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# SOME BORDERENERGETIC AND EQUIENERGETIC GRAPHS

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ABSTRACT. The sum of absolute values of eigenvalues of a graph $G$ is defined as energy of graph. If the energies of two non-isomorphic graphs are same then they are called equienergetic. The energy of complete graph with $n$ vertices is $2(n-1)$ and the graphs whose energy is equal to $2(n-1)$ are called borderenergetic graphs. It has been revealed that the graphs upto 12 vertices are borderenergetic. It is very challenging and interesting as well to search for borderenergetic graphs with more than 14 vertices. The present work is leap ahead in this direction as we have found a family of borderenergetic graphs of arbitrarily large order. We have also obtained three pairs of equienergetic graphs.

## 1. INTRODUCTION

For standard terminology and notations in graph theory we follow West [19] while the terms related to algebra are used in sense of Lang [11].

Let $G$ be a connected undirected simple graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$. The *adjacency matrix* denoted by $A(G)$ of $G$ is defined to be $A(G) = [a_{ij}]$, such that, $a_{ij} = 1$ if $v_i$ is adjacent with $v_j$, and 0 otherwise.

The eigenvalues of $A$ are called the eigenvalues of $G$. If $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of $G$ then

$$\operatorname{spec}(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ m_1 & m_2 & \cdots & m_n \end{pmatrix}.$$

The energy $E(G)$ of graph $G$ is the sum of all absolute values of eigenvalues of $G$. The concept of energy of graph was introduced by Gutman [7] in 1978. A brief account on energy of graph can be found in Cvetković [2] and Li [12].

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The graphs of order $n$, whose energy exceeds than the energy of the complete graph $K_n$ are called hyperenergetic graphs otherwise graphs of order $n$ with $E(G) \leq E(K_n)$, are called non-hyperenergetic. As mentioned in Gutman [7] $E(K_n) = 2(n-1)$. Are there any graphs other than $K_n$ with such behaviour?

This question motivated Gong et al. [6] to introduce a new concept. According to them, the graph $G$ of order $n$ satisfying $E(G) = 2(n-1)$ are called *borderenergetic*. Obviously, the complete graph $K_n$ is borderenergetic. Gong et al. [6] have proved that such graphs exist for $n = 7, 8, 9$. Li et al. [13] and Shao et al. [15] have obtained the graphs with $n = 10$ and $n = 11$ respectively while Furtula and Gutman [4] have obtained the graphs with $n = 12$. A family of non-regular and non-integral borderenergetic graphs with particular behaviour were investigated by Hou and Tao [16]. Some new families of borderenergetic graphs were obtained by Jahfar et al. [10]. Recently, a survey on borderenergetic graphs was published by Ghorbani et al. [5].

We will introduce some concepts and also state some existing results for our ready reference.

**Definition 1.1.** The *shadow graph* $D_2(G)$ of a connected graph $G$ is constructed by taking two copies of $G$ say $G'$ and $G''$. Join each vertex $u'$ in $G'$ to the neighbors of the corresponding vertex $u''$ in $G''$.

**Proposition 1.1** ([17]). *If $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of $G$, then $2n$ eigenvalues of $D_2(G)$ are $2\lambda_1, 2\lambda_2, \ldots, 2\lambda_n, 0$ ($n$ times).*

**Proposition 1.2** ([3]). *Let*

$$A = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$$

*be a symmetric block matrix. Then the spectrum of $A$ is the union of spectra of $A_0 + A_1$ and $A_0 - A_1$.*

**Definition 1.2.** The *extended shadow graph* $D_2^*(G)$ of a connected graph $G$ is constructed by taking two copies of $G$ say $G'$ and $G''$. Join each vertex $u'$ in $G'$ to the neighbours of the corresponding vertex $u''$ and with $u''$ in $G''$.

A curious question: **How the energy of a given graph $G$ can be correlated with the larger graph obtained by means of graph operations on $G$?** To quench this thirst we have considered shadow graph and extended shadow graph as these graphs are of same order but they are non isomorphic. Due to this specific characteristic, the said graphs are used to construct non-co spectral equienergetic graphs by constructing shadow graph of extended shadow graph as well as extended shadow graph of shadow graph.

## 2. Energy of Extended Shadow Graph

**Theorem 2.1.** *Let $G$ be a graph with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, with $|\lambda_i| \geq \frac{1}{2}$ for all $1 \leq i \leq n$, then $E(D_2^*(G)) = 2E(G) + n + \theta$, where $\theta$ is the difference between the number of positive and negative eigenvalues of $G$.*

*Proof.* Let $v_1, v_2, \ldots, v_n$ be the vertices of graph $G$. Then the $A(G)$ is given by

$$A(G) = \begin{array}{c} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \vdots \\ \boldsymbol{v_n} \end{array} \begin{array}{c} \begin{array}{ccccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \cdots & \boldsymbol{v_n} \end{array} \\ \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{bmatrix} \end{array}.$$

Consider a second copy of graph $G$ with vertices $u_1, u_2, u_3, \ldots, u_n$ and join $u_i$ with neighbors of $v_i$ and with $v_i$, $1 \leq i \leq n$, to obtain $D_2^*(G)$. Then the $A(D_2^*(G))$ can be written as a block matrix as follows

$$A(D_2^*(G)) = \begin{array}{c} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \vdots \\ \boldsymbol{v_n} \\ \\ \boldsymbol{u_1} \\ \boldsymbol{u_2} \\ \boldsymbol{u_3} \\ \vdots \\ \boldsymbol{u_n} \end{array} \begin{array}{c} \begin{array}{cccccccccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \cdots & \boldsymbol{v_n} & \boldsymbol{u_1} & \boldsymbol{u_2} & \boldsymbol{u_3} & \cdot & \boldsymbol{u_n} \end{array} \\ \left[\begin{array}{ccccc:ccccc} 0 & a_{12} & a_{13} & \cdots & a_{1n} & 1 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & a_{2n} & a_{21} & 1 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & 0 & \cdots & a_{3n} & a_{31} & a_{32} & 1 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 & a_{n1} & a_{n2} & a_{n3} & \cdots & 1 \\ \hdashline 1 & a_{12} & a_{13} & \cdots & a_{1n} & 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 1 & a_{23} & \cdots & a_{2n} & a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & 1 & \cdots & a_{3n} & a_{31} & a_{32} & 0 & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 1 & a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{array}\right] \end{array}$$

That is,

$$A(D_2^*(G)) = \begin{bmatrix} A(G) & A(G)+I \\ A(G)+I & A(G) \end{bmatrix}.$$

Hence, by Proposition 1.2 spectrum of $D_2^*(G)$ is union of spectra of $2A(G)+I$ and $-I$. Therefore, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of $G$, then

$$\operatorname{spec}(D_2^*(G)) = \begin{pmatrix} 2\lambda_i + 1 & -1 \\ n & n \end{pmatrix}.$$

Suppose that $|\lambda_i| \geq \frac{1}{2}$ for all $1 \leq i \leq n$, then

$$\left|\lambda_i + \frac{1}{2}\right| = \begin{cases} |\lambda_i| + \frac{1}{2}, & \text{if } \lambda_i > 0, \\ |\lambda_i| - \frac{1}{2}, & \text{if } \lambda_i < 0. \end{cases}$$

Here,

$$
\begin{aligned}
E(D_2^*(G)) &= \sum_{i=1}^{n} |2\lambda_i + 1| + \sum_{i=1}^{n} |-1| \\
&= 2\sum_{i=1}^{n} \left|\lambda_i + \frac{1}{2}\right| + n \\
&= 2\left(\sum_{\lambda_i>0} \left|\lambda_i + \frac{1}{2}\right| + \sum_{\lambda_i<0} \left|\lambda_i + \frac{1}{2}\right|\right) + n \\
&= 2\left(\sum_{\lambda_i>0} \left(|\lambda_i| + \frac{1}{2}\right) + \sum_{\lambda_i<0} \left(|\lambda_i| - \frac{1}{2}\right)\right) + n \\
&= 2\left(\left(\sum_{\lambda_i>0} |\lambda_i| + \sum_{\lambda_i<0} |\lambda_i|\right) + \frac{1}{2}\left(\sum_{\lambda_i>0} 1 - \sum_{\lambda_i<0} 1\right)\right) + n \\
&= 2E(G) + n + \theta.
\end{aligned}
$$

□

The following corollary proves the existence of borderenergetic graph of arbitrarily large order.

**Corollary 2.1.** $E(D_2^*(K_{n,n})) = E(K_{4n})$. *That is,* $D_2^*(K_{n,n})$ *is non complete borderenergetic graph.*

*Proof.* Consider complete bipartite graph $K_{n,n}$ of $2n$ vertices then

$$
\operatorname{spec}(K_{n,n}) = \begin{pmatrix} n & -n & 0 \\ 1 & 1 & 2n-2 \end{pmatrix}.
$$

Now, $D_2^*(K_{n,n})$ is a graph with $4n$ vertices and by Theorem 2.1 its spectrum is

$$
\operatorname{spec}(D_2^*(K_{n,n})) = \begin{pmatrix} 2n+1 & -2n+1 & 1 & -1 \\ 1 & 1 & 2n-2 & 2n \end{pmatrix}. \tag{2.1}
$$

Also,

$$
\operatorname{spec}(K_{4n}) = \begin{pmatrix} 4n-1 & -1 \\ 1 & 4n-1 \end{pmatrix}. \tag{2.2}
$$

Clearly from (2.1) and (2.2) $D_2^*(K_{n,n})$ and $K_{4n}$ are non co-spectral and

$$
\begin{aligned}
E(D_2^*(K_{n,n})) &= \sum_{i=1}^{4n} |\lambda_i| \\
&= (2n+1) + (2n-1) + (2n-2) + 2n \\
&= 8n - 2 = 2(4n-1) = E(K_{4n}).
\end{aligned}
$$

Thus, $E(D_2^*(K_{n,n})) = E(K_{4n})$. Hence, $D_2^*(K_{n,n})$ is non complete borderenergetic graph. □

## 3. Equienergetic Graphs

**Definition 3.1.** Two non-isomorphic graphs $G_1$ and $G_2$ of same order are said to be *equienergetic* if $E(G_1) = E(G_2)$.

In 2007 Ramane et al. have proved that there exists a pair of connected non-cospectral, equienergetic graphs with $n$ vertices for all $n \geq 9$.

**Definition 3.2.** The *line graph* $L(G)$ of a graph G is the graph whose vertex set is $E(G)$ and two vertices are adjacent in $L(G)$ whenever they are incident in $G$.

Harary [8] defined the concept of iterated line graphs. According to him if $G$ is graph and $L^1(G) = L(G)$ be its line graph, then $L^2(G) = L(L(G))$, $L^3(G) = L(L^2(G)), \dots, L^k(G) = L(L^{k-1}(G)), \dots$

Ramane et al. [14] have proved that if $G_1$ and $G_2$ are regular graphs of same order, then for $k \geq 2$, $L^k(G_1)$ and $L^k(G_2)$, $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ are equienergetic.

**Definition 3.3.** The *cartesian product* of graphs $G$ and $H$ is a graph, denoted as $G \times H$, whose vertex set is $V(G) \times V(H)$. Two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent if $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$.

The following result gives the spectra of the Cartesian product of graphs.

**Proposition 3.1** ([1])**.** *Let $G_1$ and $G_2$ are two graphs having spectra as $\mu_1, \mu_2, \dots, \mu_{n_1}$ and $\sigma_1, \sigma_2, \dots, \sigma_{n_2}$, respectively. Then spectra of $G = G_1 \times G_2$ is $\mu_i + \sigma_j$, where $i = 1, 2, \dots, n_1$ and $j = 1, 2, \dots, n_2$.*

**Theorem 3.1.** *Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of graph $G$. Then $D_2^*(G \times K_2)$ and $D_2(D_2^*(G))$ are noncospectral equienergetic if $|\lambda_i| \geq \frac{3}{2}$ for $1 \leq i \leq n$.*

*Proof.* Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be eigenvalues of graph $G$. By Proposition 3.1

$$\operatorname{spec}(G \times K_2) = \begin{pmatrix} \lambda_1 + 1 & \lambda_2 + 1 & \cdots & \lambda_n + 1 & \lambda_1 - 1 & \lambda_2 - 1 & \cdots & \lambda_n - 1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

According to Theorem 2.1,

$$\operatorname{spec}(D_2^*(G \times K_2)) = \begin{pmatrix} 2\lambda_1 + 3 & \cdots & 2\lambda_n + 3 & 2\lambda_1 - 1 & \cdots & 2\lambda_n - 1 & -1 \\ 1 & \cdots & 1 & 1 & \cdots & 1 & 2n \end{pmatrix}. \tag{3.1}$$

Moreover, by Theorem 2.1,

$$\operatorname{spec}(D_2^*(G)) = \begin{pmatrix} 2\lambda_1 + 1 & 2\lambda_2 + 1 & \cdots & 2\lambda_n + 1 & -1 \\ 1 & 1 & \cdots & 1 & n \end{pmatrix}.$$

By Proposition 1.1,

$$\operatorname{spec}(D_2(D_2^*(G))) = \begin{pmatrix} 4\lambda_1 + 2 & 4\lambda_2 + 2 & \cdots & 4\lambda_n + 2 & -2 & 0 \\ 1 & 1 & \cdots & 1 & n & 2n \end{pmatrix}. \tag{3.2}$$

If for all $1 \leq i \leq n$, $|\lambda_i| \geq \frac{3}{2}$, then

$$\left|\lambda_i + \frac{3}{2}\right| = \begin{cases} |\lambda_i| + \frac{3}{2}, & \text{if } \lambda_i > 0, \\ |\lambda_i| - \frac{3}{2}, & \text{if } \lambda_i < 0. \end{cases}$$

Also for all $1 \leq i \leq n$, $|\lambda_i| \geq \frac{3}{2} > \frac{1}{2}$,

$$\left|\lambda_i + \frac{1}{2}\right| = \begin{cases} |\lambda_i| + \frac{1}{2} & \text{if } \lambda_i > 0, \\ |\lambda_i| - \frac{1}{2} & \text{if } \lambda_i < 0, \end{cases}$$

$$\left|\lambda_i - \frac{1}{2}\right| = \begin{cases} |\lambda_i| - \frac{1}{2} & \text{if } \lambda_i > 0, \\ |\lambda_i| + \frac{1}{2} & \text{if } \lambda_i < 0. \end{cases}$$

From (3.1) and (3.2)

$$\begin{aligned}
E(D_2^*(G \times K_2)) &= \sum_{i=1}^{n} |2\lambda_i + 3| + \sum_{i=1}^{n} |2\lambda_i - 1| + 2n \\
&= 2\sum_{i=1}^{n} |\lambda_i + \frac{3}{2}| + 2\sum_{i=1}^{n} |\lambda_i - \frac{1}{2}| + 2n \\
&= 2\left(\sum_{\lambda_i>0} \left|\lambda_i + \frac{3}{2}\right| + \sum_{\lambda_i<0} \left|\lambda_i + \frac{3}{2}\right| + \sum_{\lambda_i>0} \left|\lambda_i - \frac{1}{2}\right| + \sum_{\lambda_i<0} \left|\lambda_i - \frac{1}{2}\right|\right) + 2n \\
&= 2\left(\sum_{\lambda_i>0} \left(|\lambda_i| + \frac{3}{2}\right) + \sum_{\lambda_i<0} \left(|\lambda_i| - \frac{3}{2}\right)\right) \\
&+ 2\left(\sum_{\lambda_i>0} \left(|\lambda_i| - \frac{1}{2}\right) + \sum_{\lambda_i<0} \left(|\lambda_i| + \frac{1}{2}\right)\right) + 2n \\
&= 2\left(2\left(\sum_{\lambda_i>0} |\lambda_i| + \sum_{\lambda_i<0} |\lambda_i|\right) + \left(\sum_{\lambda_i>0} 1 - \sum_{\lambda_i>0} 1\right)\right) + 2n \\
&= 4E(G) + 2\theta + 2n \qquad (3.3)
\end{aligned}$$

and

$$\begin{aligned}
E(D_2(D_2^*(G))) &= \sum_{i=1}^{n} |4\lambda_i + 2| + 2n \\
&= 4\sum_{i=1}^{n} |4\lambda_i + \frac{1}{2}| + 2n \\
&= 4\left(\sum_{\lambda_i>0} \left|\lambda_i + \frac{1}{2}\right| + \sum_{\lambda_i<0} \left|\lambda_i + \frac{1}{2}\right|\right) + 2n \\
&= 4\left(\sum_{\lambda_i>0} \left(|\lambda_i| + \frac{1}{2}\right) + \sum_{\lambda_i<0} \left(|\lambda_i| - \frac{1}{2}\right)\right) + 2n
\end{aligned}$$

$$= 4\left(\left(\sum_{\lambda_i>0}|\lambda_i| + \sum_{\lambda_i<0}|\lambda_i|\right) + \frac{1}{2}\left(\sum_{\lambda_i>0}1 - \sum_{\lambda_i<0}1\right)\right) + 2n$$

$$= 4E(G) + 2\theta + 2n. \tag{3.4}$$

Hence, from (5) and (6), $D_2^*(G \times K_2)$ and $D_2(D_2^*(G))$ are noncospectral equienergetic if $|\lambda_i| \geq \frac{3}{2}$ for $1 \leq i \leq n$ □

Let $D_2^{**}(G)$ be extended shadow graph of $D_2^*(G)$, i.e., $D_2^{**}(G) = D_2^*(D_2^*(G))$ and if $G$ be a bipartite graph, then it is well-known that the spectra of $G$ is symmetric about the origin, so half of the non-zero eigenvalues of G lies to the left and half lies to the right of the origin. Therefore if $G$ is a bipartite graph having all its eigenvalues nonzero, the number of positive and negative eigenvalues of $G$ are same. Keeping this into mind we have the following result.

**Theorem 3.2.** *Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of a bipartite graph $G$. Then $D_2^{**}(G)$ and $D_2^*(D_2(G))$ are noncospectral equienergetic if and only if $|\lambda_i| \geq \frac{3}{4}$ for $1 \leq i \leq n$.*

*Proof.* Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of bipartite graph $G$. By Theorem 2.1

$$\operatorname{spec}(D_2^*(G)) = \begin{pmatrix} 2\lambda_1+1 & 2\lambda_2+1 & \cdots & 2\lambda_n+1 & -1 \\ 1 & 1 & \cdots & 1 & n \end{pmatrix}$$

and

$$\operatorname{spec}(D_2^{**}(G)) = \begin{pmatrix} 4\lambda_1+3 & 4\lambda_2+3 & \cdots & 4\lambda_n+3 & -1 \\ 1 & 1 & \cdots & 1 & 3n \end{pmatrix}. \tag{3.5}$$

Moreover, by Proposition 1.1,

$$\operatorname{spec}(D_2(G)) = \begin{pmatrix} 2\lambda_1 & 2\lambda_2 & \cdots & 2\lambda_n & 0 \\ 1 & 1 & \cdots & 1 & n \end{pmatrix}.$$

By Theorem 2.1,

$$\operatorname{spec}(D_2^*(D_2(G))) = \begin{pmatrix} 4\lambda_1+1 & 4\lambda_2+1 & \cdots & 4\lambda_n+1 & 1 & -1 \\ 1 & 1 & \cdots & 1 & n & 2n \end{pmatrix}. \tag{3.6}$$

Clearly, from (3.5) and (3.6), $D_2^{**}(G)$ and $D_2^*(D_2(G))$ are non-co spectral graphs. As $G$ is bipartite graph we have,

$$\sum_{\lambda_i>0} 1 = \sum_{\lambda_i<0} 1.$$

Assume that for all $1 \leq i \leq n$, $|\lambda_i| \geq \frac{3}{4}$. Hence,

$$\left|\lambda_i + \frac{3}{4}\right| = \begin{cases} |\lambda_i| + \frac{3}{4}, & \text{if } \lambda_i > 0, \\ |\lambda_i| - \frac{3}{4}, & \text{if } \lambda_i < 0. \end{cases}$$

From (3.5)

$$E(D_2^{**}(G)) = \sum_{i=1}^{n}|4\lambda_i+3| + \sum_{i=1}^{3n}|-1|$$

$$
\begin{aligned}
&= 4\sum_{i=1}^{n}\left|\lambda_i + \frac{3}{4}\right| + 3n\\
&= 4\left(\sum_{\lambda_i>0}\left|\lambda_i + \frac{3}{4}\right| + \sum_{\lambda_i<0}\left|\lambda_i + \frac{3}{4}\right|\right) + 3n\\
&= 4\left(\sum_{\lambda_i>0}\left(|\lambda_i| + \frac{3}{4}\right) + \sum_{\lambda_i<0}\left(|\lambda_i| - \frac{3}{4}\right)\right) + 3n\\
&= 4\left(\left(\sum_{\lambda_i>0}|\lambda_i| + \sum_{\lambda_i<0}|\lambda_i|\right) + \frac{3}{4}\left(\sum_{\lambda_i>0}1 - \sum_{\lambda_i<0}1\right)\right) + 3n\\
&= 4E(G) + 3n. \qquad (3.7)
\end{aligned}
$$

Also if for all $1 \leq i \leq n$, $|\lambda_i| \geq \frac{3}{4} \geq \frac{1}{4}$, then

$$
\left|\lambda_i + \frac{1}{4}\right| = \begin{cases} |\lambda_i| + \frac{1}{4}, & \text{if } \lambda_i > 0,\\ |\lambda_i| - \frac{1}{4}, & \text{if } \lambda_i < 0. \end{cases}
$$

From (3.6)

$$
\begin{aligned}
E(D_2^*(D_2(G))) &= \sum_{i=1}^{n}|4\lambda_i + 1| + n + 2n\\
&= 4\sum_{i=1}^{n}\left|\lambda_i + \frac{1}{4}\right| + 3n\\
&= 4\left(\sum_{\lambda_i>0}\left|\lambda_i + \frac{1}{4}\right| + \sum_{\lambda_i<0}\left|\lambda_i + \frac{1}{4}\right|\right) + 3n\\
&= 4\left(\sum_{\lambda_i>0}\left(|\lambda_i| + \frac{1}{4}\right) + \sum_{\lambda_i<0}\left(|\lambda_i| - \frac{1}{4}\right)\right) + 3n\\
&= 4\left(\left(\sum_{\lambda_i>0}|\lambda_i| + \sum_{\lambda_i<0}|\lambda_i|\right) + \frac{1}{4}\left(\sum_{\lambda_i>0}1 - \sum_{\lambda_i<0}1\right)\right) + 3n\\
&= 4E(G) + 3n. \qquad (3.8)
\end{aligned}
$$

Thus, by (3.7) and (3.8), $D_2^{**}(G)$ and $D_2^*(D_2(G))$ are equienergetic graphs.

Conversely, suppose that the graphs $D_2^{**}(G)$ and $D_2^*(D_2(G))$ are noncospectral equienergetic. We will show that $|\lambda_i| \geq \frac{3}{4}$ for $1 \leq i \leq n$.

Assume to the contrary that let $|\lambda_i| < \frac{3}{4}$ for some $i$. Then for the same $i$, $\left|\lambda_i + \frac{3}{4}\right| = \lambda_i + \frac{3}{4}$. Without loss of generality, suppose that the eigenvalues of $G$ satisfy $|\lambda_i| \geq \frac{3}{4}$, for $i = 1, 2, \ldots, k$ and $|\lambda_i| < \frac{3}{4}$, for $i = k+1, k+2, \ldots, n$, since the eigenvalues are real and reordering does not affect the argument. We have the following cases to be considered.

*Case I* If $\lambda_i > 0$ for $i = 1, 2, \ldots, k$ and $\lambda_i \geq 0$ for $i = k+1, k+2, \ldots, n$,

$$\begin{aligned}
E(D_2^{**}(G)) &= \sum_{i=1}^{k} |4\lambda_i + 3| + \sum_{i=k+1}^{n} |4\lambda_i + 3| + \sum_{i=1}^{3n} |-1| \\
&= 4\left(\sum_{i=1}^{k} \left|\lambda_i + \frac{3}{4}\right| + \sum_{i=k+1}^{n} \left|\lambda_i + \frac{3}{4}\right|\right) + 3n \\
&= 4\left(\sum_{i=1}^{k} \left(|\lambda_i| + \frac{3}{4}\right) + \sum_{i=k+1}^{n} \left(|\lambda_i| + \frac{3}{4}\right)\right) + 3n \\
&= 4\left(\sum_{i=1}^{n} |\lambda_i| + \frac{3}{4}\sum_{i=1}^{n} 1\right) + 3n \\
&= 4\sum_{i=1}^{n} |\lambda_i| + 6n. \qquad (3.9)
\end{aligned}$$

*Case II* If $\lambda_i > 0$ for $i = 1, 2, \cdots, k$ and $\lambda_i \leq 0$ for $i = k+1, k+2, \cdots, n$. If $\theta_0$ is the number of zero eigenvalues of $G$, we have

$$\begin{aligned}
E(D_2^{**}(G)) &= \sum_{i=1}^{k} |4\lambda_i + 3| + \sum_{i=k+1}^{n} |4\lambda_i + 3| + 3n \\
&= 4\left(\sum_{i=1}^{k} \left|\lambda_i + \frac{3}{4}\right| + \sum_{i=k+1}^{n} \left|\lambda_i + \frac{3}{4}\right|\right) + 3n \\
&= 4\left(\sum_{i=1}^{k} \left(|\lambda_i| + \frac{3}{4}\right) + \sum_{i=k+1}^{n} \left(\lambda_i + \frac{3}{4}\right)\right) + 3n \\
&> 4\left(\sum_{i=1}^{k} \left(|\lambda_i| + \frac{3}{4}\right) + \sum_{i=k+1}^{n} \left(|\lambda_i| - \frac{3}{4}\right)\right) + 3n \\
&= 4\left(\sum_{i=1}^{n} |\lambda_i| - \frac{3}{4}\theta_0\right) + 3n. \qquad (3.10)
\end{aligned}$$

*Case III* If $\lambda_i < 0$ for $i = 1, 2, \ldots, k$ and $\lambda_i \geq 0$ for $i = k+1, k+2, \ldots, n$,

$$\begin{aligned}
E(D_2^{**}(G)) &= \sum_{i=1}^{k} |4\lambda_i + 3| + \sum_{i=k+1}^{n} |4\lambda_i + 3| + \sum_{i=1}^{3n} |-1| \\
&= 4\left(\sum_{i=1}^{k} \left|\lambda_i + \frac{3}{4}\right| + \sum_{i=k+1}^{n} \left|\lambda_i + \frac{3}{4}\right|\right) + 3n \\
&= 4\left(\sum_{i=1}^{k} \left(|\lambda_i| - \frac{3}{4}\right) + \sum_{i=k+1}^{n} \left(|\lambda_i| + \frac{3}{4}\right)\right) + 3n \\
&= 4\left(\sum_{i=1}^{n} |\lambda_i| + \frac{3}{4}\theta_0\right) + 3n. \qquad (3.11)
\end{aligned}$$

*Case IV* If $\lambda_i < 0$ for $i = 1, 2, \dots, k$ and $\lambda_i \leq 0$ for $i = k+1, k+2, \dots, n$,

$$\begin{aligned}
E(D_2^{**}(G)) &= \sum_{i=1}^{k} |4\lambda_i + 3| + \sum_{i=k+1}^{n} |4\lambda_i + 3| + \sum_{i=1}^{3n} |-1| \\
&= 4\left(\sum_{i=1}^{k} \left|\lambda_i + \frac{3}{4}\right| + \sum_{i=k+1}^{n} \left|\lambda_i + \frac{3}{4}\right|\right) + 3n \\
&> 4\left(\sum_{i=1}^{k} \left(|\lambda_i| - \frac{3}{4}\right) + \sum_{i=k+1}^{n} \left(|\lambda_i| - \frac{3}{4}\right)\right) + 3n \\
&= 4\left(\sum_{i=1}^{n} |\lambda_i| - \frac{3}{4}n\right) + 3n. \qquad (3.12)
\end{aligned}$$

While

$$E(D_2^{*}(D_2(G))) = 4\sum_{i=1}^{n} |\lambda_i| + 3n,$$

which remain same in each of the above cases only if $|\lambda_i| \geq \frac{1}{4}$ for $i = k+1, k+2, \dots, n$.

If $|\lambda_i| < \frac{1}{4}$ for $i = k+1, k+2, \dots, n$, then we have the following.

*Case I* If $\lambda_i > 0$ for $i = 1, 2, \dots, k$ and $\lambda_i \geq 0$ for $i = k+1, k+2, \dots, n$,

$$\begin{aligned}
E(D_2^{*}(D_2(G))) &= \sum_{i=1}^{k} |4\lambda_i + 1| + \sum_{i=k+1}^{n} |4\lambda_i + 1| + 3n \\
&= 4\left(\sum_{i=1}^{k} \left|\lambda_i + \frac{1}{4}\right| + \sum_{i=k+1}^{n} \left|\lambda_i + \frac{1}{4}\right|\right) + 3n \\
&= 4\left(\sum_{i=1}^{k} \left(|\lambda_i| + \frac{1}{4}\right) + \sum_{i=k+1}^{n} \left(|\lambda_i| + \frac{1}{4}\right)\right) + 3n \\
&= 4\left(\sum_{i=1}^{n} |\lambda_i| + \frac{1}{4}\sum_{i=1}^{n} 1\right) + 3n \\
&= 4\sum_{i=1}^{n} |\lambda_i| + 4n. \qquad (3.13)
\end{aligned}$$

*Case II* If $\lambda_i > 0$ for $i = 1, 2, \dots, k$ and $\lambda_i \leq 0$ for $i = k+1, k+2, \dots, n$, and if $\theta_0$ is the number of zero eigenvalues of $G$, then we have

$$\begin{aligned}
E(D_2^{*}(D_2(G))) &= \sum_{i=1}^{k} |4\lambda_i + 1| + \sum_{i=k+1}^{n} |4\lambda_i + 1| + 3n \\
&= 4\left(\sum_{i=1}^{k} \left|\lambda_i + \frac{1}{4}\right| + \sum_{i=k+1}^{n} \left|\lambda_i + \frac{1}{4}\right|\right) + 3n \\
&= 4\left(\sum_{i=1}^{k} \left(|\lambda_i| + \frac{1}{4}\right) + \sum_{i=k+1}^{n} \left(\lambda_i + \frac{1}{4}\right)\right) + 3n
\end{aligned}$$

$$> 4\left(\sum_{i=1}^{k}\left(|\lambda_i| + \frac{1}{4}\right) + \sum_{i=k+1}^{n}\left(|\lambda_i| - \frac{1}{4}\right)\right) + 3n$$

$$= 4\left(\sum_{i=1}^{n}|\lambda_i| - \frac{1}{4}\theta_0\right) + 3n. \tag{3.14}$$

*Case III* If $\lambda_i < 0$ for $i = 1, 2, \ldots, k$ and $\lambda_i \geq 0$ for $i = k+1, k+2, \ldots, n$,

$$\begin{aligned}
E(D_2^*(D_2(G))) &= \sum_{i=1}^{k}|4\lambda_i + 1| + \sum_{i=k+1}^{n}|4\lambda_i + 1| + 3n \\
&= 4\left(\sum_{i=1}^{k}\left|\lambda_i + \frac{1}{4}\right| + \sum_{i=k+1}^{n}\left|\lambda_i + \frac{1}{4}\right|\right) + 3n \\
&= 4\left(\sum_{i=1}^{k}\left(|\lambda_i| - \frac{1}{4}\right) + \sum_{i=k+1}^{n}\left(|\lambda_i| + \frac{1}{4}\right)\right) + 3n \\
&= 4\left(\sum_{i=1}^{n}|\lambda_i| + \frac{1}{4}\theta_0\right) + 3n.
\end{aligned}$$

*Case IV* If $\lambda_i < 0$ for $i = 1, 2, \ldots, k$ and $\lambda_i \leq 0$ for $i = k+1, k+2, \ldots, n$,

$$\begin{aligned}
E(D_2^*(D_2(G))) &= \sum_{i=1}^{k}|4\lambda_i + 1| + \sum_{i=k+1}^{n}|4\lambda_i + 1| + 3n \\
&= 4\left(\sum_{i=1}^{k}\left|\lambda_i + \frac{1}{4}\right| + \sum_{i=k+1}^{n}\left|\lambda_i + \frac{1}{4}\right|\right) + 3n \\
&> 4\left(\sum_{i=1}^{k}\left(|\lambda_i| - \frac{1}{4}\right) + \sum_{i=k+1}^{n}\left(|\lambda_i| - \frac{1}{4}\right)\right) + 3n
\end{aligned}$$

$$= 4\left(\sum_{i=1}^{n}|\lambda_i| - \frac{1}{4}n\right) + 3n. \tag{3.15}$$

Clearly, in all the cases discussed above, we have $E(D_2^{**}(G)) \neq E(D_2^*(D_2(G)))$, a contradiction. Hence, the result follows. □

**Corollary 3.1.** *If $G_1$ and $G_2$ are two equienergetic bipartite graphs with $|\lambda_i| \geq \frac{1}{2}$ and $|\mu_i| \geq \frac{1}{2}$, where $\lambda_i$ and $\mu_i$ are the eigenvalues of $G_1$ and $G_2$, respectively, for all $1 \leq i \leq n$, then $D_2^*(G_1)$ and $D_2^*(G_2)$ are non cospectral equienergetic.*

*Proof.* Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\mu_1, \mu_2, \ldots, \mu_n$ be eigenvalues of $G_1$ and $G_2$, respectively. Then by Theorem 2.1 spectrum of $G_1$ and $G_2$ are given by

$$\operatorname{spec}(D_2^*(G_1)) = \begin{pmatrix} 2\lambda_i + 1 & -1 \\ n & n \end{pmatrix}, \quad \operatorname{spec}(D_2^*(G_2)) = \begin{pmatrix} 2\mu_i + 1 & -1 \\ n & n \end{pmatrix}.$$

Suppose that $|\lambda_i| \geq \frac{1}{2}$ for $i = 1, 2, \ldots, n$. Then

$$\left|\lambda_i + \frac{1}{2}\right| = \begin{cases} |\lambda_i| + \frac{1}{2}, & \text{if } \lambda_i > 0, \\ |\lambda_i| - \frac{1}{2}, & \text{if } \lambda_i < 0. \end{cases}$$

Here,

$$\begin{aligned} E(D_2^*(G_1)) &= \sum_{i=1}^{n} |2\lambda_i + 1| + \sum_{i=1}^{n} |-1| \\ &= 2\sum_{i=1}^{n} \left|\lambda_i + \frac{1}{2}\right| + n \\ &= 2\left(\sum_{\lambda_i>0} \left|\lambda_i + \frac{1}{2}\right| + \sum_{\lambda_i<0} \left|\lambda_i + \frac{1}{2}\right|\right) + n \\ &= 2\left(\sum_{\lambda_i>0} \left(|\lambda_i| + \frac{1}{2}\right) + \sum_{\lambda_i<0} \left(|\lambda_i| - \frac{1}{2}\right)\right) + n \\ &= 2\left(\left(\sum_{\lambda_i>0} |\lambda_i| + \sum_{\lambda_i<0} |\lambda_i|\right) + \frac{1}{2}\left(\sum_{\lambda_i>0} 1 - \sum_{\lambda_i<0} 1\right)\right) + n. \end{aligned}$$

As $G_1$ is bipartite graph

$$\sum_{\lambda_i>0} 1 = \sum_{\lambda_i<0} 1.$$

Hence, $E(D_2^*(G_1)) = 2E(G_1) + n$.

Similarly, if $|\mu_i| \geq \frac{1}{2}$ for all $1 \leq i \leq n$, then

$$E(D_2^*(G_2)) = 2E(G_2) + n.$$

Since, $G_1$ and $G_2$ are equienergetic graphs $D_2^*(G_1)$ and $D_2^*(G_1)$ are equienergetic. $\square$

## 4. Extended m-Shadow Graph and Graph Energy

**Definition 4.1.** The *m-shadow graph* $D_m(G)$ of a connected graph G is constructed by taking $m$ copies of $G$, say $G_1, G_2, \ldots, G_m$, then join each vertex $u$ in $G_i$ to the neighbors of the corresponding vertex $v$ in $G_j$, $1 \leq i, j \leq m$. Vaidya and Popat [18] have proved that $E(D_m(G)) = mE(G)$.

**Definition 4.2.** The extended *m-shadow graph* $D_m^*(G)$ of a connected graph G is constructed by taking $m$ copies of $G$, say $G_1, G_2, \ldots, G_m$, then join each vertex $u$ in $G_i$ to the neighbors of the corresponding vertex $v$ and with $v$ in $G_j$, $1 \leq i, j \leq m$.

**Definition 4.3.** Let $A \in R^{m\times n}$, $B \in R^{p\times q}$. Then the *Kronecker product* (or tensor product) of A and B is defined as the matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

**Proposition 4.1** ([9]). *Let $A \in M^m$ and $B \in M^n$. Furthermore, let $\lambda$ is an eigenvalues of matrix $A$ with corresponding eigenvector $x$ and $\mu$ is an eigenvalue of matrix $B$ with corresponding eigenvector $y$. Then $\lambda\mu$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector $x \otimes y$.*

**Theorem 4.1.** *Let $G$ be a graph with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ with $|\lambda_i| \geq \frac{m-1}{m}$, for all $1 \leq i \leq n$. Then $E(D_m^*(G)) = mE(G) + (m-1)n + (m-1)\theta$, where $\theta$ is the difference between the number of positive and negative eigenvalues of $G$.*

*Proof.* Let $v_1, v_2, \dots, v_n$ be the vertices of the graph $G$. Then its adjacency matrix of $G$ is same as in the proof of Theorem 2.2. Consider $m$ copies of graph $G$ say $G_1, G_2, \dots, G_k$ with vertices $v_i^1, v_i^2, \dots, v_i^m$, $1 \leq i \leq n$, to obtain $D_m^*(G)$ such that each vertex $u$ in $G_j$ is joined to the neighbors of the corresponding vertex $v$ as well as with $v$ in $G_k$, $1 \leq j, k \leq m$. Then the $A(D_m^*(G))$ can be written as a block matrix as follow

$$A(D_m^*(G)) = \begin{bmatrix} A(G) & A(G)+I & \cdots & A(G)+I \\ A(G)+I & A(G) & \cdots & A(G)+I \\ \vdots & \vdots & \ddots & \vdots \\ A(G)+I & A(G)+I & \cdots & A(G) \end{bmatrix}_m,$$

$$\begin{aligned} A(D_m^*(G)) + I_{mn} &= \begin{bmatrix} A(G)+I & A(G)+I & \cdots & A(G)+I \\ A(G)+I & A(G)+I & \cdots & A(G)+I \\ \vdots & \vdots & \ddots & \vdots \\ A(G)+I & A(G)+I & \cdots & A(G)+I \end{bmatrix}_m \\ &= \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}_m \otimes (A(G)+I) \\ &= J_m \otimes (A(G)+I). \end{aligned}$$

Hence, by Proposition 4.1, if $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of $G$, then

$$\begin{aligned} \operatorname{spec}(D_m^*(G)+I) &= \begin{pmatrix} m(\lambda_i+1) & 0(\lambda_i+1) \\ n & mn-n \end{pmatrix} \\ &= \begin{pmatrix} m(\lambda_i+1) & 0 \\ n & mn-n \end{pmatrix}, \\ \operatorname{spec}(D_m^*(G)) &= \begin{pmatrix} m\lambda_i + (m-1) & -1 \\ n & mn-n \end{pmatrix}. \end{aligned}$$

Suppose that $|\lambda_i| \geq \frac{m-1}{m}$ for all $1 \leq i \leq n$. Then

$$\left|\lambda_i + \frac{m-1}{m}\right| = \begin{cases} |\lambda_i| + \frac{m-1}{m}, & \text{if } \lambda_i > 0,\\ |\lambda_i| - \frac{m-1}{m}, & \text{if } \lambda_i < 0.\end{cases}$$

Here,

$$\begin{aligned}
E(D_m^*(G)) &= \sum_{i=1}^{n} |m\lambda_i + (m-1)| + \sum_{i=1}^{(m-1)n} |-1|\\
&= m\sum_{i=1}^{n}\left|\lambda_i + \frac{m-1}{m}\right| + (m-1)n\\
&= m\left(\sum_{\lambda_i>0}\left|\lambda_i + \frac{m-1}{m}\right| + \sum_{\lambda_i<0}\left|\lambda_i + \frac{m-1}{m}\right|\right) + (m-1)n\\
&= m\left(\sum_{\lambda_i>0}\left(|\lambda_i| + \frac{m-1}{m}\right) + \sum_{\lambda_i<0}\left(|\lambda_i| - \frac{m-1}{m}\right)\right) + (m-1)n\\
&= m\left(\left(\sum_{\lambda_i>0}|\lambda_i| + \sum_{\lambda_i<0}|\lambda_i|\right) + \frac{m-1}{m}\left(\sum_{\lambda_i>0}1 - \sum_{\lambda_i<0}1\right)\right) + (m-1)n\\
&= mE(G) + (m-1)n + (m-1)\theta. \qquad \square
\end{aligned}$$

## 5. Concluding Remarks

The energy of extended shadow graph has been obtained and using it a new family of non complete borderenergetic graphs and new pairs of non cospectral equienergetic graphs have been investigated.

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