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SPECTRA OF THE LOWER TRIANGULAR MATRIX $\mathbb{B}(r_1, \dots, r_l; s_1, \dots, s_{l'})$ OVER c_0

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ABSTRACT. The infinite lower triangular matrix $\mathbb{B}(r_1, \dots, r_l; s_1, \dots, s_{l'})$ is considered over the sequence space c_0, where l and l' are positive integers. The diagonal and sub-diagonal entries of the matrix consist of the oscillatory sequences $r = (r_{k(\text{mod } l)+1})$ and $s = (s_{k(\text{mod } l')+1})$, respectively. The rest of the entries of the matrix are zero. It is shown that the matrix represents a bounded linear operator. Then the spectrum of the matrix is evaluated and partitioned into its fine structures: point spectrum, continuous spectrum, residual spectrum, etc. In particular, the spectra of the matrix $\mathbb{B}(r_1, \dots, r_4; s_1, \dots, s_6)$ are determined. Finally, an example is taken in support of the results.

1. INTRODUCTION

The study of the spectrum of a bounded linear operator has received much attention in recent years due to its wide application in functional analysis, classical quantum mechanics, etc. Let A be an infinite matrix that is bounded and linear in a Banach space U. Then many dynamical systems can be reformulated as the system of linear equations $Ax = \lambda x$, where λ is a complex number and x is a nonzero vector in U. The stability of this system can be explained by the spectrum of A. In this course, spectrum localization of an infinite matrix over a sequence space is viewed as an important problem by many authors [10, 14–16, 23, 26]. An extensive study of most of the research done in this direction can be found in the review articles [25] and [17].

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For a sequence $x = (x_k)$, the backward difference operator Δ is defined by $\Delta x = x_k - x_{k-1}$, where $x_{-1} = 0$. The matrix representation of this operator is as follows:

$$\Delta = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

In short, Δ is an infinite matrix whose diagonal entries and subdiagonal entries are the constant sequences $(1, 1, \dots)$ and $(-1, -1, \dots)$, respectively. Akhmedov and Başar [1] determined the spectral decompositions of this operator over bv_p $(1 \le p < \infty)$, whereas Altay and Başar [3] evaluated the spectra of the same operator over the spaces c and c_0. Altay and Başar [4] then considered the difference operator $B(r, s)$ over c_0 and c, which is a generalization of the operator Δ. The diagonal and subdiagonal entries of $B(r, s)$ contain the sequences $(r, r, \dots)$ and $(s, s, \dots)$, where r and $s \neq 0$ are real numbers. Furkan and Bilgiç studied $B(r, s)$ in the same direction over ℓ_p and bv_p in [6]. For more study, we refer [2, 7, 8, 12, 13, 18, 19, 22, 24] etc. Now if one considers the more generalized difference matrix whose diagonal and subdiagonal entries are the oscillatry sequences $(r_1, r_2, \dots, r_l, r_1, \dots)$ and $(s_1, s_2, \dots, s_{l'}, s_1, \dots)$, where l and l' are some positive integers, then the number of limit points of both the sequences will be different and it will be interesting to study the spectral property of the matrix.

In this paper, we have determined the spectra and fine spectra of the generalized difference matrix $\mathbb{B}(r_1, \dots, r_l; s_1, \dots, s_{l'})$ in which the diagonal entries consist of a sequence whose terms are oscillating between the points $r_1, r_2, \dots, r_l$ and the subdiagonal entries consist of an oscillatory sequence whose terms are oscillating between the points $s_1, s_2, \dots, s_{l'}$. Furthermore, the spectra and fine spectra of the matrix $\mathbb{B}(r_1, \dots, r_4; s_1, \dots, s_6)$ are also discussed.

2. Preliminaries

Let U and V be Banach spaces. Then the space of all bounded linear operators from U into V is denoted by $B(U, V)$. If $U = V$, then the space is denoted by $B(U)$. Let $L \in B(U)$ and U^* be dual of U. Then the adjoint $L^* \in B(U^*)$ of L is defined by $(L^* f)(t) = f(Lt)$ for all $f \in U^*$. Let $J : D(J) \to U$ be a linear operator defined over a subset $D(J)$ of U. Then the operator $(J - \lambda I)^{-1}$ is called the resolvent operator of J, where λ is a complex number and I is the identity operator.

A complex number λ is said to be a *regular value* [11] of a linear operator $J : D(J) \to U$ if and only if the operator $(J - \lambda I)^{-1}$ exists, bounded and is defined on a set which is dense in U. The set of all regular values of the linear operator J is called *resolvent set* and is denoted by $\rho(J)$. The complement $\sigma(J) = \mathbb{C} - \rho(J)$ is called the *spectrum* of J. The spectrum $\sigma(J)$ is further partitioned into the following three disjoint sets.

(a) $\sigma_p(J) = \{\lambda \in \mathbb{C} : (J - \lambda I)^{-1}$ does not exist$\}$. This set is called the *point spectrum* (*discrete spectrum*) of the operator J. The members of this set are called *eigenvalues* of J.

(b) $\sigma_c(T)$, which is defined as the set of all complex numbers λ for which $(J-\lambda I)^{-1}$ exists and defined on a set which is dense in U, but it is not a bounded operator in U. This set is called *continuous spectrum* of J.

(c) $\sigma_r(T)$, which contains all those complex numbers for which $(J - \lambda I)^{-1}$ exists, defined on a set which is not dense in U. This set is called the *residual spectrum* of J.

Let $R(J - \lambda I)$ denotes the range of the operator $J - \lambda I$. Goldberg [9] has classified the spectrum using the following six properties of $R(J - \lambda I)$ and $(J - \lambda I)^{-1}$:

(I) $R(J - \lambda I) = U$;
(II) $R(J - \lambda I) \neq U$ but $\overline{R(J - \lambda I)} = U$;
(III) $\overline{R(J - \lambda I)} \neq U$

and

(1) $(J - \lambda I)^{-1}$ exists and it is bounded;
(2) $(J - \lambda I)^{-1}$ exists but it is not bounded;
(3) $(J - \lambda I)^{-1}$ does not exist.

Based on the above six properties, the Goldberg's classification of the spectrum can be given as shown in the Table 1.

Table 1. Subdivisions of spectrum of a bounded linear operator

	(I)	(II)	(III)
1	$\rho(J,U)$	—	$\sigma_r(J,U)$
2	$\sigma_c(J,U)$	$\sigma_c(J,U)$	$\sigma_r(J,U)$
3	$\sigma_p(J,U)$	$\sigma_p(J,U)$	$\sigma_p(J,U)$

Theorem 2.1 ([21]). *Let L be a bounded linear operator on a normed linear space U. Then L has a bounded inverse if and only if L^* is onto.*

Lemma 2.1 ([20]). *An infinite matrix $A = (a_{nk}) \in B(c_0)$ if and only if*

(*a*) $(a_{nk})_k \in \ell_1$ *for all n and* $\sup_n \sum_k |a_{nk}| < \infty$;
(*b*) $(a_{nk})_n \in c_0$ *for all k.*

Moreover, the norm $\|A\| = \sup_n \sum_k |a_{nk}|$.

Throughout the paper, we denote the set of natural numbers by $\mathbb{N}$, the set of complex numbers by $\mathbb{C}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We assume that $x_{-n} = 0$ for all $n \in \mathbb{N}$.

3. Main Results

Let l and l' be two natural numbers. Suppose that H is the least common multiple of l and l'. Let r_i, $i = 1, \ldots, l$, and $s_i \neq 0$, $i = 1, \ldots, l'$, be complex numbers. Then the matrix $\mathbb{B}(r_1, \ldots, r_l; s_1, \ldots, s_{l'})$ is defined as $\mathbb{B} = (b_{ij})_{i,j\geq 0}$, where

$$b_{ij} = \begin{cases} r_{j(\text{mod } l)+1}, & \text{when } i = j, \\ s_{j(\text{mod } l')+1}, & \text{when } i = j+1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

That is

$$\mathbb{B} = \begin{bmatrix} r_1 & & & & \\ s_1 & \ddots & & \text{\Large 0} & \\ & \ddots & r_l & & \\ & & s_{l'} & \ddots & \\ \text{\Large 0} & & & \ddots & \end{bmatrix}.$$

If the matrix $\mathbb{B}$ transforms a sequence $x = (x_k)$ into $y = (y_k)$, then

$$y_k = \sum_{j=0}^{\infty} b_{kj} x_j = b_{k,k-1} x_{k-1} + b_{kk} x_k = s_{(k-1)(\text{mod } l')+1} x_{k-1} + r_{k(\text{mod } l)+1} x_k, \tag{3.2}$$

for all $k \in \mathbb{N}_0$.

Theorem 3.1. *$\mathbb{B} \in B(c_0)$ and $\|\mathbb{B}\|_{c_0} \leq \max_{i,j}\{|r_i| + |s_j| \colon 1 \leq i \leq l, 1 \leq j \leq l'\}$.*

Suppose that a is an integer and n is a natural number. We denote, by $[a_n]$, the set of all non-negative integers x for which n divides $x - a$. Then $a(\text{mod } n)$ is the least member of $[a_n]$. Let α and β be the mappings which are defined on the set of integers as follows:

$$\alpha(k) = k(\text{mod } l) + 1$$

and

$$\beta(k) = k(\text{mod } l') + 1.$$

Without loss of generality, we assume that $s_{\beta(k)} s_{\beta(k+1)} \cdots s_{\beta(k+j)} = 1$ and $(r_{\alpha(k)} - \lambda)(r_{\alpha(k+1)} - \lambda) \cdots (r_{\alpha(k+j)} - \lambda) = 1$, when $k + j < k$. If λ is a complex number such that $(\mathbb{B} - \lambda I)^{-1}$ exists, then the entries of the matrix $(\mathbb{B} - \lambda I)^{-1} = (z_{nk})$, $n \geq 0$, and $k \geq 0$, are given by

$$z_{nk} = \begin{cases} \dfrac{(-1)^{n-k} s_{\beta(k)} \cdots s_{\beta(k+\zeta''-1)}}{(r_{\alpha(k)} - \lambda) \cdots (r_{\alpha(k+\zeta')} - \lambda)} \cdot \dfrac{(s_1 \ldots s_{l'})^{m''}}{\{(r_1 - \lambda) \cdots (r_l - \lambda)\}^{m'}} & \\ \times \left\{ \dfrac{(s_1 \cdots s_{l'})^{\frac{H}{l'}}}{\{(r_1 - \lambda) \cdots (r_l - \lambda)\}^{\frac{H}{l}}} \right\}^m, & \text{when } n > k, \\ \dfrac{1}{r_{\alpha(k)} - \lambda}, & \text{when } n = k, \\ 0, & \text{otherwise,} \end{cases} \tag{3.3}$$

where ζ, ζ' and ζ'' are the least non-negative integers such that

$$\begin{cases} n-k = mH+\zeta, \\ \qquad \zeta = m'l+\zeta', \\ \qquad \zeta = m''l'+\zeta'', \end{cases} \tag{3.4}$$

for some non-negative integers m, m' and m''.

Lemma 3.1. *If* $(|\lambda - r_1|\cdots|\lambda - r_l|)^{1/l} > (|s_1|\cdots|s_{l'}|)^{1/l'}$, *then* $(\mathbb{B}-\lambda I)^{-1} \in B(c_0)$.

Proof. Since $(|\lambda - r_1|\cdots|\lambda - r_l|)^{1/l} > (|s_1|\cdots|s_{l'}|)^{1/l'}$ and $s_1, s_2, \ldots, s_{l'}$ are non-zero, therefore $\lambda \neq r_i$ for all $i = 1, 2, \ldots, l$. Then the matrix $\mathbb{B} - \lambda I$ is a triangle and hence $(\mathbb{B}-\lambda I)^{-1} = (z_{nk})$ exists, which is given by (3.3). We first consider a row of $(\mathbb{B}-\lambda I)^{-1}$ which is a multiple of H, that is $n = \widetilde{m}H$ for some $\widetilde{m} \in \mathbb{N}_0$. Now, let $k = \hat{m}H$ for $\hat{m} = 0, 1, \ldots, \widetilde{m}$. Then (3.4) implies that $n-k = (\widetilde{m}-\hat{m})H$ and $m' = m'' = \zeta = \zeta' = \zeta'' = 0$. Thus, from (3.3), we have

$$z_{nk} = \frac{(-1)^{n-k}}{r_{\alpha(k)}-\lambda}\left\{\frac{(s_1\cdots s_{l'})^{\frac{H}{l'}}}{\{(r_1-\lambda)\cdots(r_l-\lambda)\}^{\frac{H}{l}}}\right\}^{\widetilde{m}-\hat{m}},$$

for all $\hat{m} = 0, 1, \ldots, \widetilde{m}$. Therefore,

$$\sum_{k\in[0_H]}|z_{nk}| = \frac{1}{|r_{\alpha(k)}-\lambda|}\sum_{j=0}^{\tilde{m}}\left\{\frac{(|s_1|\cdots|s_{l'}|)^{\frac{H}{l'}}}{\{|r_1-\lambda|\cdots|r_l-\lambda|\}^{\frac{H}{l}}}\right\}^{j},$$

where $[0_H]$ denotes the set of all non-negative integers which are multiple of H. For the same row, if we consider $k = \hat{m}H+1$ for $\hat{m} = 0, 1, \ldots, \tilde{m}-1$, then $n-k = (\widetilde{m}-\hat{m}-1)H + H - 1$. Let m_1' and m_1'' be quotients and ζ_1' and ζ_1'' be remainders when $H-1$ is divided by l and l' respectively, that is

$$H-1 = m_1'l+\zeta_1',$$
$$H-1 = m_1''l'+\zeta_1''.$$

Then, from (3.3), we obtain that

$$\begin{aligned} z_{nk} =& \frac{(-1)^{n-k}s_{\beta(k)}\cdots s_{\beta(k+\zeta_1''-1)}}{(r_{\alpha(k)}-\lambda)\cdots(r_{\alpha(k+\zeta_1')}-\lambda)}\cdot\frac{(s_1\cdots s_{l'})^{m_1''}}{\{(r_1-\lambda)\cdots(r_l-\lambda)\}^{m_1'}} \\ &\times\left\{\frac{(s_1\cdots s_{l'})^{\frac{H}{l'}}}{\{(r_1-\lambda)\cdots(r_l-\lambda)\}^{\frac{H}{l}}}\right\}^{\widetilde{m}-\hat{m}-1}, \end{aligned}$$

for all $\hat{m} = 0, 1, \ldots, \widetilde{m}-1$. Hence,

$$\begin{aligned} \sum_{k\in[1_H]}|z_{nk}| =& \frac{|s_{\beta(k)}|\cdots|s_{\beta(k+\zeta_1''-1)}|}{|r_{\alpha(k)}-\lambda|\cdots|r_{\alpha(k+\zeta_1')}-\lambda|}\cdot\frac{(|s_1|\cdots|s_{l'}|)^{m_1''}}{\{|r_1-\lambda|\cdots|r_l-\lambda|\}^{m_1'}} \\ &\times\sum_{j=0}^{\widetilde{m}-1}\left\{\frac{(|s_1|\cdots|s_{l'}|)^{\frac{H}{l'}}}{\{|r_1-\lambda|\cdots|r_l-\lambda|\}^{\frac{H}{l}}}\right\}^{j}, \end{aligned}$$

where $[1_H]$ denotes the set of all nonnegative integers x such that H divides $x-1$. Similarly, for $k = \hat{m}H+2, \ldots, \hat{m}H+H-1$, we have

$$\sum_{k\in[2_L]} |z_{nk}| = \frac{|s_{\beta(k)}|\cdots|s_{\beta(k+\zeta_2''-1)}|}{|r_{\alpha(k)}-\lambda|\cdots|r_{\alpha(k+\zeta_2')}-\lambda|} \cdot \frac{(|s_1|\ldots|s_{l'}|)^{m_2''}}{\{|r_1-\lambda|\cdots|r_l-\lambda|\}^{m_2'}} \times \sum_{j=0}^{\widetilde{m}-1} \left\{ \frac{(|s_1|\cdots|s_{l'}|)^{\frac{H}{l'}}}{\{|r_1-\lambda|\cdots|r_l-\lambda|\}^{\frac{H}{l}}} \right\}^j,$$

$$\vdots$$

$$\sum_{k\in[(H-1)_L]} |z_{nk}| = \frac{|s_{\beta(k)}|\cdots|s_{\beta(k+\zeta_{H-1}''-1)}|}{|r_{\alpha(k)}-\lambda|\cdots|r_{\alpha(k+\zeta_{H-1}')}-\lambda|} \cdot \frac{(|s_1|\cdots|s_{l'}|)^{m_{H-1}''}}{\{|r_1-\lambda|\cdots|r_l-\lambda|\}^{m_{H-1}'}} \times \sum_{j=0}^{\widetilde{m}-1} \left\{ \frac{(|s_1|\cdots|s_{l'}|)^{\frac{H}{l'}}}{\{|r_1-\lambda|\cdots|r_l-\lambda|\}^{\frac{H}{l}}} \right\}^j,$$

for some integers ζ_i', ζ_i'', m_i' and m_i'' for all $i \in \{2,3,\ldots,H-1\}$. Thus,

$$\sum_{k=0}^{\infty} |z_{nk}| = \frac{1}{|r_{\alpha(k)}-\lambda|} \sum_{j=0}^{\widetilde{m}} \left\{ \frac{(|s_1|\cdots|s_{l'}|)^{\frac{H}{l'}}}{\{|r_1-\lambda|\cdots|r_l-\lambda|\}^{\frac{H}{l}}} \right\}^j + M \sum_{j=0}^{\widetilde{m}-1} \left\{ \frac{(|s_1|\cdots|s_{l'}|)^{\frac{H}{l'}}}{\{|r_1-\lambda|\cdots|r_l-\lambda|\}^{\frac{H}{l}}} \right\}^j, \tag{3.5}$$

where

$$M = \frac{|s_{\beta(k)}|\cdots|s_{\beta(k+\zeta_1''-1)}|}{|r_{\alpha(k)}-\lambda|\cdots|r_{\alpha(k+\zeta_1')}-\lambda|} \cdot \frac{(|s_1|\cdots|s_{l'}|)^{m_1''}}{\{|r_1-\lambda|\cdots|r_l-\lambda|\}^{m_1'}} + \cdots + \frac{|s_{\beta(k)}|\cdots|s_{\beta(k+\zeta_{L-1}''-1)}|}{|r_{\alpha(k)}-\lambda|\cdots|r_{\alpha(k+\zeta_{L-1}')}-\lambda|} \cdot \frac{(|s_1|\cdots|s_{l'}|)^{m_{H-1}''}}{\{|r_1-\lambda|\cdots|r_l-\lambda|\}^{m'_{H-1}}}.$$

Let $M_0 = \max\left\{\frac{1}{|r_{\alpha(k)}-\lambda|}, M\right\}$. Then

$$\sum_{k=0}^{\infty} |z_{nk}| \le \frac{2M_0(|r_1-\lambda||r_2-\lambda|\cdots|r_l-\lambda|)^{\frac{H}{l}}}{(|r_1-\lambda||r_2-\lambda|\cdots|r_l-\lambda|)^{\frac{H}{l}} - (|s_1||s_2|\cdots|s_{l'}|)^{\frac{H}{l'}}}.$$

Therefore, $\sup_{n\in[0_H]} \sum_{k=0}^{\infty} |z_{nk}| < \infty$. Similarly, we prove that

$$\sup_{n\in[1_H]} \sum_{k=0}^{\infty} |z_{nk}| < \infty,$$

$$\sup_{n\in[2_H]} \sum_{k=0}^{\infty} |z_{nk}| < \infty,$$

$$\vdots$$

$$\sup_{n\in[(H-1)_H]}\sum_{k=0}^{\infty}|z_{nk}|<\infty.$$

Thus,

$$\sup_n\sum_{k=0}^{\infty}|z_{nk}|=\max\left\{\sup_{n\in[0_H]}\sum_{k=0}^{\infty}|z_{nk}|,\ \sup_{n\in[1_H]}\sum_{k=0}^{\infty}|z_{nk}|,\dots,\sup_{n\in[(H-1)_H]}\sum_{k=0}^{\infty}|z_{nk}|\right\}.$$

This implies that $\sup_n\sum_{k=0}^{\infty}|z_{nk}|<\infty$. Likewise, for an arbitrary column of $(\mathbb{B}-\lambda I)^{-1}$, adding the entries separately whose rows n belong to $[0_H],[1_H],\dots,[(H-1)_H]$ respectively, we get $\sum_{n=0}^{\infty}|z_{nk}|<\infty$. Therefore, $\lim_{n\to\infty}|z_{nk}|=0$ for all $k\in\mathbb{N}_0$. Hence, by Lemma 2.1, the matrix $(\mathbb{B}-\lambda I)^{-1}\in B(c_0)$. ☐

Consider the set $S=\left\{\lambda\in\mathbb{C}:(|\lambda-r_1|\cdots|\lambda-r_l|)^{\frac{1}{l}}\le(|s_1|\cdots|s_{l'}|)^{\frac{1}{l'}}\right\}$. Then we have the following theorem.

Theorem 3.2. *$\sigma(\mathbb{B},c_0)=S$.*

Proof. First, we prove that $\sigma(\mathbb{B},c_0)\subseteq S$. Let λ be a complex number that does not belong to S. Then $(|\lambda-r_1|\cdots|\lambda-r_l|)^{1/l}>(|s_1|\cdots|s_{l'}|)^{1/l'}$. In that case, from Lemma 3.1, it follows that $(\mathbb{B}-\lambda I)^{-1}\in B(c_0)$. That is, $\lambda\notin\sigma(\mathbb{B},c_0)$. Hence, $\sigma(\mathbb{B},c_0)\subseteq S$.

Next, we show that $S\subseteq\sigma(\mathbb{B},c_0)$. Let $\lambda\in S$. Then, $(|\lambda-r_1|\cdots|\lambda-r_l|)^{1/l}\le(|s_1|\cdots|s_{l'}|)^{1/l'}$. If λ equals any of the r_i for all $i\in\{1,2,\dots,l\}$, then the range of the operator $\mathbb{B}-\lambda I$ is not dense in c_0, and hence $\lambda\in\sigma(\mathbb{B},c_0)$. Therefore, we assume that $\lambda\ne r_i$ for all $i\in\{1,2,\dots,l\}$. In that case, $\mathbb{B}-\lambda I$ is a triangle and $(\mathbb{B}-\lambda I)^{-1}=(z_{nk})$ exists, which is given by (3.3). Let $y=(1,0,0,\dots)\in c_0$ and let $x=(x_k)$ be the sequence such that $(\mathbb{B}-\lambda I)^{-1}y=x$. It follows, from (3.3), that

$$x_{nH}=z_{nH,0}=\frac{(-1)^{nH}}{r_1-\lambda}\left\{\frac{(s_1s_2\cdots s_{l'})^{\frac{H}{l'}}}{\{(r_1-\lambda)\cdots(r_l-\lambda)\}^{\frac{H}{l}}}\right\}^n,\tag{3.6}$$

for all $n\in N_0$. Since $\{(r_1-\lambda)\cdots(r_l-\lambda)\}^{1/l}\le(s_1\cdots s_{l'})^{\frac{1}{l'}}$, the subsequence (x_{nH}) of x does not converge to 0. Consequently, the sequence $x=(x_k)\notin c_0$. Therefore, $(B-\lambda I)^{-1}\notin B(c_0)$. Thus, $\lambda\in\sigma(\mathbb{B},c_0)$ and hence $S\subseteq\sigma(\mathbb{B},c_0)$. This proves the theorem. ☐

Theorem 3.3. *$\sigma_p(\mathbb{B},c_0)=\emptyset$.*

Proof. Let $\lambda\in\sigma_p(\mathbb{B},c_0)$. Then there exists a nonzero sequence $x=(x_k)$ such that $\mathbb{B}x=\lambda x$. This implies that

$$s_{(k-1)(\text{mod } l')+1}x_{k-1}+r_{k(\text{mod } l)+1}x_k=\lambda x_k.\tag{3.7}$$

Let x_{k_0} be the first non-zero term of the sequence $x=(x_k)$. Then from the relation (3.7), we find that $\lambda=r_{k_0(\text{mod } l)+1}$. Next, for $k=k_0+l$, (3.7) becomes

$$s_{(k_0+l-1)(\text{mod } l')+1}x_{k_0+l-1}+r_{(k_0+l)(\text{mod } l)+1}x_{k_0+l}=\lambda x_{k_0+l}.$$

That is,

$$s_{(k_0+l-1)(\text{mod } l')+1}x_{k_0+l-1} + r_{k_0(\text{mod } l)+1}x_{k_0+l} = \lambda x_{k_0+l}. \tag{3.8}$$

Putting $\lambda = r_{k_0(\text{mod } l)+1}$ in (3.8), we find that

$$s_{(k_0+l-1)(\text{mod } l')+1}x_{k_0+l-1} = 0.$$

As $s_{(k_0+l-1)(\text{mod } l')+1} \neq 0$, therefore $x_{k_0+l-1} = 0$. Similarly, using (3.7) for $k = k_0+l-1$ and putting the value $x_{k_0+l-1} = 0$, we obtain $x_{k_0+l-2} = 0$. Repeating the same step for $k = k_0 + l - 2, k_0 + l - 3, \ldots, k_0 + 1$, we deduce that $x_{k_0} = 0$, which is a contradiction. Hence, $\sigma_p(\mathbb{B}, c_0) = \emptyset$. □

Let $\mathbb{B}^* = (b^*_{ij})$ denote the adjoint of the operator $\mathbb{B}$. Then the matrix representation of $\mathbb{B}^*$ is equal to the transpose of the matrix $\mathbb{B}$. It follows that

$$b^*_{ij} = \begin{cases} r_{i(\text{mod } l)+1}, & \text{when } i = j, \\ s_{i(\text{mod } l')+1}, & \text{when } i + 1 = j, \\ 0, & \text{otherwise.} \end{cases} \tag{3.9}$$

That is,

$$\mathbb{B}^* = \begin{bmatrix} r_1 & s_1 & & & & \\ & r_2 & \ddots & & \mathbf{0} & \\ & & \ddots & s_{l'} & & \\ & & & r_1 & \ddots & \\ \mathbf{0} & & & & \ddots & \ddots \end{bmatrix}.$$

The next theorem gives the point spectrum of the operator B^*.

Theorem 3.4. $\sigma_p(\mathbb{B}^*, c_0^*) = \left\{\lambda \in \mathbb{C} : (|\lambda - r_1| \cdots |\lambda - r_l|)^{\frac{1}{l}} < (|s_1| \cdots |s_{l'}|)^{\frac{1}{l'}}\right\}$.

Proof. Let $\lambda \in \sigma_p(\mathbb{B}^*, c_0^* \cong \ell_1)$. Then there exists a nonzero sequence $x = (x_k) \in \ell_1$ such that $\mathbb{B}^*x = \lambda x$. From this relation, the subsequences $(x_{kH}), (x_{kH+1}), \ldots, (x_{kH+H-1})$ of $x = (x_k)$ are given by

$$\begin{aligned} x_{kH} &= \left\{ \frac{((\lambda - r_1) \cdots (\lambda - r_l))^{\frac{H}{l}}}{(s_1 \cdots s_{l'})^{\frac{H}{l'}}} \right\}^k x_0 \\ x_{kH+1} &= \frac{(\lambda - r_1)}{s_1} \left\{ \frac{((\lambda - r_1) \cdots (\lambda - r_l))^{\frac{H}{l}}}{(s_1 \cdots s_{l'})^{\frac{H}{l'}}} \right\}^k x_0 \\ &\vdots \\ x_{kH+H-1} &= \frac{(\lambda - r_1)^{\frac{H}{l}} \cdots (\lambda - r_{l-1})^{\frac{H}{l}} (\lambda - r_l)^{\frac{H}{l}-1}}{s_1^{\frac{H}{l'}} \cdots s_{l'-1}^{\frac{H}{l'}} s_{l'}^{\frac{H}{l'}-1}} \end{aligned}$$

$$\times \left\{ \frac{((\lambda - r_1)\cdots(\lambda - r_l))^{\frac{H}{l}}}{(s_1 \cdots s_{l'})^{\frac{H}{l'}}} \right\}^k x_0.$$

Thus,

$$\begin{aligned} \sum_{k=0}^{\infty} |x_k| &= \sum_{k=0}^{\infty} |x_{kH}| + \sum_{k=0}^{\infty} |x_{kH+1}| + \cdots + \sum_{n=0}^{\infty} |x_{kH+H-1}| \\ &= \left(1 + \left| \frac{\lambda - r_1}{s_1} \right| + \cdots + \left| \frac{(\lambda - r_1)^{\frac{H}{l}} \cdots (\lambda - r_{l-1})^{\frac{H}{l}} (\lambda - r_l)^{\frac{H}{l}-1}}{s_1^{\frac{H}{l'}} \cdots s_{l'-1}^{\frac{H}{l'}} s_{l'}^{\frac{H}{l'}-1}} \right| \right) \\ &\quad \times \sum_{k=0}^{\infty} \left| \frac{((\lambda - r_1)\cdots(\lambda - r_l))^{\frac{H}{l}}}{(s_1 \cdots s_{l'})^{\frac{H}{l'}}} \right|^k |x_0|. \end{aligned}$$

Clearly, the sequence $x = (x_k) \in \ell_1$ if and only if $(|\lambda - r_1| \cdots |\lambda - r_l|)^{\frac{1}{l}} < (|s_1| \cdots |s_{l'}|)^{\frac{1}{l'}}$. This proves the theorem. □

Theorem 3.5. $\sigma_r(\mathbb{B}, c_0) = \left\{ \lambda \in \mathbb{C} : (|\lambda - r_1| \cdots |\lambda - r_l|)^{\frac{1}{l}} < (|s_1| \cdots |s_{l'}|)^{\frac{1}{l'}} \right\}$.

Proof. The residual spectrum of a bounded linear operator L on a Banach space U is given by the relation $\sigma_r(L, U) = \sigma_p(L^*, U^*) \setminus \sigma_p(L, U)$. Therefore, $\sigma_r(\mathbb{B}, c_0) = \sigma_p(\mathbb{B}^*, c_0^*) \setminus \sigma_p(\mathbb{B}, c_0)$. Then the proof of this theorem is an easy consequence of the Theorems 3.3 and 3.4. □

Theorem 3.6. $\sigma_c(\mathbb{B}, c_0) = \left\{ \lambda \in \mathbb{C} : (|\lambda - r_1| \cdots |\lambda - r_l|)^{\frac{1}{l}} = (|s_1| \cdots |s_{l'}|)^{\frac{1}{l'}} \right\}$.

Proof. Since spectrum of an operator on a Banach space is disjoint union of point, residual and continuous spectrum, therefore from Theorems 3.2, 3.3 and 3.5, we deduce that

$$\sigma_c(\mathbb{B}, c_0) = \left\{ \lambda \in \mathbb{C} : (|\lambda - r_1| \cdots |\lambda - r_l|)^{\frac{1}{l}} = (|s_1| \cdots |s_{l'}|)^{\frac{1}{l'}} \right\}. \qquad \square$$

Theorem 3.7. $\{r_1, r_2, \ldots, r_l\} \subseteq III_1(\mathbb{B}, c_0)$.

Proof. Theorem 3.5 shows that $r_1 \in \sigma_r(\mathbb{B}, c_0)$. However, $\sigma_r(\mathbb{B}, c_0) = III_1(\mathbb{B}, c_0) \cup III_2(\mathbb{B}, c_0)$. Therefore, to prove $r_1 \in III_1\sigma(\mathbb{B}, c_0)$, we shall show that the matrix $\mathbb{B} - r_1 I$ has bounded inverse and from Theorem 2.1, it will be sufficient to show that $(\mathbb{B} - r_1 I)^*$ is onto. For this, let $y = (y_k) \in \ell_1$. Then $(\mathbb{B} - r_1 I)^* x = y$ implies that

$$(r_{i(\text{mod } l)+1} - r_1) x_i + s_{i(\text{mod } l')+1} x_{i+1} = y_i, \tag{3.10}$$

for all $i \in \mathbb{N}_0$. Solving (3.10) for $x = (x_i)$, we obtain that

$$x_{mH+k} = \sum_{j=0}^{k-2} \frac{1}{s_{j(\text{mod } l')+1}} \prod_{i=j+1}^{k-1} \frac{r_1 - r_{i(\text{mod } l)+1}}{s_{i(\text{mod } l')+1}} y_{mH+j} + \frac{y_{mH+k-1}}{s_{(k-1)(\text{mod } l')+1}}, \tag{3.11}$$

for $k = 1, \ldots, H$, and $m = 0, \ldots, \infty$. Let

$$C_j = \frac{1}{s_{j(\text{mod } l')+1}} \prod_{i=j+1}^{k-1} \frac{r_1 - r_{i(\text{mod } l)+1}}{s_{i(\text{mod } l')+1}},$$

for $j = 0, \dots, k-2$, and

$$C_{k-1} = \frac{1}{s_{(k-1)(\text{mod } l')+1}}.$$

Then (3.11) can be written as

$$x_{mH+k} = C_0 y_{mH} + C_1 y_{mH+1} + \cdots + C_{k-1} y_{mH+k-1}. \tag{3.12}$$

Taking summation from $m = 0$ to ∞ of the absolute values of x_{mH+k}, we obtain

$$\sum_{m=0}^{\infty} |x_{mH+k}| \leq |C_0| \sum_{m=0}^{\infty} |y_{mH}| + |C_1| \sum_{m=0}^{\infty} |y_{mH+1}| + \cdots + |C_{k-1}| \sum_{m=0}^{\infty} |y_{mH+k-1}|. \tag{3.13}$$

Since $y = (y_k) \in \ell_1$, therefore the right hand side of the inequality (3.13) is a sum of k finite terms. Thus, $\sum_{m=0}^{\infty} |x_{mH+k}| < \infty$ for $k \in \{1, 2, \dots, H\}$. This implies that the series

$$\sum_i |x_i| = |x_0| + \sum_{m=0}^{\infty} |x_{mH+1}| + \sum_{m=0}^{\infty} |x_{mH+2}| + \cdots + \sum_{m=0}^{\infty} |x_{mH+H}| \tag{3.14}$$

is a sum of $H + 1$ finite terms. Hence, $x = (x_i) \in \ell_1$. We have shown that for every $y = (y_i) \in \ell_1$ there exists a sequence $x = (x_i) \in \ell_1$ such that $(\mathbb{B} - r_1 I)^* x = y$. That is, $(B - r_1 I)^*$ is onto. Similarly, we can show that $r_i \in III_1(\mathbb{B}, c_0)$ for $i = 2, \dots, l$. This proves the theorem. □

Theorem 3.8. $\sigma_r(\mathbb{B}, c_0) \setminus \{r_1, r_2, \dots, r_l\} \subseteq III_2(\mathbb{B}, c_0)$.

Proof. Let λ belongs to the set $\sigma_r(\mathbb{B}, c_0) \setminus \{r_1, r_2, \dots, r_l\}$. Then $(|\lambda - r_1| \cdots |\lambda - r_l|)^{\frac{1}{l}} < (|s_1| \dots |s_{l'}|)^{\frac{1}{l'}}$ and $\lambda \notin r_i$ for all $i \in \{1, 2, \dots, l\}$. This inequality shows that the series $\sum_{k=0}^{\infty} |z_{nk}|$ in (3.5) is not convergent when n goes to infinity. In that case, $\mathbb{B} - \lambda I$ does not have bounded inverse. Then from Table 1, we find that $\lambda \in III_2(\mathbb{B}, c_0)$. Hence $\sigma_r(\mathbb{B}, c_0) \setminus \{r_1, r_2, \dots, r_l\} \subseteq III_2(\mathbb{B}, c_0)$. □

Theorem 3.9. $III_1(\mathbb{B}, c_0) = \{r_1, r_2, \dots, r_l\}$.

Proof. From Table 1, we have $\sigma_r(\mathbb{B}, c_0) = III_1(\mathbb{B}, c_0) \cup III_2(, c_0)$ and the union is disjoint. Then taking complement of the inclusion of Theorem 3.8 in $\sigma_r(\mathbb{B}, c_0)$, we obtain that $\sigma_r(\mathbb{B}, c_0) \setminus III_2(\mathbb{B}, c_0) \subseteq \{r_1, r_2, \dots, r_l\}$. That is, $III_1(\mathbb{B}, c_0) \subseteq \{r_1, r_2, \dots, r_l\}$. This inclusion together with Theorem 3.7 implies that $III_1(\mathbb{B}, c_0) = \{r_1, r_2, \dots, r_l\}$. □

Theorem 3.10. $III_2(\mathbb{B}, c_0) = \sigma_r(\mathbb{B}, c_0) \setminus \{r_1, r_2, \dots, r_l\}$.

Proof. Taking complement of the result of Theorem 3.9 in $\sigma_r(\mathbb{B}, c_0)$, we obtain that $III_2(\mathbb{B}, c_0) = \sigma_r(\mathbb{B}, c_0) \setminus \{r_1, r_2, \dots, r_l\}$. □

4. Fine Spectra of the Matrix $\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6)$

We consider the matrix

$$\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6) = \begin{bmatrix} r_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ s_1 & r_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & s_2 & r_3 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & s_3 & r_4 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & s_4 & r_1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & s_5 & r_2 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & s_6 & r_3 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & s_1 & r_4 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & s_2 & r_1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}.$$

Now, consider the following sets:

$$D = \left\{\lambda \in \mathbb{C} : (|\lambda - r_1||\lambda - r_2||\lambda - r_3||\lambda - r_4|)^{\frac{1}{4}} \leq (|s_1||s_2||s_3||s_4||s_5||s_6|)^{\frac{1}{6}}\right\},$$
$$D_1 = \left\{\lambda \in \mathbb{C} : (|\lambda - r_1||\lambda - r_2||\lambda - r_3||\lambda - r_4|)^{\frac{1}{4}} < (|s_1||s_2||s_3||s_4||s_5||s_6|)^{\frac{1}{6}}\right\},$$
$$D_2 = \left\{\lambda \in \mathbb{C} : (|\lambda - r_1||\lambda - r_2||\lambda - r_3||\lambda - r_4|)^{\frac{1}{4}} = (|s_1||s_2||s_3||s_4||s_5||s_6|)^{\frac{1}{6}}\right\}.$$

From the discussion of the previous section, we deduce the following results:

(a) $\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6) \in B(c_0)$;
(b) $\|\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6)\|_{c_0} \leq \max_{i,j}\{|r_i| + |s_j|\colon 1 \leq i \leq 4, 1 \leq j \leq 6\}$;
(c) $\sigma(\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6), c_0) = D$;
(d) $\sigma_p(\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6), c_0) = \emptyset$;
(e) $\sigma_p(\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6)^*, c_0^* \cong \ell_1) = D_1$;
(f) $\sigma_r(\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6), c_0) = D_1$;
(g) $\sigma_c(\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6), c_0) = D_2$;
(h) $III_1\sigma(\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6), c_0) = \{r_1, r_2, r_3, r_4\}$;
(i) $III_2\sigma(\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6), c_0) = D_1 \setminus \{r_1, r_2, r_3, r_4\}$.

In particular, if we take $r_1 = 1 - i$, $r_2 = -i$, $r_3 = -1.5$, $r_4 = -i$ and $s_1 = i$, $s_2 = 1 + i$, $s_3 = -2$, $s_4 = -1.5$, $s_5 = 1 - i$, $s_6 = -1$, then the spectrum is given by

$$\sigma(\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6), c_0) = \left\{\lambda \in \mathbb{C} : (|\lambda - 1 + i||\lambda + i|^2|\lambda + 1.5|)^{\frac{1}{4}} \leq 6^{\frac{1}{6}}\right\},$$

which is shown by the shaded region in Figure 1.

5. Conclusions

We have studied the spectral decomposition of the matrix $\mathbb{B}(r_1, \ldots, r_l; s_1, \ldots, s_{l'})$, which generalizes the following matrices.

- The backward difference operator Δ [3] for $l = 1$, $l' = 1$, $r_1 = 1$ and $s_1 = -1$.
- The Right shift operator for $l = 1$, $l' = 1$, $r_1 = 0$ and $s_1 = 1$.

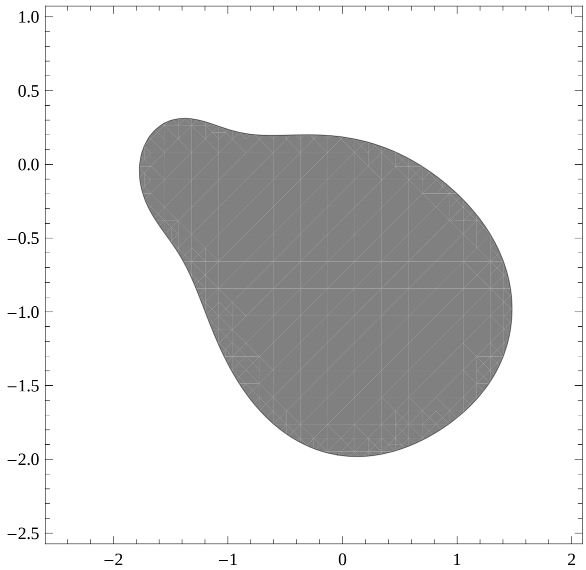

FIGURE 1. Spectrum of $\mathbb{B}(r_1, \ldots, r_4; s_1, \ldots, s_6)$.

- The Zweier matrix [5] for $l = 1$, $l' = 1$, $r_1 = s$ and $s_1 = 1 - s$ for some complex numbers $s \neq 0, 1$.
- The generalized difference operator $B(r, s)$ [4] for $l = 1$, $l' = 1$, $r_1 = r$ and $s_1 = s$ for some complex numbers r and $s \neq 0$.

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