

SOME PROPERTIES OF NEW HYPERGEOMETRIC FUNCTIONS IN FOUR VARIABLES

MAGED G. BIN-SAAD¹, JIHAD A. YOUNIS¹, AND KOTTAKKARAN S. NISAR^{2*}

ABSTRACT. In this paper, we introduce ten new quadruple hypergeometric series. We also obtain their various properties such that integral representations, fractional derivatives, N-fractional connections, operational relations and generating functions.

1. INTRODUCTION

In recent years, several interesting and useful properties of certain multiple hypergeometric functions have been investigated by many authors (see, e.g., [1, 3–9, 11, 12, 14, 15, 17, 21, 22, 25, 26]). In a sequel of such type of works mentioned above in this paper, we introduce ten new hypergeometric series of four variables as below

$$(1.1) \quad X_{70}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_2, c_3, c_4; x, y, z, u) \\
 = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+n+q}(a_3)_q}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!},$$

$$(1.2) \quad X_{71}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_1, c_2, c_3; x, y, z, u) \\
 = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+n+q}(a_3)_q}{(c_1)_{m+n}(c_2)_p(c_3)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!},$$

$$X_{72}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_2, c_1, c_3; x, y, z, u)$$

Key words and phrases. Gamma functions, Laplace-type integrals, fractional derivatives, N-fractional operator, operational relations, generating functions, Exton's functions, quadruple hypergeometric series.

2010 *Mathematics Subject Classification.* Primary: 33C05, 33C20, 33C45, 33C50, 33C65, 26A33.
 DOI

Received: August 22, 2020.

Accepted: March 09, 2021.

$$\begin{aligned}
(1.3) \quad &= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+n+q}(a_3)_q}{(c_1)_{m+p}(c_2)_n(c_3)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \\
&X_{73}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_2, c_1, c_1, c_3; x, y, z, u) \\
(1.4) \quad &= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+n+q}(a_3)_q}{(c_1)_{n+p}(c_2)_m(c_3)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \\
&X_{74}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_1, c_1, c_2; x, y, z, u) \\
(1.5) \quad &= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+n+q}(a_3)_q}{(c_1)_{m+n+p}(c_2)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \\
&X_{75}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_2, c_3, c_4; x, y, z, u) \\
(1.6) \quad &= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p+q}(a_3)_p(a_4)_q}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \\
&X_{76}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_1, c_2, c_3; x, y, z, u) \\
(1.7) \quad &= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p+q}(a_3)_p(a_4)_q}{(c_1)_{m+n}(c_2)_p(c_3)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \\
&X_{77}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_2, c_1, c_3; x, y, z, u) \\
(1.8) \quad &= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p+q}(a_3)_p(a_4)_q}{(c_1)_{m+p}(c_2)_n(c_3)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \\
&X_{78}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_2, c_1, c_1, c_3; x, y, z, u) \\
(1.9) \quad &= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p+q}(a_3)_p(a_4)_q}{(c_1)_{n+p}(c_2)_m(c_3)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!}, \\
&X_{79}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_1, c_1, c_2; x, y, z, u) \\
(1.10) \quad &= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p+q}(a_3)_p(a_4)_q}{(c_1)_{m+n+p}(c_2)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!},
\end{aligned}$$

where $(a)_m$ is the Pochhammer symbol defined by

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = a(a+1) \cdots (a+m-1),$$

for $m \geq 1$, $(a)_0 = 1$, Γ being the well-known Gamma function.

The present paper aims at introducing and investigating certain properties of hypergeometric series $X_{70}^{(4)}, X_{72}^{(4)}, \dots, X_{79}^{(4)}$. The structure of this paper is as follows. In Section 2, integral representations of Laplace-type are obtained. In Section 3, we establish some fractional derivatives for our series. Section 4 presents certain connections by means of N-fractional operator. Section 5 deals with the derivation of operational relations between the quadruple functions $X_{70}^{(4)}, X_{72}^{(4)}, \dots, X_{79}^{(4)}$ and triple

hypergeometric functions. The generating functions are given in the last section of this paper.

2. INTEGRAL REPRESENTATIONS OF LAPLACE-TYPE

In this section, we present certain integrals of Laplace-type involving the quadruple series $X_i^{(4)}$, $i = 70, 71, \dots, 79$. For our purpose, we begin by recalling the following confluent hypergeometric functions [23]:

$$(2.1) \quad {}_0F_1(-; c; x) = \sum_{m=0}^{\infty} \frac{1}{(c)_m} \cdot \frac{x^m}{m!},$$

$$(2.2) \quad {}_1F_1(a; c; x) = \sum_{m=0}^{\infty} \frac{(a)_m}{(c)_m} \cdot \frac{x^m}{m!},$$

$$(2.3) \quad \Phi_3(a; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_m}{(c)_{m+n}} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!},$$

$$(2.4) \quad \mathbf{H}_6(a; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}}{(c)_{m+n}} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!},$$

$$(2.5) \quad \mathbf{H}_7(a; b, c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}}{(b)_m(c)_n} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!}.$$

Now, if we consider the definitions of the confluent hypergeometric functions ${}_0F_1$, ${}_1F_1$, Φ_3 , \mathbf{H}_6 and \mathbf{H}_7 , we can derive the following integral representations:

$$(2.6) \quad \begin{aligned} & X_{70}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_2, c_3, c_4; x, y, z, u) \\ &= \frac{1}{\Gamma(a_2)} \int_0^{\infty} e^{-s} s^{a_2-1} \mathbf{H}_7(a_1; c_1, c_2; x, sy) {}_0F_1(-; c_3; s^2 z) {}_1F_1(a_3; c_4; su) ds, \\ & \operatorname{Re}(a_2) > 0, \end{aligned}$$

$$(2.7) \quad \begin{aligned} & X_{71}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_1, c_2, c_3; x, y, z, u) \\ &= \frac{1}{\Gamma(a_2)} \int_0^{\infty} e^{-s} s^{a_2-1} \mathbf{H}_6(a_1; c_1; x, sy) {}_0F_1(-; c_2; s^2 z) {}_1F_1(a_3; c_3; su) ds, \\ & \operatorname{Re}(a_2) > 0, \end{aligned}$$

$$(2.8) \quad \begin{aligned} & X_{72}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_2, c_1, c_3; x, y, z, u) \\ &= \frac{1}{\Gamma(a_1)\Gamma(a_2)} \int_0^{\infty} \int_0^{\infty} e^{-(s+t)} s^{a_1-1} t^{a_2-1} {}_0F_1(-; c_1; s^2 x + t^2 z) \\ & \times {}_0F_1(-; c_2; sty) {}_1F_1(a_3; c_3; tu) ds dt, \quad \operatorname{Re}(a_1) > 0, \operatorname{Re}(a_2) > 0, \end{aligned}$$

$$(2.9) \quad \begin{aligned} & X_{73}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_2, c_1, c_1, c_3; x, y, z, u) \\ &= \frac{1}{\Gamma(a_1)\Gamma(a_2)} \int_0^{\infty} \int_0^{\infty} e^{-(s+t)} s^{a_1-1} t^{a_2-1} {}_0F_1(-; c_1; sty + t^2 z) \\ & \times {}_0F_1(-; c_2; s^2 x) {}_1F_1(a_3; c_3; tu) ds dt, \quad \operatorname{Re}(a_1) > 0, \operatorname{Re}(a_2) > 0, \end{aligned}$$

$$\begin{aligned}
& X_{74}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_1, c_1, c_2; x, y, z, u) \\
&= \frac{1}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \int_0^\infty e^{-(s+t)} s^{a_1-1} t^{a_2-1} {}_0F_1(-; c_1; s^2x + sty + t^2z) \\
(2.10) \quad & \times {}_1F_1(a_3; c_2; tu) dsdt, \quad \operatorname{Re}(a_1) > 0, \operatorname{Re}(a_2) > 0,
\end{aligned}$$

$$\begin{aligned}
& X_{75}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_2, c_3, c_4; x, y, z, u) \\
&= \frac{1}{\Gamma(a_2)} \int_0^\infty e^{-s} s^{a_2-1} \mathbf{H}_7(a_1; c_1, c_2; x, sy) {}_1F_1(a_3; c_3; sz) {}_1F_1(a_4; c_4; su) ds, \\
(2.11) \quad & \operatorname{Re}(a_2) > 0,
\end{aligned}$$

$$\begin{aligned}
& X_{76}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_1, c_2, c_3; x, y, z, u) \\
&= \frac{1}{\Gamma(a_2)} \int_0^\infty e^{-s} s^{a_2-1} \mathbf{H}_6(a_1; c_1; x, sy) {}_1F_1(a_3; c_2; sz) {}_1F_1(a_4; c_3; su) ds, \\
(2.12) \quad & \operatorname{Re}(a_2) > 0,
\end{aligned}$$

$$\begin{aligned}
& X_{77}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_2, c_1, c_3; x, y, z, u) \\
&= \frac{1}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \int_0^\infty e^{-(s+t)} s^{a_1-1} t^{a_2-1} \Phi_3(a_3; c_1; tz, s^2x) {}_0F_1(-; c_2; sty) \\
(2.13) \quad & \times {}_1F_1(a_4; c_3; tu) dsdt, \quad \operatorname{Re}(a_1) > 0, \operatorname{Re}(a_2) > 0,
\end{aligned}$$

$$\begin{aligned}
& X_{78}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_2, c_1, c_1, c_3; x, y, z, u) \\
&= \frac{1}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty \int_0^\infty e^{-(s+t)} s^{a_1-1} t^{a_2-1} \Phi_3(a_3; c_1; tz, sty) {}_0F_1(-; c_2; s^2x) \\
(2.14) \quad & \times {}_1F_1(a_4; c_3; tu) dsdt, \quad \operatorname{Re}(a_1) > 0, \operatorname{Re}(a_2) > 0,
\end{aligned}$$

$$\begin{aligned}
& X_{79}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_1, c_1, c_2; x, y, z, u) \\
&= \frac{1}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \int_0^\infty \int_0^\infty \int_0^\infty e^{-(s+t+v)} s^{a_1-1} t^{a_2-1} v^{a_3-1} \\
& \quad \times {}_0F_1(-; c_1; s^2x + sty + tvz) {}_1F_1(a_4; c_2; tu) dsdt dv, \\
(2.15) \quad & \operatorname{Re}(a_1) > 0, \operatorname{Re}(a_2) > 0, \operatorname{Re}(a_3) > 0.
\end{aligned}$$

Proof. To establish (2.6), denote by \mathcal{J} the right side of the relation (2.6). Then, by substituting the expression of the confluent hypergeometric functions (2.1), (2.2) and (2.5) into the right hand side of (2.6), we have

$$\mathcal{J} = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_3)_q}{(c_1)_m(c_1)_n(c_1)_p(c_1)_q\Gamma(a_2)} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!} \int_0^\infty e^{-s} s^{a_2+2p+n+q-1} ds,$$

by using the known equality (see [23])

$$\Gamma(a) = \int_0^\infty e^{-s} s^{a-1} ds, \quad \operatorname{Re}(a) > 0,$$

we get the result after some simplifications. Similarly, one can proof the relations (2.6) to (2.15). \square

3. FRACTIONAL DERIVATIVES

The fractional derivative operator D_w^k that was introduced by Miller and Ross [16] is given as

$$(3.1) \quad D_w^k w^a = \frac{\Gamma(a+1)}{\Gamma(a-k+1)} w^{a-k}, \quad \operatorname{Re}(a) > -1.$$

Now, by using the above operator, we aim in this section at establishing the following fractional derivative formulae:

$$(3.2) \quad \begin{aligned} & D_w^{a_1-c} \left[w^{a_1-1} X_{70}^{(4)} \left(c, c, a_2, a_2, c, a_2, a_2, a_3; c_1, c_2, c_3, c_4; w^2 x, wy, z, u \right) \right] \\ &= \frac{\Gamma(a_1)}{\Gamma(c)} w^{c-1} X_{70}^{(4)} \left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_2, c_3, c_4; w^2 x, wy, z, u \right), \end{aligned}$$

$$(3.3) \quad \begin{aligned} & D_w^{a_2-c} \left[w^{a_2-1} X_{71}^{(4)} \left(a_1, a_1, c, c, a_1, c, c, a_3; c_1, c_1, c_2, c_3; x, wy, w^2 z, wu \right) \right] \\ &= \frac{\Gamma(a_2)}{\Gamma(c)} w^{c-1} X_{71}^{(4)} \left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_1, c_2, c_3; x, wy, w^2 z, wu \right), \end{aligned}$$

$$(3.4) \quad \begin{aligned} & D_{w_1}^{a_1-c} D_{w_2}^{a_2-c'} \left[w_1^{a_1-1} w_2^{a_2-1} X_{72}^{(4)} \left(c, c, c', c', c, c', c', a_3; c_1, c_2, c_1, c_3; w_1^2 x, w_1 w_2 y, \right. \right. \\ & \quad \left. \left. w_2^2 z, w_2 u \right) \right] \\ &= \frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(c)\Gamma(c')} w_1^{c-1} w_2^{c'-1} X_{72}^{(4)} \left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_2, c_1, c_3; w_1^2 x, w_1 w_2 y, \right. \end{aligned}$$

$$(3.5) \quad \begin{aligned} & \left. w_2^2 z, w_2 u \right), \\ & D_{w_1}^{a_1-c} D_{w_2}^{a_2-c'} \left[w_1^{a_1-1} w_2^{a_2-1} X_{73}^{(4)} \left(c, c, a_2, a_2, c, a_2, a_2, c'; c_2, c_1, c_1, c_3; w_1^2 x, w_1 y, z, w_2 u \right) \right] \\ &= \frac{\Gamma(a_1)\Gamma(a_3)}{\Gamma(c)\Gamma(c')} w_1^{c-1} w_2^{c'-1} X_{73}^{(4)} \left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_2, c_1, c_1, c_3; w_1^2 x, w_1 y, z, w_2 u \right), \end{aligned}$$

$$(3.6) \quad \begin{aligned} & D_{w_1}^{a_1-c} D_{w_2}^{a_2-c'} D_{w_3}^{a_3-c''} \left[w_1^{a_1-1} w_2^{a_2-1} w_3^{a_3-1} X_{74}^{(4)} \left(c, c, c', c', c, c', c', c''; c_1, c_1, c_1, c_2; w_1^2 x, \right. \right. \\ & \quad \left. \left. w_1 w_2 y, w_2^2 z, w_2 w_3 u \right) \right] \\ &= \frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}{\Gamma(c)\Gamma(c')\Gamma(c'')} w_1^{c-1} w_2^{c'-1} w_3^{c''-1} \end{aligned}$$

$$(3.6) \quad \begin{aligned} & \times X_{74}^{(4)} \left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_1, c_1, c_2; w_1^2 x, w_1 w_2 y, w_2^2 z, w_2 w_3 u \right), \\ & D_{w_1}^{a_3-c} D_{w_2}^{a_4-c'} \left[w_1^{a_3-1} w_2^{a_4-1} X_{75}^{(4)} \left(a_1, a_1, a_2, a_2, a_1, a_2, c, c'; c_1, c_2, c_3, c_4; x, y, w_1 z, w_2 u \right) \right] \end{aligned}$$

$$(3.7) \quad = \frac{\Gamma(a_3)\Gamma(a_4)}{\Gamma(c)\Gamma(c')} w_1^{c-1} w_2^{c'-1} X_{75}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_2, c_1, c_1, c_3; x, y, w_1 z, w_2 u),$$

$$D_w^{a_2-c} \left[w^{a_2-1} X_{76}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, c, c'; c_1, c_1, c_2, c_3; x, wy, wz, wu) \right]$$

$$(3.8) \quad = \frac{\Gamma(a_2)}{\Gamma(c)} w^{c-1} X_{76}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_2, c_1, c_1, c_3; x, wy, wz, wu),$$

$$(3.9) \quad D_{w_1}^{a_1-c} D_{w_2}^{a_2-c'} D_{w_3}^{a_3-c''} D_{w_4}^{a_4-c'''} \left[w_1^{a_1-1} w_2^{a_2-1} w_3^{a_3-1} w_4^{a_4-1} X_{77}^{(4)}(c, c, c', c', c, c', c'', c'''; c_1, c_2, c_1, c_3; w_1^2 x, w_1 w_2 y, w_2 w_3 z, w_2 w_4 u) \right]$$

$$= \frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}{\Gamma(c)\Gamma(c')\Gamma(c'')\Gamma(c''')} w_1^{c-1} w_2^{c'-1} w_3^{c''-1} w_4^{c'''-1}$$

$$(3.10) \quad \times X_{77}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_2, c_1, c_3; w_1^2 x, w_1 w_2 y, w_2 w_3 z, w_2 w_4 u),$$

$$D_{w_1}^{a_2-c} D_{w_2}^{a_3-c'} D_{w_3}^{a_4-c''} \left[w_1^{a_2-1} w_2^{a_3-1} w_3^{a_4-1} X_{78}^{(4)}(a_1, a_1, c, c, a_1, c, c', c''; c_2, c_1, c_1, c_3; x, w_1 y, w_1 w_2 z, w_1 w_3 u) \right]$$

$$= \frac{\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}{\Gamma(c)\Gamma(c')\Gamma(c'')} w_1^{c-1} w_2^{c'-1} w_3^{c''-1}$$

$$(3.11) \quad \times X_{78}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_2, c_1, c_1, c_3; x, w_1 y, w_1 w_2 z, w_1 w_3 u),$$

$$D_{w_1}^{a_1-c} D_{w_2}^{a_2-c'} \left[w_1^{a_1-1} w_2^{a_2-1} X_{79}^{(4)}(c, c, c', c', c, c', a_3, a_4; c_1, c_1, c_1, c_2; w_1^2 x, w_1 w_2 y, w_2 z, w_2 u) \right]$$

$$= \frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(c)\Gamma(c')} w_1^{c-1} w_2^{c'-1}$$

$$(3.12) \quad \times X_{79}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_1, c_1, c_2; w_1^2 x, w_1 w_2 y, w_2 z, w_2 u).$$

Proof. We have

$$D_w^{a_1-c} \left[w^{a_1-1} X_{70}^{(4)}(c, c, a_2, a_2, c, a_2, a_2, a_3; c_1, c_2, c_3, c_4; w^2 x, wy, z, u) \right]$$

$$= \sum_{m,n,p,q=0}^{\infty} \frac{(c)_{2m+n} (a_2)_{2p+n+q} (a_3)_q}{(c_1)_m (c_2)_n (c_3)_p (c_4)_q} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!} D_w^{a_1-c} w^{a_1+2m+n-1}.$$

Now, with the help of (3.1) and Definition 1.1, the proof of the first fractional derivative formula is completed. The proofs of the assertions (3.3) to (3.12) run parallel to that of the assertion (3.2) then we skip the details. \square

4. N-FRACTIONAL CONNECTIONS

First, by recalling the N-fractional operator due to Bin-Saad [4]:

$$(4.1) \quad \mathcal{M}_w^{a,c,b} = [w^{a-1}(1-w)^{-b}]_{a-c} = \frac{\Gamma(a-c)}{2\pi i} \int_C \frac{\eta^{a-1}(1-\eta)^{-b}}{(\eta-z)^{a-c}} d\eta,$$

where $a, b, c \in \mathbb{C}$ and $(a-c) \notin \mathbb{Z}$, we aim in this section to investigate the following relationships:

$$(4.2) \quad \begin{aligned} & \mathcal{M}_u^{a,c,b} X_{12} \left(a_1, b; c_1, c_2, c_3; x, \frac{y}{(1-u)}, \frac{z}{(1-u)^2} \right) \\ &= AX_{70}^{(4)} (a_1, a_1, b, b, a_1, b, b, a; c_1, c_2, c_3, c; x, y, z, u), \end{aligned}$$

$$(4.3) \quad \begin{aligned} & \mathcal{M}_u^{a,c,b} X_{10} \left(a_1, b; c_1, c_2; x, \frac{y}{(1-u)}, \frac{z}{(1-u)^2} \right) \\ &= AX_{71}^{(4)} (a_1, a_1, b, b, a_1, b, b, a; c_1, c_1, c_2, c; x, y, z, u), \end{aligned}$$

$$(4.4) \quad \begin{aligned} & \mathcal{M}_u^{a,c,b} X_{11} \left(a_1, b; c_1, c_2; x, \frac{y}{(1-u)}, \frac{z}{(1-u)^2} \right) \\ &= AX_{72}^{(4)} (a_1, a_1, b, b, a_1, b, b, a; c_1, c_2, c_1, c; x, y, z, u), \end{aligned}$$

$$(4.5) \quad \begin{aligned} & \mathcal{M}_u^{a,c,b} X_{10} \left(b, a_1; c_1, c_2; \frac{x}{(1-u)^2}, \frac{y}{(1-u)}, z \right) \\ &= AX_{73}^{(4)} (a_1, a_1, b, b, a_1, b, b, a; c_2, c_1, c_1, c; z, y, x, u), \end{aligned}$$

$$(4.6) \quad \begin{aligned} & \mathcal{M}_u^{a,c,b} X_9 \left(a_1, b; c_1; x, \frac{y}{(1-u)}, \frac{z}{(1-u)^2} \right) \\ &= AX_{74}^{(4)} (a_1, a_1, b, b, a_1, b, b, a; c_1, c_1, c_1, c; x, y, z, u), \end{aligned}$$

$$(4.7) \quad \begin{aligned} & \mathcal{M}_u^{a,c,b} X_{17} \left(a_1, b, a_2; c_1, c_2, c_3; x, \frac{y}{(1-u)}, \frac{z}{(1-u)} \right) \\ &= AX_{75}^{(4)} (a_1, a_1, b, b, a_1, b, a_2, a; c_1, c_2, c_3, c; x, y, z, u), \end{aligned}$$

$$(4.8) \quad \begin{aligned} & \mathcal{M}_u^{a,c,b} X_{14} \left(a_1, b, a_2; c_1, c_2; x, \frac{y}{(1-u)}, \frac{z}{(1-u)} \right) \\ &= AX_{76}^{(4)} (a_1, a_1, b, b, a_1, b, a_2, a; c_1, c_1, c_2, c; x, y, z, u), \end{aligned}$$

$$(4.9) \quad \begin{aligned} & \mathcal{M}_u^{a,c,b} X_{16} \left(a_1, b, a_2; c_1, c_2; x, \frac{y}{(1-u)}, \frac{z}{(1-u)} \right) \\ &= AX_{77}^{(4)} (a_1, a_1, b, b, a_1, b, a_2, a; c_1, c_2, c_1, c; x, y, z, u), \end{aligned}$$

$$\mathcal{M}_u^{a,c,b} X_{15} \left(a_1, b, a_2; c_2, c_1; x, \frac{y}{(1-u)}, \frac{z}{(1-u)} \right)$$

$$(4.10) \quad = AX_{78}^{(4)}(a_1, a_1, b, b, a_1, b, a_2, a; c_2, c_1, c_1, c; x, y, z, u),$$

$$\mathcal{M}_u^{a,c,b} X_{13} \left(a_1, b, a_2; c_1; x, \frac{y}{(1-u)}, \frac{z}{(1-u)} \right)$$

$$(4.11) \quad = AX_{79}^{(4)}(a_1, a_1, b, b, a_1, b, a_2, a; c_1, c_1, c_1, c; x, y, z, u),$$

where $A = e^{-\pi i(a-c)} \frac{\Gamma(1-c)}{\Gamma(1-a)} u^{c-1}$ and $X_9, X_{10}, \dots, X_{17}$ are Exton's hypergeometric functions of three variables [10] defined by

$$(4.12) \quad X_9(a_1, a_2; c; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n} (a_2)_{n+2p}}{(c)_{m+n+p}} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!},$$

$$(4.13) \quad X_{10}(a_1, a_2; c_1, c_2; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n} (a_2)_{n+2p}}{(c_1)_{m+n} (c_2)_p} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!},$$

$$(4.14) \quad X_{11}(a_1, a_2; c_1, c_2; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n} (a_2)_{n+2p}}{(c_1)_{m+p} (c_2)_n} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!},$$

$$(4.15) \quad X_{12}(a_1, a_2; c_1, c_2, c_3; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n} (a_2)_{n+2p}}{(c_1)_m (c_2)_n (c_3)_p} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!},$$

$$(4.16) \quad X_{13}(a_1, a_2, a_3; c; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n} (a_2)_{n+p} (a_3)_p}{(c)_{m+n+p}} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!},$$

$$(4.17) \quad X_{14}(a_1, a_2, a_3; c_1, c_2; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n} (a_2)_{n+p} (a_3)_p}{(c_1)_{m+n} (c_2)_p} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!},$$

$$(4.18) \quad X_{15}(a_1, a_2, a_3; c_2, c_1; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n} (a_2)_{n+p} (a_3)_p}{(c_1)_{n+p} (c_2)_m} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!},$$

$$(4.19) \quad X_{16}(a_1, a_2, a_3; c_1, c_2; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n} (a_2)_{n+p} (a_3)_p}{(c_1)_{m+p} (c_2)_n} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!},$$

$$(4.20) \quad X_{17}(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n} (a_2)_{n+p} (a_3)_p}{(c_1)_m (c_2)_n (c_3)_p} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!}.$$

Proof. To prove (4.2), from the equality (4.15), we can write

$$\begin{aligned} & \mathcal{M}_u^{a,c,b} X_{12} \left(a_1, b; c_1, c_2, c_3; x, \frac{y}{(1-u)}, \frac{z}{(1-u)^2} \right) \\ &= \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n} (b)_{n+2p}}{(c_1)_m (c_2)_n (c_3)_p} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \mathcal{M}_u^{a,c,b} (1-u)^{-(n+2p)}. \end{aligned}$$

By applying the formula (4.1) and in view of the relation (1.1) one can get the result with direct calculations. The proofs of the remaining relations run in the same way. \square

5. OPERATIONAL RELATIONS

Here, in this section, we shall discuss some operational relations by means of the following operational formulas (see [3, 20]):

$$(5.1) \quad D_{\alpha}^k \alpha^a = \frac{\Gamma(a+1)}{\Gamma(a-k+1)} \alpha^{a-k},$$

$$(5.2) \quad D_{\alpha}^{-k} \alpha^a = \frac{\Gamma(a+1)}{\Gamma(a+k+1)} \alpha^{a+k},$$

$k \in \mathbb{N} \cup \{0\}$, $a \in \mathbb{C} - \{-1, -2, \dots\}$, where D_{α} denotes the derivative operator and D_{α}^{-1} denotes the inverse of the derivative.

In the following, certain operational connections among the hypergeometric series of three and four variables as:

$$(5.3) \quad \left[1 - \left(D_{\alpha}^2 \beta^{-1} D_{\beta}^{-1} \gamma^{-1} D_{\gamma}^{-1} \alpha^2\right) u\right]^{-a} X_8(a_1, a_2, a_3; c_1, c_2, c_3; x, \alpha y, z) \left(\alpha^{a_2-1} \beta^{c_4-1} \gamma^{a-1}\right) \\ = \alpha^{a_2-1} \beta^{c_4-1} \gamma^{a-1} X_{70}^{(4)}(a_2, a_2, a_1, a_1, a_2, a_1, a_1, a_3; c_4, c_2, c_1, c_3; u, \alpha y, x, z),$$

$$(5.4) \quad \left[1 - \left(D_{\alpha}^2 \beta^{-1} D_{\beta}^{-1} \gamma^{-1} D_{\gamma}^{-1} \alpha^2\right) u\right]^{-a} X_{14}(a_1, a_2, a_3; c_1, c_2; x, \alpha y, \alpha z) \left(\alpha^{a_2-1} \beta^{c_3-1} \gamma^{a-1}\right) \\ = \alpha^{a_2-1} \beta^{c_3-1} \gamma^{a-1} X_{71}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_3; c_1, c_1, c_3, c_2; x, \alpha y, u, \alpha z),$$

$$(5.5) \quad \left[1 - \left(D_{\alpha_1} D_{\alpha_2} \beta^{-1} D_{\beta}^{-1} \gamma^{-1} D_{\gamma}^{-1} \alpha_1 \alpha_2\right) u\right]^{-a} X_{20}\left(a_1, a_2, \frac{a_3}{2}, \frac{a_3+1}{2}; c_1, c_2; \alpha_1^2 x, \alpha_1 y, 4\alpha_2^2 z\right) \left(\alpha_1^{a_1-1} \alpha_2^{a_3-1} \beta^{c_3-1} \gamma^{a-1}\right) \\ = \alpha_1^{a_1-1} \alpha_2^{a_3-1} \beta^{c_3-1} \gamma^{a-1} X_{72}^{(4)}(a_3, a_3, a_1, a_1, a_3, a_1, a_1, a_2; c_1, c_3, c_1, c_2; \alpha_2^2 z, u, \alpha_1^2 x, \alpha_1 y),$$

$$(5.6) \quad \left[1 - \left(D_{\alpha}^2 \beta^{-1} D_{\beta}^{-1} \gamma^{-1} D_{\gamma}^{-1} \alpha^2\right) u\right]^{-a} X_6(a_1, a_2, a_3; c_1, c_2; x, \alpha y, z) \left(\alpha^{a_2-1} \beta^{c_3-1} \gamma^{a-1}\right) \\ = \alpha^{a_2-1} \beta^{c_3-1} \gamma^{a-1} X_{73}^{(4)}(a_2, a_2, a_1, a_1, a_2, a_1, a_1, a_3; c_3, c_1, c_1, c_2; u, \alpha y, x, z),$$

$$(5.7) \quad \left[1 - \left(D_{\alpha_1} D_{\alpha_2} \beta^{-1} D_{\beta}^{-1} \gamma^{-1} D_{\gamma}^{-1} \alpha_1 \alpha_2\right) u\right]^{-a} X_{20}\left(a_1, a_2, \frac{a_3}{2}, \frac{a_3+1}{2}; c_1, c_2; \alpha_1^2 \beta x, \alpha_1 y, 4\alpha_2^2 \beta z\right) \left(\alpha_1^{a_1-1} \alpha_2^{a_3-1} \beta^{c_1-1} \gamma^{a-1}\right) \\ = \alpha_1^{a_1-1} \alpha_2^{a_3-1} \beta^{c_1-1} \gamma^{a-1} \\ \times X_{74}^{(4)}(a_3, a_3, a_1, a_1, a_3, a_1, a_1, a_2; c_1, c_1, c_1, c_2; \alpha_2^2 \beta z, u, \alpha_1^2 \beta x, \alpha_1 y),$$

$$\begin{aligned}
(5.8) \quad & \left[1 - \left(D_{\alpha_1} D_{\alpha_2} \beta^{-1} D_{\beta}^{-1} \gamma^{-1} D_{\gamma}^{-1} \alpha_1 \alpha_2 \right) u \right]^{-a} X_{17} (a_1, a_2, a_3; c_1, c_2, c_3; x, \alpha_1 y, \alpha_1 z) \\
& \times \left(\alpha_1^{a_2-1} \alpha_2^{a_4-1} \beta^{c_4-1} \gamma^{a-1} \right) \\
= & \alpha_1^{a_2-1} \alpha_2^{a_4-1} \beta^{c_4-1} \gamma^{a-1} X_{75}^{(4)} (a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_2, c_3, c_4; x, \alpha_1 y, \alpha_1 z, u), \\
& \left[1 - \left(D_{\alpha}^2 \beta^{-1} D_{\beta}^{-1} \gamma^{-1} D_{\gamma}^{-1} \alpha^2 \right) u \right]^{-a} F(3)_A (a_1, a_2, a_3, a_4; c_1, c_2, c_3; \alpha \beta x, y, z) \\
& \times \left(\alpha^{a_2-1} \beta^{c_1-1} \gamma^{a-1} \right) \\
= & \alpha^{a_2-1} \beta^{c_1-1} \gamma^{a-1} X_{76}^{(4)} (a_2, a_2, a_1, a_1, a_2, a_1, a_3, a_4; c_1, c_1, c_2, c_3; u, \alpha \beta x, y, z),
\end{aligned}$$

$$\begin{aligned}
(5.9) \quad & \left[1 - \left(D_{\alpha_1} D_{\alpha_2} \beta^{-1} D_{\beta}^{-1} \gamma^{-1} D_{\gamma}^{-1} \alpha_1 \alpha_2 \right) u \right]^{-a} X_{16} (a_1, a_2, a_3; c_1, c_2; x, \alpha_1 y, \alpha_1 z) \\
& \times \left(\alpha_1^{a_2-1} \alpha_2^{a_4-1} \beta^{c_3-1} \gamma^{a-1} \right) \\
= & \alpha_1^{a_2-1} \alpha_2^{a_4-1} \beta^{c_3-1} \gamma^{a-1} X_{77}^{(4)} (a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_2, c_1, c_3; x, \alpha_1 y, \alpha_1 z, u),
\end{aligned}$$

$$\begin{aligned}
(5.10) \quad & \left[1 - \left(D_{\alpha}^2 \beta^{-1} D_{\beta}^{-1} \gamma^{-1} D_{\gamma}^{-1} \alpha^2 \right) u \right]^{-a} F_G (a_1, a_1, a_1, a_2, a_3, a_4; c_1, c_2, c_2; x, \alpha y, z) \\
& \times \left(\alpha^{a_3-1} \beta^{c_3-1} \gamma^{a-1} \right) \\
= & \alpha^{a_3-1} \beta^{c_3-1} \gamma^{a-1} X_{78}^{(4)} (a_3, a_3, a_1, a_1, a_3, a_1, a_4, a_2; c_3, c_2, c_2, c_1; u, \alpha y, z, x),
\end{aligned}$$

$$\begin{aligned}
(5.11) \quad & \left[1 - \left(D_{\alpha_1} D_{\alpha_2} \beta^{-1} D_{\beta}^{-1} \gamma^{-1} D_{\gamma}^{-1} \alpha_1 \alpha_2 \right) u \right]^{-a} \\
& \times F_N \left(a_1, \frac{a_2}{2}, a_3, a_4, \frac{a_2+1}{2}, a_4; c_1, c_2, c_2; \alpha_2 x, 4\alpha_1^2 \beta y, \alpha_2 \beta z \right) \left(\alpha_1^{a_2-1} \alpha_2^{a_4-1} \beta^{c_2-1} \gamma^{a-1} \right) \\
= & \alpha_1^{a_2-1} \alpha_2^{a_4-1} \beta^{c_2-1} \gamma^{a-1} \\
& \times X_{79}^{(4)} (a_2, a_2, a_4, a_4, a_2, a_4, a_3, a_1; c_2, c_2, c_2, c_1; \alpha_1^2 \beta y, u, \alpha_2 \beta z, \alpha_2 x),
\end{aligned}$$

where X_6 , X_8 and X_{20} are the Exton's triple hypergeometric series defined by [10]

$$(5.12) \quad X_6 (a_1, a_2, a_3; c_1, c_2; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n+p} (a_2)_n (a_3)_p}{(c_1)_{m+n} (c_2)_p} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!},$$

$$(5.13) \quad X_8 (a_1, a_2, a_3; c_1, c_2, c_3; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n+p} (a_2)_n (a_3)_p}{(c_1)_m (c_2)_n (c_3)_p} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!},$$

$$(5.14) \quad X_{20} (a_1, a_2, a_3, a_4; c_1, c_2; x, y, z) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n} (a_2)_n (a_3)_p (a_4)_p}{(c_1)_{m+p} (c_2)_n} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!},$$

and $F_A^{(3)}$, F_G , F_N denote the Lauricella's series of three variables (see [13]).

Proof. To solve equation (5.3), first we assume the left hand side of (5.3) by the notation \mathcal{J} , then expressing the Exton's function X_8 as a series in the left hand side of (5.3) and using the binomial theorem, it follows that:

$$\begin{aligned} \mathcal{J} = & \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n+p}(a_2)_n(a_3)_p(a)_q}{(c_1)_m(c_2)_n(c_3)_p} \cdot \frac{x^m}{m!} \cdot \frac{y^n}{n!} \cdot \frac{z^p}{p!} \cdot \frac{u^q}{q!} \\ & \times \beta^{-q}\gamma^{-q}D_{\alpha}^{2q}D_{\beta}^{-q}D_{\gamma}^{-q} \left(\alpha^{a_2+n+2q-1}\beta^{c_4-1}\gamma^{a-1} \right). \end{aligned}$$

Now, we use the above formulas in (5.1) and (5.2), then in view of Definition 1.1, we arrive at the desired result (5.3). In a similar manner, one can prove the relations (5.4) to (5.11). \square

6. GENERATING FUNCTIONS

In this section, we will consider some generating functions for our quadruple series. Because the proofs of the following relations are similar to the proofs of results in [2, 18, 19, 23, 24], we omit these proofs.

The generating relations of series $X_{70}^{(4)}, X_{72}^{(4)}, \dots, X_{79}^{(4)}$ given as below

$$\begin{aligned} (6.1) \quad & \sum_{k=0}^{\infty} \frac{(a_1)_k}{k!} X_{70}^{(4)}(a_1+k, a_1+k, a_2, a_2, a_1+k, a_2, a_2, a_3; c_1, c_2, c_3, c_4; x, y, z, u) t^k \\ & = (1-t)^{-a_1} X_{70}^{(4)} \left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_2, c_3, c_4; \frac{x}{(1-t)^2}, \frac{y}{(1-t)}, z, u \right), \end{aligned}$$

$$\begin{aligned} (6.2) \quad & \sum_{k=0}^{\infty} \frac{(a_2)_k}{k!} X_{71}^{(4)}(a_1, a_1, a_2+k, a_2+k, a_1, a_2+k, a_2+k, a_3; c_1, c_1, c_2, c_3; x, y, z, u) t^k \\ & = (1-t)^{-a_2} X_{71}^{(4)} \left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_1, c_2, c_3; x, \frac{y}{(1-t)}, \frac{z}{(1-t)^2}, \frac{u}{(1-t)} \right), \end{aligned}$$

$$\begin{aligned} (6.3) \quad & \sum_{k_1, k_2=0}^{\infty} \frac{(a_1)_{k_1}(a_2)_{k_2}}{k_1!k_2!} X_{72}^{(4)}(a_1+k_1, a_1+k_1, a_2+k_2, a_2+k_2, a_1+k_1, a_2+k_2, a_2+k_2, \\ & a_3; c_1, c_2, c_1, c_3; x, y, z, u) t_1^{k_1} t_2^{k_2} \\ & = (1-t_1)^{-a_1} (1-t_2)^{-a_2} X_{72}^{(4)} \left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_2, c_1, c_3; \frac{x}{(1-t_1)^2}, \right. \\ & \left. \frac{y}{(1-t_1)(1-t_2)}, \frac{z}{(1-t_2)^2}, \frac{u}{(1-t_2)} \right), \end{aligned}$$

$$(6.4) \quad \sum_{k_1, k_2=0}^{\infty} \frac{(a_1)_{k_1}(a_3)_{k_2}}{k_1!k_2!} X_{73}^{(4)}(a_1+k_1, a_1+k_1, a_2, a_2, a_1+k_1, a_2, a_2, a_3+k_2;$$

$$\begin{aligned}
& c_2, c_1, c_1, c_3; x, y, z, u) t_1^{k_1} t_2^{k_2} \\
& = (1-t_1)^{-a_1} (1-t_2)^{-a_3} X_{73}^{(4)} \left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_2, c_1, c_1, c_3; \frac{x}{(1-t_1)^2}, \right. \\
& \quad \left. \frac{y}{(1-t_1)}, z, \frac{u}{(1-t_2)} \right),
\end{aligned}$$

(6.5)

$$\begin{aligned}
& \sum_{k=0}^{\infty} \frac{(a_3)_k}{k!} X_{74}^{(4)} (a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3 + k; c_1, c_1, c_1, c_2; x, y, z, u) t^k \\
& = (1-t)^{-a_3} X_{74}^{(4)} \left(a_1, a_1, a_2, a_2, a_1, a_2, a_2, a_3; c_1, c_1, c_1, c_2; x, y, z, \frac{u}{(1-t)} \right),
\end{aligned}$$

(6.6)

$$\begin{aligned}
& \sum_{k=0}^{\infty} \frac{(a_2)_k}{k!} X_{75}^{(4)} (a_1, a_1, a_2 + k, a_2 + k, a_1, a_2 + k, a_3, a_4; c_1, c_2, c_3, c_4; x, y, z, u) t^k \\
& = (1-t)^{-a_2} X_{75}^{(4)} \left(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_2, c_3, c_4; x, \frac{y}{(1-t)}, \frac{z}{(1-t)}, \frac{u}{(1-t)} \right),
\end{aligned}$$

(6.7)

$$\begin{aligned}
& \sum_{k_1, k_2=0}^{\infty} \frac{(a_1)_{k_1} (a_2)_{k_2}}{k_1! k_2!} X_{76}^{(4)} (a_1 + k_1, a_1 + k_1, a_2 + k_2, a_2 + k_2, a_1 + k_1, a_2 + k_2, a_3, a_4; \\
& \quad c_1, c_1, c_2, c_3; x, y, z, u) t_1^{k_1} t_2^{k_2} \\
& = (1-t_1)^{-a_1} (1-t_2)^{-a_2} X_{76}^{(4)} \left(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_1, c_2, c_3; \frac{x}{(1-t_1)^2}, \right. \\
& \quad \left. \frac{y}{(1-t_1)(1-t_2)}, \frac{z}{(1-t_2)}, \frac{u}{(1-t_2)} \right),
\end{aligned}$$

(6.8)

$$\begin{aligned}
& \sum_{k_1, k_2=0}^{\infty} \frac{(a_3)_{k_1} (a_4)_{k_2}}{k_1! k_2!} X_{77}^{(4)} \left(a_1, a_1, a_2, a_2, a_1, a_2, a_3 + k_1, a_4 + k_2; c_1, c_2, c_1, c_3; \right. \\
& \quad \left. x, y, z, u \right) t_1^{k_1} t_2^{k_2} \\
& = (1-t_1)^{-a_3} (1-t_2)^{-a_4} X_{77}^{(4)} \left(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_1, c_2, c_1, c_3; x, y, \right. \\
& \quad \left. \frac{z}{(1-t_1)}, \frac{u}{(1-t_2)} \right),
\end{aligned}$$

(6.9)

$$\sum_{k_1, k_2, k_3=0}^{\infty} \frac{(a_1)_{k_1} (a_2)_{k_2} (a_3)_{k_3}}{k_1! k_2! k_3!} X_{78}^{(4)} (a_1 + k_1, a_1 + k_1, a_2 + k_2, a_2 + k_2, a_1 + k_1, a_2 + k_2,$$

$$\begin{aligned}
& a_3 + k_3, a_4; c_2, c_1, c_1, c_3; x, y, z, u) t_1^{k_1} t_2^{k_2} t_3^{k_3} \\
& = (1-t_1)^{-a_1} (1-t_2)^{-a_2} (1-t_3)^{-a_3} X_{78}^{(4)} \left(a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; c_2, c_1, c_1, c_3; \right. \\
& \quad \left. \frac{x}{(1-t_1)^2}, \frac{y}{(1-t_1)(1-t_2)}, \frac{z}{(1-t_2)(1-t_3)}, \frac{u}{(1-t_2)} \right), \\
(6.10) \quad & \sum_{k_1, k_2, k_3, k_4=0}^{\infty} \frac{(a_1)_{k_1} (a_2)_{k_2} (a_3)_{k_3} (a_4)_{k_4}}{k_1! k_2! k_3! k_4!} X_{79}^{(4)} (a_1 + k_1, a_1 + k_1, a_2 + k_2, a_2 + k_2, a_1 + k_1, \\
& \quad a_2 + k_2, a_3 + k_3, a_4 + k_4; c_1, c_1, c_1, c_2; x, y, z, u) t_1^{k_1} t_2^{k_2} t_3^{k_3} t_4^{k_4} \\
& = (1-t_1)^{-a_1} (1-t_2)^{-a_2} (1-t_3)^{-a_3} (1-t_4)^{-a_4} X_{79}^{(4)} (a_1, a_1, a_2, a_2, a_1, a_2, a_3, a_4; \\
& \quad c_1, c_1, c_1, c_2; \frac{x}{(1-t_1)^2}, \frac{y}{(1-t_1)(1-t_2)}, \frac{z}{(1-t_2)(1-t_3)}, \frac{u}{(1-t_2)(1-t_4)}).
\end{aligned}$$

Acknowledgements. The authors should express their deep gratitude to all the referees for their helpful and critical comments.

REFERENCES

- [1] P. Agarwal, J. Choi and S. Jain, *Extended hypergeometric functions of two and three variables*, Commun. Korean Math. Soc. **30** (2015), 403–414. <https://doi.org/10.4134/CKMS.2015.30.4.403>
- [2] D. Baleanu, P. Agarwal, R. K. Parmar, M. M. Alqurashi and S. Salahshour, *Extension of the fractional derivative operator of the Riemann-Liouville*, J. Nonlinear Sci. Appl. **10** (2017), 2914–2924. <http://dx.doi.org/10.22436/jnsa.010.06.06>
- [3] M. G. Bin-Saad, *Symbolic operational images and decomposition formulas of hypergeometric functions*, J. Math. Anal. Appl. **376**(2) (2011), 451–468. <https://doi.org/10.1016/j.jmaa.2010.10.073>
- [4] M. G. Bin-Saad, *Relations among certain generalized hypergeometric functions suggested by N-fractional calculus*, Math. Letters **2** (2016), 47–57. <https://doi.org/10.11648/j.ml.20160206.12>
- [5] M. G. Bin-Saad and J. A. Younis, *Certain integrals associated with hypergeometric functions of four variables*, Earthline J. Math. Sci. **2** (2019), 325–341. <https://doi.org/10.34198/ejms.2219.325341>
- [6] M. G. Bin-Saad and J. A. Younis, *Certain generating functions of some quadruple hypergeometric series*, Eurasian Bulletin Math. **2** (2019), 56–62.
- [7] M. G. Bin-Saad, J. A. Younis and R. Aktas, *Integral representations for certain quadruple hypergeometric series*, Far East J. Math. Sci. **103** (2018), 21–44.
- [8] M. G. Bin-Saad, J. A. Younis and R. Aktas, *New quadruple hypergeometric series and their integral representations*, Sarajevo Math. J. **14** (2018), 45–57. <https://doi.org/10.5644/SJM.14.1-05>
- [9] P. Deepthi, J. C. Prajapati and A. K. Rathie, *New Laplace transforms of the ${}_2F_2$ hypergeometric function*, J. Fract. Calc. Appl. **8** (2017), 150–155. <http://fcag-egypt.com/Journals/JFCA/>
- [10] H. Exton, *Hypergeometric functions of three variables*, J. Indian Acad. Math. **4** (1982), 113–119.

- [11] S. Jun, I. Kim and A. K. Rathie, *On a new class of Eulerian's type integrals involving generalized hypergeometric functions*, Aust. J. Math. Anal. Appl. **16** (2019), 1–15.
- [12] W. Koepf, I. Kim and A. K. Rathie, *On a new class of Laplace-type integrals involving generalized hypergeometric functions*, Axioms **8**(87) (2019), 21 pages. <https://doi.org/10.3390/axioms8030087>
- [13] G. Lauricella, *Sull funzioni ipergeometric a più variabili*, Rend. Cric. Mat. Palermo **7** (1893), 111–158. <https://doi.org/10.1007/BF03012437>
- [14] M.-J. Luo, G. V. Milovanović and P. Agarwal, *Some results on the extended beta and extended hypergeometric functions*, Appl. Math. Comput. **248**(2014), 631–651. <https://doi.org/10.1016/j.amc.2014.09.110>
- [15] M. Masjed-Jamei and W. Koepf, *Some summation theorems for generalized hypergeometric functions*, Axioms **7**(2) (2018), 20 pages. <https://doi.org/10.3390/axioms7020038>
- [16] K. S. Miller and B. Ross, *An Introduction to Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.
- [17] E. Özergin, *Some properties of hypergeometric functions*, Ph.D. Thesis, Eastern Mediterranean University, North Cyprus, 2011.
- [18] M. A. Özarslan and E. Özergin, *Some generating relations for extended hypergeometric functions via generalized fractional derivative operator*, Math. Comput. Model. **52** (2010), 1825–1833. <https://doi.org/10.1016/j.mcm.2010.07.011>
- [19] R. K. Parmar, *Some generating relations for generalized extended hypergeometric functions involving generalized fractional derivative operator*, J. Concr. Appl. Math. **12** (2014), 217–228.
- [20] B. Ross, *Fractional Calculus and its Applications*, Proceedings of the International Conference held at the University of New Haven, June 1974, Lecture Notes in Mathematics **457**, Springer-Verlag, 1975.
- [21] M. Shadab and J. Choi, *Extensions of Appell and Lauricella hypergeometric functions*, Far East J. Math. Sci. **102** (2017), 1301–1317. <http://dx.doi.org/10.17654/MS102061301>
- [22] M. Singh, S. Pundhir and M. P. Singh, *Generating function of certain hypergeometric functions by means of fractional calculus*, International J. Comput. Eng. Res. **7** (2017), 40–47.
- [23] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood Lt1., Chichester, 1984.
- [24] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press, Bristone, London, New York, Toronto, 1985.
- [25] H. M. Srivastava, A. Cetinkaya and I. O. Kiyamaz, *A certain generalized Pochhammer symbol and its applications to hypergeometric functions*, Appl. Math. Comput. **226** (2014), 484–491. <https://doi.org/10.1016/j.amc.2013.10.032>
- [26] J. A. Younis and M. G. Bin-Saad, *Integral representations involving new hypergeometric functions of four variables*, J. Frac. Calc. Appl. **10** (2019), 77–91. <http://math-frac.oreg/Journals/JFCA/>

¹DEPARTMENT OF MATHEMATICS,
ADEN UNIVERSITY,
ADEN, KHORMAKSAR, P.O.BOX 6014, YEMEN
Email address: mgbinsaad@yahoo.com

Email address: jihadalsaqqaf@gmail.com

²DEPARTMENT OF MATHEMATICS,
COLLEGE OF ARTS AND SCIENCE,
WADI ALDAWASER, PRINCE SATTAM BIN ABDULAZIZ UNIVERSITY, SAUDI ARABIA
Email address: sooppy@psau.edu.sa