzzy metrizable.*

Proof. Since $(\mathtt{X},*_{\mathtt{X}},0_{\mathtt{X}})$ is a countable Q-algebra, based on Theorem 3.2, $(\mathtt{X},*_{\mathtt{X}},0_{\mathtt{X}})$ is a metrizable Q-algebra and so there is a map $\mathtt{D}:\mathtt{X}\times\mathtt{X}\to\mathbb{R}^{\geq 0}$. Define a map

	$0_{\mathtt{X}}$	1	2	3	4	5	6	7
ξ:	1	0.99	0.98	0.97	0.96	0.95	0.94	0.93

Table 5. Fuzzy subset A

$\mathtt{L}:\mathtt{X}^2\times\mathbb{R}^{\geq 0}\to\mathtt{I}$ by

$$\mathtt{L}(x,y,t)=\begin{cases}0, & \text{if } t=0,\\ \mathtt{L}(x,y,t)=\frac{t}{t+\mathtt{D}(x,y)}, & \text{if } t>0.\end{cases}$$

Routine computations demonstrate that $(\mathtt{X},\mathtt{L},\mathtt{T}_{\min})$ is a KM-fuzzy metric space. Now, consider a map $\xi:\mathtt{X}\to[0,1]$ such that for any $x\in\mathtt{X}$, $\xi(x)=\xi(0_{\mathtt{X}})$ and for any $y\in\mathtt{X}$, $t>0$, $\xi(x)\leq\mathtt{L}(x,y,t)$. Clearly, ξ is a fuzzy sub-Q-algebra and so $(\xi,\mathtt{L},\mathtt{T}_{\min})$ is a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$. □

Theorem 4.2. *Let $(\xi,\mathtt{L},\mathtt{T}_{\min})$ be a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$ and $\mathtt{T}$ be any arbitrary triangular norm. Then $(\xi,\mathtt{L},\mathtt{T})$ is a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$.*

Proof. Let $x,y\in\mathtt{X}$. Since

$$\mathtt{T}\Big(\mathtt{T}(x*y),\mathtt{T}(\xi(x),\xi(y))\Big)\leq\mathtt{T}_{\min}\Big(\mathtt{T}_{\min}(x*y),\mathtt{T}_{\min}(\xi(x),\xi(y))\Big),$$

understood $\mathtt{T}_{\min}(\xi(x*y),\mathtt{T}_{\min}(\xi(x),\xi(y)))\leq\mathtt{L}(x,y,t)$, implies that

$$\mathtt{T}(\xi(x*y),\mathtt{T}(\xi(x),\xi(y)))\leq\mathtt{L}(x,y,t),$$

Hence, if $\mathtt{X}=(\xi,\mathtt{L},\mathtt{T}_{\min})$ is a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$, then for any another arbitrary triangular norms $\mathtt{T}$, $\mathtt{X}=(\xi,\mathtt{L},\mathtt{T})$ is a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$. □

Let ξ be a fuzzy subset of $\mathtt{X}$. We say that ξ is trivial constant, if $\xi\equiv 0_{\mathtt{X}}$ and $\xi\equiv 1$.

Theorem 4.3. *Let $(\xi,\mathtt{L},\mathtt{T}_{\min})$ be a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$.*

(*i*) *If ξ is a fuzzy sub-Q-algebra, then for any $x,y\in\mathtt{X},\mathtt{L}(x,y,t)\geq\mathtt{T}_{\min}(\xi(x),\xi(y))$.*
(*ii*) *In starting time, ξ is almost everywhere zero.*
(*iii*) *In starting time, ξ can not be a non-trivial fuzzy sub-Q-algebra.*

Proof. Let $x,y\in\mathtt{X}$.

(*i*) Since $(\xi,\mathtt{L},\mathtt{T}_{\min})$ is a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$, get

$$\mathtt{L}(x,y,t)\geq\mathtt{T}_{\min}(\xi(x*y),\mathtt{T}_{\min}(\xi(x),\xi(y)))\geq\mathtt{T}_{\min}(\xi(x),\xi(y)).$$

Hence, for any $x,y\in\mathtt{X},\mathtt{L}(x,y,t)\leq\xi(x)$.

(*ii*) Based item (*i*), for any $x,y\in\mathtt{X}$, $\mathtt{L}(x,y,t)\geq\mathtt{T}_{\min}(\xi(x*y),\mathtt{T}_{\min}(\xi(x),\xi(y)))$, so in starting time,

$$0_{\mathtt{X}}=\mathtt{L}(x,y,0_{\mathtt{X}})\geq\mathtt{T}_{\min}(\xi(x*y),\mathtt{T}_{\min}(\xi(x),\xi(y)))$$

implies $\xi(x)=0_{\mathtt{X}}$ or $\xi(y)$ or $\xi(x*y)=0_{\mathtt{X}}=0_{\mathtt{X}}$. It follows that ξ is almost everywhere zero.

(*iii*) Based item (*ii*), $0_{\mathtt{X}}=\mathtt{L}(0_{\mathtt{X}},0_{\mathtt{X}},0_{\mathtt{X}})\geq\mathtt{T}_{\min}(\xi(0_{\mathtt{X}}),\xi(0_{\mathtt{X}}))$, so $\xi(x)=\xi(y)=0_{\mathtt{X}}$ and so $\xi\equiv 0_{\mathtt{X}}$. □

Corollary 4.1. *Let $(\xi,\mathtt{L},\mathtt{T}_{\min})$ be a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$. If ξ is a fuzzy sub-Q-algebra, then in non-starting time, for any $x\in\mathtt{X}$, $\mathtt{L}(x,0_{\mathtt{X}},t)\geq\xi(x)$.*

	$0_{\mathtt{X}}$	a	b	c
ξ:	1	0.9	0.8	0.65

TABLE 6. A

$\mathtt{T}_P$	$\xi(0_{\mathtt{X}})$	$\xi(a)$	$\xi(b)$	$\xi(c)$
$\xi(0_{\mathtt{X}})$	1	0.9	0.8	0.65
$\xi(a)$	0.9	0.81	0.72	0.585
$\xi(b)$	0.8	0.72	0.64	0.52
$\xi(c)$	0.65	0.585	0.52	0.4225

TABLE 7. $\mathtt{T}_P$

$\mathtt{L}$	$0_{\mathtt{X}}$	a	b	c
$0_{\mathtt{X}}$	1	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{1}{2}$
a	$\frac{3}{4}$	1	$\frac{3}{5}$	$\frac{1}{2}$
b	$\frac{3}{5}$	$\frac{3}{5}$	1	$\frac{1}{2}$
c	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1

TABLE 8. $\mathtt{L}$

Let $(\mathtt{X}, *)$ be a Q-algebra and ξ be a fuzzy subset of $\mathtt{X}$. We will call a triangular norm $\mathtt{T}$, ξ-dominates the KM-fuzzy metric $\mathtt{L}$ and is denoted by $\mathtt{T} \succeq \mathtt{L}$, if in any time and for any $x \neq y \in \mathtt{X}$, $\mathtt{T}(\xi(x), \xi(y)) \geq \mathtt{L}(x, y, t)$.

Theorem 4.4. *Let* $(\xi, \mathtt{L}, \mathtt{T})$ *be a strong* KM*-fuzzy metric* Q*-subalgebra of* $\mathtt{X}$. *Then,* $\mathtt{T} \succeq \mathtt{L}$.

Proof. Let $x, y \in \mathtt{X}$. Since $(\xi, \mathtt{L}, \mathtt{T})$ is a strong KM-fuzzy metric Q-subalgebra of $\mathtt{X}$,

$$\mathtt{L}(x, y, t) = \mathtt{T}(\xi(x * y), \mathtt{T}(\xi(x), \xi(y))) \leq \mathtt{T}(\xi(x), \xi(y)).$$

Hence, $\mathtt{T}(\xi(x), \xi(y)) \geq \mathtt{L}(x, y, t)$ and so $\mathtt{T}$, ξ-dominates the KM-fuzzy metric $\mathtt{L}$. ☐

Example 4.2, shows that the converse of Theorem 4.4, is not necessarily true. Let $x, y \in \mathtt{I}$. We will consider $\mathtt{T}_P(x, y) = xy$ as the product triangular norm and $\mathtt{T}_H(x, y) = \frac{xy}{x+y-xy}$ as Hamacher product triangular norm will apply in the following examples.

Example 4.2. Consider the Q-algebra $(\mathtt{X}, *)$ in Table 1. Then, $(\xi, \mathtt{L}, \mathtt{T}_P)$ is a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$ as Tables 6, 7 and 8.

Computations demonstrate that $\mathtt{T} \succeq \mathtt{L}$, while $(\xi, \mathtt{L}, \mathtt{T})$ is not a strong KM-fuzzy metric Q-subalgebra of $\mathtt{X}$.

Corollary 4.2. *Let* $(\xi, \mathtt{L}, \mathtt{T})$ *be a* KM*-fuzzy metric* Q*-subalgebra of* $\mathtt{X}$. *If* ξ *is a strong fuzzy sub-*Q*-algebra, then* $\lim_{t \to +\infty} \mathtt{T}(\xi(x, y), \xi(x, y)) = 1$.

Theorem 4.5. *Let* $(\xi, \mathtt{L}, \mathtt{T})$ *be a* KM*-fuzzy metric* Q*-subalgebra of* $\mathtt{X}$. *If* ξ *is a strong fuzzy sub-*Q*-algebra, then* ξ *is constant.*

Proof. Let $x \in \mathtt{X}$. Since ξ is a strong fuzzy subset of $\mathtt{X}$, we understood $\xi(0_{\mathtt{X}}) = \xi(x)$. Now, $(\xi, \mathtt{L}, \mathtt{T})$ is a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$, so $\mathtt{T}(\xi(x * 0_{\mathtt{X}}), \mathtt{T}(\xi(x), \xi(0_{\mathtt{X}}))) \leq \mathtt{L}(x, 0_{\mathtt{X}}, t)$. Hence, $\mathtt{T}(\xi(x), \xi(x)) \leq \mathtt{L}(x, 0_{\mathtt{X}}, t)$ and so $\mathtt{T}(\xi(0_{\mathtt{X}}), \xi(0_{\mathtt{X}})) = 0_{\mathtt{X}}$ and so for any $x, y \in \mathtt{X}, \mathtt{T}(\xi(x), \xi(y)) \leq \mathtt{T}(\xi(0_{\mathtt{X}}), \xi(0_{\mathtt{X}})) = 0_{\mathtt{X}}$. It concludes that $\mathtt{T}(\xi(x), \xi(y)) = 0_{\mathtt{X}}$, which means that $\xi(x) = \xi(y) = 0_{\mathtt{X}}$, because of $\xi(0_{\mathtt{X}}) = 0_{\mathtt{X}}$ and $\xi(0_{\mathtt{X}}) \geq \xi(x) \vee \xi(y)$. Therefore, $\xi \equiv 0_{\mathtt{X}}$. ☐

$*$	$0_{\mathtt{X}}$	1	2	3
$0_{\mathtt{X}}$	$0_{\mathtt{X}}$	1	2	3
1	1	$0_{\mathtt{X}}$	3	2
2	2	3	$0_{\mathtt{X}}$	1
3	3	2	1	$0_{\mathtt{X}}$

Table 9. $(\mathtt{X}, *, 0_{\mathtt{X}})$

	$0_{\mathtt{X}}$	1	2	3
ξ:	0.8	0.7	0.6	0.6

Table 10. A

We recall that a triangular norm $\mathtt{T}$, is nilpotent, if for some $n \in \mathbb{N}$, $\mathtt{T}(a^{\lfloor n \rfloor}) = 0_{\mathtt{X}}$, which $a \neq 1$ and $\mathtt{T}(a^{\lfloor n \rfloor}) = \mathtt{T}(\underbrace{a, a, a \ldots, a}_{n-\text{times}})$.

Theorem 4.6. *Let $(\xi, \mathtt{L}, \mathtt{T})$ be a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$ and ξ be a strong fuzzy sub-Q-algebra of $\mathtt{X}$. Then, in any non-starting time, $\mathtt{T}$ can not be nilpotent.*

Proof. Let $x \in \mathtt{X}$. If time is not in starting and $\mathtt{T}$ is a nilpotent triangular norm, then for $x \neq 1$ and for some $n \in \mathbb{N}, \mathtt{T}(x^{\lfloor n \rfloor}) = 0_{\mathtt{X}}$. Since ξ is an fuzzy subset of $\mathtt{X}$ and $(\xi, \mathtt{L}, \mathtt{T})$ is a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$, we understood

$$\begin{aligned} 0 =& \mathtt{T}(\xi(x^n), 0)) = \mathtt{T}(\xi(x^n), \mathtt{T}(\xi(x)^{\lfloor n \rfloor}) = \mathtt{T}(\xi(\underbrace{x * x * \ldots * x}_{n-times})), \mathtt{T}(\xi(x)^{\lfloor n \rfloor})) \\ =& \mathtt{T}(\xi(x), \mathtt{T}(\xi(x), \xi(x))) = \mathtt{L}(x, x, t) = 1, \end{aligned}$$

which it is a contradiction, so $t = 0$. ☐

Let $x, y \in \mathtt{I}$. Then, $\mathtt{T}_{Luk}(x, y) = \max\{0, x + y - 1\}$ is called the Lukasiewicz t-norm, which will apply it in the following results.

Example 4.3. Consider the Q-algebra $(\mathtt{X}, *, 0_{\mathtt{X}})$ in Table 9 and define the fuzzy subset A of $\mathtt{X}$ as shown in Table 10 and KM-fuzzy metric

$$\mathtt{L}(x, y, t) = \begin{cases} 0, & \text{if } t = 0, \\ \mathtt{L}(x, y, t) = \frac{t}{t + |x - y|}, & \text{if } t > 0. \end{cases}$$

If $(\xi, \mathtt{L}, \mathtt{T}_{Luk})$ is a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$, then $t \geq \frac{2}{9}$, since $x * x = 0_{\mathtt{X}}$ and $\mathtt{L}(x, x, t) = 1$, we understood

$$\bigvee_{x, y \in \mathtt{X}} \Big(\mathtt{T}_{Luk}(\xi(x * y), \mathtt{T}_{Luk}(\xi(x), \xi(y))\Big) \leq \bigwedge_{x, y \in \mathtt{X}} \frac{t}{|x - y| + t}$$

implies $\frac{2}{10} \leq \frac{t}{1+t}$, i.e., $t \geq \frac{2}{9}$.

We see that ξ is a not a strong fuzzy sub-Q-algebra, while $\mathtt{T}_{Luk}$ is a nilpotent triangular norm.

5. Operations on KM-fuzzy Metric Q-subalgebra

In this section, we extend the KM-fuzzy metric Q-subalgebras based on the some operations such as intersection, product and fuzzy homomorphism.

Let $f : \mathtt{X} \to \mathtt{Y}$ be a map, ξ be a fuzzy subset of $\mathtt{X}$ and $\mathtt{L}$ be a fuzzy metric on $\mathtt{X}$. We recall that $f(\xi)(y) = \bigvee\{\xi(x) \mid f(x) = y\}$ and define $f(\mathtt{L})(y, y', t) = \bigvee\{\mathtt{L}(x, x', t) \mid f(x) = y, f(x') = y'\}$ and have the following results.

Theorem 5.1. *Let $f : (\mathtt{X}, *_{\mathtt{X}}) \to (\mathtt{Y}, *_{\mathtt{Y}})$ be a homomorphism of Q-subalgebras. If $(\xi, \mathtt{L}, \mathtt{T})$ is a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$, then $f(\xi, \mathtt{L}, \mathtt{T})$ is a KM-fuzzy metric Q-subalgebra of $\mathtt{Y}$.*

Proof. Since ξ is a fuzzy subset of $\mathtt{X}$, we find $f(\xi)$ is a fuzzy subset of $\mathtt{Y}$. Moreover, $f : \mathtt{X} \to \mathtt{Y}$ is a map, so for each $x, x' \in \mathtt{X}, y, y' \in \mathtt{Y}$ and $t, s \geq 0$, we have:

(*i*) $f(\mathtt{L})(y, y', 0) = \bigvee\{\mathtt{L}(x, x', 0) \mid f(x) = y, f(x') = y'\} = 0$;

(*ii*) $f(\mathtt{L})(y, y, t) = \bigvee\{\mathtt{L}(x, x, t) \mid f(x) = y\} = 1$ for all $t > 0$;

(*iii*)
$$\begin{aligned} f(\mathtt{L})(y, y', t) &= \bigvee\{\mathtt{L}(x, x', t) \mid f(x) = y, f(x') = y'\} \\ =&\bigvee\{\mathtt{L}(x', x, t) \mid f(x) = y, f(x') = y'\} = f(\mathtt{L}(y', y, t)); \end{aligned}$$

(*iv*)
$$\begin{aligned} \mathtt{T}(f(\mathtt{L}(y, y', t)), f(\mathtt{L}(y', y'', s))) =&\mathtt{T}\Big(\bigvee\{\mathtt{L}(x, x', t) \mid f(x) = y, f(x') = y'\}, \\ &\bigvee\{\mathtt{L}(x', x'', s) \mid f(x') = y', f(x'') = y''\}\Big) \\ =&\bigvee\Big(\{\mathtt{T}(\mathtt{L}(x, x', t), \mathtt{L}(x', x'', s)) \mid f(x) = y, f(x') = y', \\ &f(x'') = y''\}\Big) \\ \leq&\bigvee\{\mathtt{L}(x, x'', t + s) \mid f(x) \\ =&y, f(x'') = y''\} = f(\mathtt{L}(y, y'', t + s)). \end{aligned}$$

(*v*) Since $\mathtt{L}(x, y, -) : \mathbb{R}^{\geq 0} \to [0, 1]$ is a left-continuous map, we obtain $\bigvee\{\mathtt{L}(x, x', t) \mid f(x) = y, f(x') = y'\} = f(\mathtt{L})(y, y', -) : \mathbb{R}^{\geq 0} \to [0, 1]$ is a left-continuous map.

(*vi*) Let $t \to +\infty$. Since $\mathtt{L}(x, x', t) \to 1$, when $t \to +\infty$, we get $f(\mathtt{L}(y, y', t)) = \bigvee\{\mathtt{L}(x, x', t) \mid f(x) = y, f(x') = y'\} \to 1$.

(*vii*) Let for all $t > 0$, $f(\mathtt{L}(y, y', t)) = \bigvee\{\mathtt{L}(x, x', t) \mid f(x) = y, f(x') = y'\} = 1$. Then, for all $t > 0$, $\mathtt{L}(x, x', t) = 1$. Since for all $t > 0$, $\mathtt{L}(x, x', t) = 1$ implies that $x = x'$, get $y = f(x) = f(x') = y$.

$f(\mathtt{L})$ is a fuzzy metric of $\mathtt{Y}$. □

Let $f : \mathtt{X} \to \mathtt{Y}$ be a map, ν be a fuzzy subset of $\mathtt{Y}$ and $\mathtt{L}$ be a fuzzy metric on $\mathtt{Y}$. We recall that $f^{-1}(\nu)(x) = \nu(f(x))$ and define $f^{-1}(\mathtt{L})(x, x', t) = \mathtt{L}(f(x), f(x'), t)$ and have the following results.

Theorem 5.2. *Let* $f : (\mathtt{X}, *_{\mathtt{X}}) \to (\mathtt{Y}, *_{\mathtt{Y}})$ *be a homomorphism of* Q*-subalgebras. If* $(\nu, \mathtt{L}, \mathtt{T})$ *is a* KM*-fuzzy metric* Q*-subalgebra of* $\mathtt{Y}$*, then* $f^{-1}(\nu, \mathtt{L}, \mathtt{T})$ *is a* KM*-fuzzy metric* Q*-subalgebra of* $\mathtt{X}$*.*

Proof. Since ν is a fuzzy subset of $\mathtt{Y}$, we understood $f^{-1}(\nu)$ is a fuzzy subset of $\mathtt{X}$. Moreover, $f : \mathtt{X} \to \mathtt{Y}$ is a map, so for each $x, x' \in \mathtt{X}, y, y' \in \mathtt{Y}$ and $t, s \geq 0$, we have:

(*i*) $f^{-1}(\mathtt{L})(x, x', 0) = \mathtt{L}(f(x), f(x'), 0) = 0$,
(*ii*) $f^{-1}(\mathtt{L})(x, x, t) = \mathtt{L}(f(x), f(x), t) = 1$ for all $t > 0$,
(*iii*) $f^{-1}(\mathtt{L})(x, x', t) = \mathtt{L}(f(x), f(x'), t) = \mathtt{L}(f(x'), f(x), t) = f^{-1}(\mathtt{L}(x', x, t))$,
(*iv*)

$$\begin{aligned}\mathtt{T}(f^{-1}(\mathtt{L})(x, x', t)), f^{-1}(\mathtt{L})(x', x'', s))) =& \mathtt{T}(\mathtt{L}(f(x), f(x'), t), \mathtt{L}(f(x'), f(x''), s)) \\ \leq& \mathtt{L}(f(x), f(x''), t + s) = f^{-1}(\mathtt{L}(x, x'', t + s)).\end{aligned}$$

(*v*) Since $\mathtt{L}(f(x), f(x'), -) : \mathbb{R}^{\geq 0} \to [0, 1]$ is a left-continuous map, we find that $f^{-1}(\mathtt{L})(x, x', -) = \mathtt{L}(f(x), f(x'), -) : \mathbb{R}^{\geq 0} \to [0, 1]$ is a left-continuous map.
(*vi*) Let $t \to +\infty$. Since $\mathtt{L}(f(x), f(x'), t) \to 1$, when $t \to +\infty$, we get $f^{-1}(\mathtt{L})(x, x', t)) \to 1$.
(*vii*) Let for all $t > 0$, $f^{-1}(\mathtt{L})(x, x', t) = 1$. Then, for all $t > 0$, $\mathtt{L}(f(x), f(x'), t) = 1$. Since for all $t > 0$, $\mathtt{L}(f(x), f(x'), t) = 1$ implies that $f(x) = f(x')$, and f is one to one, get $x = x'$.

$f(\mathtt{L})$ is a fuzzy metric of $\mathtt{Y}$. □

Let $(\mathtt{X}, \mathtt{L}_1, \mathtt{T}_1)$ and $(\mathtt{X}, \mathtt{L}_2, \mathtt{T}_2)$ be KM-fuzzy metric spaces. Define $\mathtt{L}_1 \cap \mathtt{L}_2 : \mathtt{X}^2 \times \mathbb{R}^{\geq 0} \to \mathtt{I}$, by $(\mathtt{L}_1 \cap \mathtt{L}_2)(x, y, t) = \min\{\mathtt{L}_1(x, y, t), \mathtt{L}_2(x, y, t)\}$ and $\mathtt{T}_1 \cap \mathtt{T}_2 : \mathtt{I}^2 \to [0, 1]$ by $(\mathtt{T}_1 \cap \mathtt{T}_2)(x, y) = \min\{\mathtt{T}_1(x, y), \mathtt{T}_2(x, y)\}$.

Theorem 5.3. *Let* $(\mathtt{X}, \mathtt{L}_1, \mathtt{T}_1)$ *and* $(\mathtt{X}, \mathtt{L}_2, \mathtt{T}_2)$ *be* KM*-fuzzy metric spaces. Then,* $(\mathtt{X}, \mathtt{L}_1 \cap \mathtt{L}_2, \mathtt{T}_1 \cap \mathtt{T}_2)$ *is a* KM*-fuzzy metric space.*

Proof. Let $x, y \in \mathtt{X}$ and $t, s \in \mathbb{R}^{\geq 0}$. Then, the following hold.
(*i*) $(\mathtt{L}_1 \cap \mathtt{L}_2)(x, y, 0) = \min\{\mathtt{L}_1(x, y, 0), \mathtt{L}_2(x, y, 0)\} = 0$.
(*ii*) For all $t > 0, (\mathtt{L}_1 \cap \mathtt{L}_2)(x, x, t) = \min\{\mathtt{L}_1(x, x, t), \mathtt{L}_2(x, x, t)\} = 1$.
(*iii*) $(\mathtt{L}_1 \cap \mathtt{L}_2)(x, y, t) = \min\{\mathtt{L}_1(x, y, t), \mathtt{L}_2(x, y, t)\} = \min\{\mathtt{L}_1(y, x, t), \mathtt{L}_2(y, x, t)\} = (\mathtt{L}_1 \cap \mathtt{L}_2)(y, x, t)$.

(iv)

$$\begin{aligned}
&(\mathtt{T}_1\cap\mathtt{T}_2)((\mathtt{L}_1\cap\mathtt{L}_2)(x,y,t),(\mathtt{L}_1\cap\mathtt{L}_2)(y,z,s))\\
=&\min\Big\{\mathtt{T}_1\Big((\mathtt{L}_1\cap\mathtt{L}_2)(x,y,t),(\mathtt{L}_1\cap\mathtt{L}_2)(y,z,s)\Big),\mathtt{T}_2\Big((\mathtt{L}_1\cap\mathtt{L}_2)(x,y,t),(\mathtt{L}_1\cap\mathtt{L}_2)(y,z,s)\Big)\Big\}\\
=&\min\Big\{\mathtt{T}_1\Big(\min\{\mathtt{L}_1(x,y,t),\mathtt{L}_2(x,y,t)\},\min\{\mathtt{L}_1(y,z,s),\mathtt{L}_2(y,z,s)\}\Big),\\
&\mathtt{T}_2\Big(\min\{\mathtt{L}_1(x,y,t),\mathtt{L}_2(x,y,t)\},\min\{\mathtt{L}_1(y,z,s),\mathtt{L}_2(y,z,s)\}\Big)\Big\}\\
=&\min\Big\{\mathtt{T}_1\Big(\min\{\mathtt{L}_1(x,y,t),\mathtt{L}_1(y,z,s)\},\min\{\mathtt{L}_2(x,y,t),\mathtt{L}_2(y,z,s)\}\Big),\\
&\mathtt{T}_2\Big(\min\{\mathtt{L}_1(x,y,t),\mathtt{L}_1(y,z,s)\},\min\{\mathtt{L}_2(x,y,t),\mathtt{L}_2(y,z,s)\}\Big)\Big\}\\
\leq&\min\Big\{\mathtt{L}_1(x,z,t+s),\mathtt{L}_2(x,z,t+s)\Big\}\\
=&(\mathtt{L}_1\cap\mathtt{L}_2)(x,z,t+s).
\end{aligned}$$

(v) Since $\mathtt{L}_1(x,y,-):\mathbb{R}^{\geq 0}\to[0,1]$ and $\mathtt{L}_1(x,y,-):\mathbb{R}^{\geq 0}\to[0,1]$ are left-continuous map, we understood $(\mathtt{L}_1\cap\mathtt{L}_2)(x,y,-):\mathbb{R}^{\geq 0}\to[0,1]$, that $(\mathtt{L}_1\cap\mathtt{L}_2)(x,y,-)=\min\{\mathtt{L}_1(x,y,-),\mathtt{L}_2(x,y,-)\}$, is a left-continuous map.

(vi) Since $\mathtt{L}_1(x,y,t)\to 1$ and $\mathtt{L}_2(x,y,t)\to 1$, when $t\to+\infty$, we understood $(\mathtt{L}_1\cap\mathtt{L}_2)(x,y,t)=\min\{\mathtt{L}_1(x,y,t),\mathtt{L}_2(x,y,t)\}\to\min\{1,1\}=1$, when $t\to+\infty$.

(vii) Since for all $t>0$, $\mathtt{L}(x,y,t)=1$, implies that $x=y$, for all $t>0$, understood $(\mathtt{L}_1\cap\mathtt{L}_2)(x,y,t)=\min\{\mathtt{L}_1(x,y,t),\mathtt{L}_2(x,y,t)\}=1$, implies that $\mathtt{L}_1(x,y,t)=1$ and $\mathtt{L}_2(x,y,t)=1$ and so $x=y$. Hence, $(\mathtt{X},\mathtt{L}_1\cap\mathtt{L}_2,\mathtt{T}_1\cap\mathtt{T}_2)$ is a KM-fuzzy metric space. ☐

Theorem 5.4. *Let* $(\xi_1,\mathtt{L}_1,\mathtt{T}_1)$ *and* $(\xi_1,\mathtt{L}_1,\mathtt{T}_1)$ *be KM-fuzzy metric Q-subalgebra of* $\mathtt{X}$ *and* $(\mathtt{X},*)$ *be a Q-algebra. Then,* $(\xi_1\cap\xi_2,\mathtt{L}_1\cap\mathtt{L}_2,\mathtt{T}_1\cap\mathtt{T}_2)$ *is a KM-fuzzy metric Q-subalgebra of* $\mathtt{X}$.

Proof. By Theorem 5.3, $(\xi_1\cap\xi_2,\mathtt{L}_1\cap\mathtt{L}_2,\mathtt{T}_1\cap\mathtt{T}_2)$ is a KM-fuzzy metric Q-subset of $\mathtt{X}$. Let $x,y\in\mathtt{X}$. Since $(\xi_1,\mathtt{L}_1,\mathtt{T}_1)$ and $(\xi_1,\mathtt{L}_1,\mathtt{T}_1)$ are KM-fuzzy metric Q-subalgebra of $\mathtt{X}$,

$$\begin{aligned}
&(\mathtt{T}_1\cap\mathtt{T}_2)((\xi_1\cap\xi_2)(x*y),(\mathtt{T}_1\cap\mathtt{T}_2)((\xi_1\cap\xi_2)(x),(\xi_1\cap\xi_2)(y)))\\
=&\min\Big\{\mathtt{T}_1((\xi_1\cap\xi_2)(x*y),(\mathtt{T}_1\cap\mathtt{T}_2)((\xi_1\cap\xi_2)(x),(\xi_1\cap\xi_2)(y))),\\
&\mathtt{T}_2((\xi_1\cap\xi_2)(x*y),(\mathtt{T}_1\cap\mathtt{T}_2)((\xi_1\cap\xi_2)(x),(\xi_1\cap\xi_2)(y)))\Big\}\\
\leq&\min\Big\{\Big(\mathtt{T}_1(\xi_1(x*y),\mathtt{T}_1(\xi_1(x),\xi_1(y)))\Big),(\mathtt{T}_2(\xi_2(x*y),\mathtt{T}_2(\xi_2(x),\xi_2(y))))\Big\}\\
\leq&\mathtt{L}_1(x,y,t)\wedge\mathtt{L}_2(x,y,t).
\end{aligned}$$

Hence, $(\xi_1\cap\xi_2,\mathtt{L}_1\cap\mathtt{L}_2,\mathtt{T}_1\cap\mathtt{T}_2)$ is a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$. ☐

Example 5.1. Consider the Q-algebra in Example 3.1 and is defined in Table 1. Then, $(\xi_1,\mathtt{L}_1,\mathtt{T}_{Prod})$ and $(\xi_2,\mathtt{L}_2,\mathtt{T}_{Hom})$ are KM-fuzzy metric, which fuzzy subsets A and B are defined in Tables 11 and 12, fuzzy metric $\mathtt{L}_1$ and fuzzy metric $\mathtt{L}_2$ are defined in

Tables 13 and 14, respectively. Then, for instance, by $t = 2$, $(\xi_1 \cap \xi_2, \mathtt{L}_1 \cap \mathtt{L}_2, \mathtt{T}_1 \cap \mathtt{T}_2)$ is a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$ as Tables 15 and 16.

	$0_{\mathtt{X}}$	a	b	c
ξ_1:	0.9	0.8	0.6	0.6

TABLE 11. Fuzzy subset A

	$0_{\mathtt{X}}$	a	b	c
ξ_2:	0.8	0.7	0.5	0.5

TABLE 12. Fuzzy subset B

$\mathtt{L}_1$	$0_{\mathtt{X}}$	a	b	c
$0_{\mathtt{X}}$	1	$\frac{t}{t+1}$	$\frac{t}{t+2}$	$\frac{t}{t+3}$
a	$\frac{t}{t+1}$	1	$\frac{t}{t+2}$	$\frac{t}{t+3}$
b	$\frac{t}{t+2}$	$\frac{t}{t+2}$	1	$\frac{t}{t+3}$
c	$\frac{t}{t+3}$	$\frac{t}{t+3}$	$\frac{t}{t+3}$	1

TABLE 13. $\mathtt{L}_1$

Let $(\mathtt{X}, \mathtt{L}_1, \mathtt{T})$ and $(\mathtt{X}, \mathtt{L}_2, \mathtt{T})$ be KM-fuzzy metric spaces. Define $\mathtt{L}_1 \times \mathtt{L}_2 : \mathtt{X}^2 \times \mathtt{X}^2 \times \mathbb{R}^{\geq 0} \to \mathtt{I}$, by $(\mathtt{L}_1 \times \mathtt{L}_2)((x, x'), (y, y'), t)) = \mathtt{T}(\mathtt{L}_1(x, y, t), \mathtt{L}_2(x', y', t))$. Moreover, for fuzzy subsets ξ_1, ξ_2 of $\mathtt{X}$, define $\xi_1 \times \xi_2 : \mathtt{X} \times \mathtt{X} \to \mathtt{I}$ by $(\xi_1 \times \xi_2)(x, y) = \xi_1(x) \wedge \xi_2(y)$. It is clear that $\xi_1 \times \xi_2$ is a fuzzy subset of $\mathtt{X} \times \mathtt{X}$ and have the following results.

Theorem 5.5. *Let $(\xi_1, \mathtt{L}_1, \mathtt{T}_{\min})$ and $(\xi_1, \mathtt{L}_1, \mathtt{T}_{\min})$ be KM-fuzzy metric Q-subalgebras of $\mathtt{X}$. Then, $(\xi_1 \times \xi_2, \mathtt{L}_1 \times \mathtt{L}_2, \mathtt{T}_{\min})$ is a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$*

Proof. (*i*) $(\mathtt{L}_1 \times \mathtt{L}_2)((x, x'), (y, y'), 0)) = \mathtt{T}(\mathtt{L}_1(x, y, 0), \mathtt{L}_2(x', y', 0)) = \mathtt{T}(0, 0) = 0$.

(*ii*) $(\mathtt{L}_1 \times \mathtt{L}_2)((x, x'), (x, x'), t)) = \mathtt{T}(\mathtt{L}_1(x, x, t), \mathtt{L}_2(x', x', t)) = \mathtt{T}(1, 1) = 1$ for all $t > 0$.

(*iii*)

$$\begin{aligned}(\mathtt{L}_1 \times \mathtt{L}_2)((x, x'), (y, y'), t)) &= \mathtt{T}(\mathtt{L}_1(x, y, t), \mathtt{L}_2(x', y', t)) \\ &= \mathtt{T}(\mathtt{L}_1(y, x, t), \mathtt{L}_2(y', x', t)) \\ &= (\mathtt{L}_1 \times \mathtt{L}_2)((y, y'), (x, x'), t)).\end{aligned}$$

$\mathtt{L}_2$	$0_{\mathtt{X}}$	a	b	c
$0_{\mathtt{X}}$	1	$e^{-\frac{1}{t}}$	$e^{-\frac{2}{t}}$	$e^{-\frac{3}{t}}$
a	$e^{-\frac{1}{t}}$	1	$e^{-\frac{1}{t}}$	$e^{-\frac{2}{t}}$
b	$e^{-\frac{2}{t}}$	$e^{-\frac{1}{t}}$	1	$e^{-\frac{1}{t}}$
c	$e^{-\frac{3}{t}}$	$e^{-\frac{2}{t}}$	$e^{-\frac{1}{t}}$	1

TABLE 14. $\mathtt{L}_2$

	$0_{\mathtt{X}}$	a	b	c
$\xi_1 \cap \xi_2$:	0.8	0.7	0.5	0.5

TABLE 15. Fuzzy subset $A \cap B$

$\mathtt{L}_1 \cap \mathtt{L}_2$	$0_{\mathtt{X}}$	a	b	c
$0_{\mathtt{X}}$	1	$e^{-\frac{1}{2}}$	$e^{-\frac{2}{2}}$	$e^{-\frac{3}{2}}$
a	$e^{-\frac{1}{2}}$	1	$\frac{1}{2}$	$e^{-\frac{2}{t}}$
b	$e^{-\frac{2}{t}}$	$\frac{1}{2}$	1	$\frac{2}{5}$
c	$e^{-\frac{3}{t}}$	$e^{-\frac{2}{t}}$	$\frac{2}{5}$	1

TABLE 16. $\mathtt{L}_1 \cap \mathtt{L}_2$

(*iv*)

$$\begin{aligned}
&\mathtt{T}((\mathtt{L}_1 \times \mathtt{L}_2)((x,x'),(y,y'),t)),(\mathtt{L}_1 \times \mathtt{L}_2)((y,y'),(z,z'),s)))\\
=&\mathtt{T}(\mathtt{T}(\mathtt{L}_1(x,y,t),\mathtt{L}_2(x',y',t)),\mathtt{T}(\mathtt{L}_1(y,z,s),\mathtt{L}_2(y',z',s)))\\
=&\mathtt{T}(\mathtt{T}(\mathtt{L}_1(x,y,t),\mathtt{L}_1(y,z,s)),\mathtt{T}(\mathtt{L}_2(x',y',t),\mathtt{L}_2(y',z',s)))\\
\leq&\mathtt{T}(\mathtt{L}_1(x,z,t+s),\mathtt{L}_2(x',z',t+s))\\
=&(\mathtt{L}_1 \times \mathtt{L}_2)((x,x'),(z,z'),t+s)).
\end{aligned}$$

(*v*) Since $\mathtt{L}_1(x,y,-) : \mathbb{R}^{\geq 0} \to [0,1]$ and $\mathtt{L}_2(x',y',-) : \mathbb{R}^{\geq 0} \to [0,1]$ are left-continuous maps, get $(\mathtt{L}_1 \times \mathtt{L}_2)((x,x'),(y,y'),-)) = \mathtt{T}(\mathtt{L}_1(x,y,-),\mathtt{L}_2(x',y',-))$ is a left-continuous map.

(*vi*) Let $t \to +\infty$. Since $\mathtt{L}_1(x,y,t) \to 1$ and $\mathtt{L}_2(x',y',t) \to 1$, we get $(\mathtt{L}_1 \times \mathtt{L}_2)((x,x'),(y,y'),t)) = \mathtt{T}(\mathtt{L}_1(x,y,t),\mathtt{L}_2(x',y',t)) \to \mathtt{T}(1,1) = 1$.

(*vii*) Let $t > 0$, $(\mathtt{L}_1 \times \mathtt{L}_2)((x,x'),(y,y'),t)) = 1$. Then, $\mathtt{T}(\mathtt{L}_1(x,y,t),\mathtt{L}_2(x',y',t)) = 1$. Since $1 = \mathtt{T}(1,1)$, we get $\mathtt{L}_1(x,y,t) = 1$ and $\mathtt{L}_2(x',y',t) = 1$. Thus, $x = y$ and $x' = y'$ and so, $(x,x'),(y,y')$.

Now, $(\xi_1,\mathtt{L}_1,\mathtt{T})$ and $(\xi_1,\mathtt{L}_1,\mathtt{T})$ are KM-fuzzy metric Q-subalgebras of $\mathtt{X}$. Then,

$$\begin{aligned}
&\mathtt{T}((\xi_1 \times \xi_2)((x,x') * (y,y')),\mathtt{T}((\xi_1 \times \xi_2)(x,x'),(\xi_1 \times \xi_2)(y,y'))\\
=&\mathtt{T}(\xi_1(x*y) \wedge \xi_2(x'*y'),\mathtt{T}(\xi_1(x) \wedge \xi_2(x'),\xi_1(y) \wedge \xi_2(y'))\\
\leq&\mathtt{T}(\xi_1(x*y),\mathtt{T}(\xi_1(x),\xi_1(y)) \wedge \mathtt{T}(\xi_2(x'*y'),\mathtt{T}(\xi_2(x'),\xi_2(y'))\\
\leq&\mathtt{L}_1(x,y,t) \wedge \mathtt{L}_2(x',y',t) = \mathtt{T}(\mathtt{L}_1(x,y,t),\mathtt{L}_2(x',y',t)) = (\mathtt{L}_1 \times \mathtt{L}_2)((x,x'),(y,y'),t).
\end{aligned}$$

□

Based Theorem 4.2 and Theorem 5.5, we have the following corollary.

Corollary 5.1. *Let* $(\xi_1,\mathtt{L}_1,\mathtt{T}_{\min})$ *and* $(\xi_1,\mathtt{L}_1,\mathtt{T}_{\min})$ *be* KM*-fuzzy metric* Q*-subalgebras of* $\mathtt{X}$ *and* $\mathtt{T}$ *be any arbitrary triangular norm. Then,* $(\xi_1 \times \xi_2,\mathtt{L}_1 \times \mathtt{L}_2,\mathtt{T})$ *is a* KM*-fuzzy metric* Q*-subalgebra of* $\mathtt{X}$.

	$0_{\mathtt{X}}$	x	y	z
ξ:	0.111	0.11	0.1	0.1111

Table 17. A

$*$	$0_{\mathtt{X}}$	x	y	z
0	$0_{\mathtt{X}}$	x	y	z
x	x	$0_{\mathtt{X}}$	z	y
y	y	z	$0_{\mathtt{X}}$	x
z	z	y	x	$0_{\mathtt{X}}$

Table 18. $(\mathtt{X}, *, 0)$

$\mathtt{L}$	$0_{\mathtt{X}}$	a	b	c
$0_{\mathtt{X}}$	1	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{1}{2}$
a	$\frac{3}{4}$	1	$\frac{3}{5}$	$\frac{1}{2}$
b	$\frac{3}{5}$	$\frac{3}{5}$	1	$\frac{1}{2}$
c	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1

Table 19. $\mathtt{L}$

Let $(\xi, \mathtt{L}, \mathtt{T})$ be a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$ and $\alpha \in \mathtt{I}$. Define $\xi^{\alpha} = \{x \in \mathtt{X} \mid \xi(x) \geq \alpha\}$ and $\mathtt{L}^{\alpha} = \{(x, y) \in \mathtt{X}^2, t \in \mathbb{R}^{\geq 0} \mid \mathtt{L}(x, y, t) \geq \alpha\}$.

Theorem 5.6. *Let $(\xi, \mathtt{L}, \mathtt{T})$ be a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$ and $\alpha \in \mathtt{I}$.*

- *(i) ξ is a fuzzy sub-Q-algebra of $\mathtt{X}$ if and only if $(\xi^{\alpha}, *)$ is a Q-subalgebra of $\mathtt{X}$.*
- *(ii) If ξ is a fuzzy sub-Q-algebra of $\mathtt{X}$, then $\mathtt{T}(\alpha, \mathtt{T}(\alpha, \alpha)) \leq \mathtt{L}$.*

Proof. (i) Since for any $x \in \mathtt{X}$, $\xi(0_{\mathtt{X}}) = \xi(x * x) \geq \xi(x)$, we conclude that for any $\alpha \in \mathtt{I}$, $\xi^{\alpha} \neq \emptyset$, because ξ is a fuzzy sub-Q-algebra of $\mathtt{X}$. Assume that $x, y \in \xi^{\alpha}$. Then, $\xi(x) \geq \alpha$ and $\xi(y) \geq \alpha$ and so $\xi(x * y) \geq \xi(x) \wedge \xi(y) = \alpha$. It follows that $x * y \in \xi^{\alpha}$. Hence, $(\xi^{\alpha}, *)$ is a Q-algebra.

Conversely, let $(\xi^{\alpha}, *)$ is a Q-subalgebra of $\mathtt{X}$, whence $\alpha \in \mathtt{I}$. Assume that $x, y \in \mathtt{X}$ and $\xi(x) \wedge \xi(y) = \alpha$. It follows that $\xi(x) \geq \alpha$, $\xi(y) \geq \alpha$ and so $x, y \in \xi^{\alpha}$. Since $(\xi^{\alpha}, *)$ is a Q-subalgebra of $\mathtt{X}$, get $x * y \in \xi^{\alpha}$ and so $\xi(x * y) \geq \alpha = \xi(x) \wedge \xi(y)$. Hence, ξ is a fuzzy sub-Q-algebra of $\mathtt{X}$.

(ii) Since $\mathtt{L}(x, x, t) = 1$, we understood $\mathtt{L}^{\alpha} \neq \emptyset$. Let $x, x', y, y' \in \mathtt{X}$. Then,

$$\mathtt{T}(\xi(x) \wedge \xi(x), \mathtt{T}(\xi(x) \wedge \xi(x)) \leq \mathtt{T}(\xi(x * x), \mathtt{T}(\xi(x), \xi(x))) \leq \mathtt{L}(x, x, t).$$

Now, $\xi(x) = \alpha$ and so, $\mathtt{T}(\alpha, \mathtt{T}(\alpha, \alpha)) \leq \mathtt{L}$. □

Example 5.2, shows that the converse of Theorem 5.6 (ii), is not necessarily true.

Example 5.2. Consider $(\mathtt{X}, *)$ in Table 18. Then, $(\xi, \mathtt{L}, \mathtt{T}_{\min})$ is a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$ as Tables 17, 19. Clearly, for any $x \in \mathtt{X}, \mathtt{T}_{\min}(\xi(x), \mathtt{T}_{\min}(\xi(x), \xi(x))) =$

$\xi(x) \leq \mathtt{L}(x, y, t)$. Since $= 0.111 = \xi(y) = \xi(x * z) \leq \xi(x) \wedge \xi(z)$, we have ξ is not a fuzzy sub-Q-algebra of $\mathtt{X}$.

Theorem 5.7. *Let $(\xi, \mathtt{L}, \mathtt{T}_{\min})$ be a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$, ξ be a fuzzy sub-Q-algebra of $\mathtt{X}$ and $\alpha \in \mathtt{I}$. Then, $\mathtt{L}^\alpha$ contains at least a Q-subalgebra of $\mathtt{X}^2$.*

Proof. Let $(x, y) \in \xi^\alpha \times \xi^\alpha$. Applying Theorem 5.6,

$$\alpha = \mathtt{T}_{\min}(\alpha, \alpha) = \mathtt{T}_{\min}(\alpha, \mathtt{T}_{\min}(\alpha, \alpha)) \leq \mathtt{L}(x, y, t),$$

and so $(x, y) \in \mathtt{L}^\alpha$ and it follows that $\xi^\alpha \times \xi^\alpha \subseteq \mathtt{L}^\alpha$. Using Theorem 3.3, $\xi^\alpha \times \xi^\alpha$ is a Q-algebra, since ξ^α is a Q-algebra based on Theorem 5.6. □

Example 5.3, shows that the converse of Theorem 5.7, is not necessarily true.

Example 5.3. Consider the KM-fuzzy metric Q-subalgebra of $\mathtt{X}$ in Example 5.2. Clearly $\mathtt{L}^{\frac{1}{2}}$ is a Q-subalgebra of $\mathtt{X}^2$, while ξ is not a fuzzy sub-Q-algebra of $\mathtt{X}$.

Corollary 5.2. *Let $(\xi, \mathtt{L}, \mathtt{T}_{\min})$ be a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$, ξ be a fuzzy sub-Q-algebra of $\mathtt{X}$ and $\alpha \in \mathtt{I}$. Then, $\mathtt{L}^\alpha$ is a Q-subalgebra of $\mathtt{X}^2$.*

Proof. Let $(x, y), (x', y') \in \mathtt{L}^\alpha$. Based Theorem 5.7, $\alpha \leq \mathtt{L}(x * x', y * y', t)$ and so $(x, y) *_{\mathtt{X}^2} (x', y') = (x *_X x, y *_X y') \in \mathtt{L}^\alpha$. Therefore, $\mathtt{L}^\alpha$ is a Q-subalgebra of $\mathtt{X}^2$. □

Corollary 5.3. *Let $(\xi, \mathtt{L}, \mathtt{T})$ be a KM-fuzzy metric Q-subalgebra of $\mathtt{X}$, ξ be a fuzzy sub-Q-algebra of $\mathtt{X}$ and $\alpha \in \mathtt{I}$. If $\mathtt{T} \neq \mathtt{T}_{\min}$ and $\mathtt{L}^\alpha$ is a Q-subalgebra of $\mathtt{X}^2$, then $\alpha = 0$ or $\alpha = 1$.*

Proof. Let $\mathtt{T} \neq \mathtt{T}_{\min}$ and $\mathtt{L}^\alpha$ be a Q-subalgebra of $\mathtt{X}^2$. Since $\mathtt{T}(\alpha, \mathtt{T}(\alpha, \alpha)) \leq \alpha$, we understood $\mathtt{L}(x, y, t) \geq \alpha$ if and only if $\mathtt{T}(\alpha, \mathtt{T}(\alpha, \alpha)) = \alpha$ if and only if $\alpha = 0$ or $\alpha = 1$. □

Example 5.4. Let $x, y \in \mathtt{I}$. Consider the Dombi t-norm as

$$\mathtt{T}_{Dom}(x, y) = \frac{1}{1 + \left(\left(\frac{1-x}{x}\right)^r + \left(\frac{1-y}{y}\right)^r\right)^{\frac{1}{r}}},$$

which $r > 0$. Since

$$\begin{aligned}\mathtt{T}_{Dom}(\alpha, \alpha) =& \frac{1}{1 + \left(\left(\frac{1-\alpha}{\alpha}\right)^r + \left(\frac{1-\alpha}{\alpha}\right)^r\right)^{\frac{1}{r}}} = \frac{1}{1 + \left(2\left(\frac{1-\alpha}{\alpha}\right)^r\right)^{\frac{1}{r}}} = \frac{1}{1 + 2^{\frac{1}{r}}\left(\frac{1-\alpha}{\alpha}\right)} \\ =& \frac{\alpha}{2^{\frac{1}{r}} + \alpha(1 - 2^{\frac{1}{r}})},\end{aligned}$$

we get

$$\begin{aligned}
\mathtt{T}_{Dom}(\alpha, \mathtt{T}_{Dom}(\alpha,\alpha)) =& \mathtt{T}_{Dom}\left(\alpha, \frac{\alpha}{2^{\frac{1}{r}}+\alpha(1-2^{\frac{1}{r}})}\right)\\
=& \frac{1}{1+\left(\left(\frac{1-\alpha}{\alpha}\right)^r+\left(\dfrac{1-\dfrac{\alpha}{2^{\frac{1}{r}}+\alpha(1-2^{\frac{1}{r}})}}{\dfrac{\alpha}{2^{\frac{1}{r}}+\alpha(1-2^{\frac{1}{r}})}}\right)^r\right)^{\frac{1}{r}}}\\
=& \frac{1}{1+\left(\left(\frac{1-\alpha}{\alpha}\right)^r+\left(\frac{2^{\frac{1}{r}}(1-\alpha)}{\alpha}\right)^r\right)^{\frac{1}{r}}} = \frac{1}{1+3^{\frac{1}{r}}\left(\frac{1-\alpha}{\alpha}\right)}\\
=& \frac{\alpha}{\alpha+3^{\frac{1}{r}}(1-\alpha)}.
\end{aligned}$$

It follows that $\mathtt{T}_{Dom}(\alpha, \mathtt{T}_{Dom}(\alpha,\alpha)) = \alpha$ if and only if $\alpha = 0$ or $\alpha = 1$, because of $3^{\frac{1}{r}} \neq 1$.

Theorem 5.8. *Let* $(\xi, \mathtt{L}, \mathtt{T})$ *be a strong* KM*-fuzzy metric* Q*-subalgebra of* $\mathtt{X}$*,* ξ *be a fuzzy sub-*Q*-algebra of* $\mathtt{X}$ *and* $\alpha \in \mathtt{I}$*. Then,* $\xi^\alpha \times \xi^\alpha$ *contains at least a* Q*-subalgebra of* $\mathtt{X}^2$*.*

Proof. Let $(x, y) \in \mathtt{L}^\alpha$. Then, $\mathtt{L}(x, y, t) \geq \alpha$ and so $\mathtt{T}(\xi(x * y), \mathtt{T}(\xi(x), \xi(y))) \geq \alpha$. Hence, $\xi(x * y) \geq \alpha, \xi(x) \geq \alpha$ and $\xi(y) \geq \alpha$, because $(\xi, \mathtt{L}, \mathtt{T})$ is a strong KM-fuzzy metric Q-subalgebra of $\mathtt{X}$. Thus, $(x, y) \in \xi^\alpha \times \xi^\alpha$ and so $\mathtt{L}^\alpha \subseteq \xi^\alpha \times \xi^\alpha$. Hence $\xi^\alpha \times \xi^\alpha$ contains at least a Q-subalgebra of $\mathtt{X}^2$. ☐

6. **Conclusion**

Combination and integration the notion of logic algebras and fuzzy subsets is an important in the modifying real problems in the world. We consider the Q-algebras and integrate them to KM-fuzzy pseudometric spaces such that for any given non-empty universal subsets, one can construct a metric space equipped to fuzzy problems. In this study, we established the notion of KM-fuzzy metric Q-algebra and want to applied these concepts to the applied problems in the future works. Also we investigated that countable Q-algebras are convertible to the well-ordered Q-algebras and showed there is a correspondence between the class of metrizable Q-algebra and countable Q-algebra. Indeed, we consider an arbitrary Q-algebra $(\mathtt{X}, *)$ and with respect to the its binary operation $*$, extracted a map $\mathtt{D} : \mathtt{X} \times \mathtt{X} \to \mathbb{R}^{\geq 0}$ such that $\mathtt{D}$ is related to the $*$ and $(\mathtt{X}, \mathtt{D})$ is a metric space. The main idea of this research is the extension of the class of Q-algebras to fuzzy metrizable Q-algebras and so are extracted the fundamental results for the logic algebras. We need to construct the topological Q-algebras and fuzzy topological Q-algebras in the future works and so in this study, converted Q-algebras to the metric Q-algebras. Indeed, the notions of metrizable Q-algebras and fuzzy metrizable Q-algebras are fundamental for the class of the metric spaces and the fuzzy metric spaces.

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