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ANALYSIS OF A WEAK GALERKIN MIXED FORMULATION FOR MAXWELL'S EQUATIONS

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Abstract. In this paper we introduce and analyse a mixed weak Galekin finite element method for the Maxwell equations in the primary electric field-Lagrange multiplier. Our weak Galerkin method is equipped with stable finite elements composed of habitual polynomials of degree k for the electric field and polynomials of degree $k+1$ for the Lagrange multiplier. Optimal order error estimations for the proposed weak Galerkin mixed finite element formulation are demonstrated and are confirmed numerically on a two dimensional bounded domain.

1. Introduction

The idea of the weak Galerkin finite element method introduced by [13] consists in the approximation of the differential operators in the partial differential equation by weak forms as distributions over the space of discontinuous functions including boundary information. Compared to the discontinuous Galerkin methods [11, 16–19], the weak Galerkin methods also use discontinuous functions in the finite element procedure which gives a great flexibility to the WG-FEM in dealing with boundary conditions and different geometric complexities, while weak Galerkin methods require only weak continuity of variables through well-defined discrete differential operators and are absolutely stable when correctly constructed. Ever since it was introduced, the WG-method was used by several authors for the resolution of various partial differential equations such as linear parabolic problems [3, 20, 21], Helmoltz equations with large wave numbers [12] and elliptic interface problems [7, 8]. Recently, Lin

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Mu and his collaborators [9] construct a new WG-FEM which discretizes the second order elliptic equation in non-mixed form directly, and admit general finite element partitions consisting of arbitrary polytopal elements. The Weak Galerkin mixed finite element method is an extension of weak Galerkin finite element method [14] and it was used for the numerical resolution of partial differential equations [4, 6, 10, 15], In [14], a mixed WG-FEM has been introduced and analysed for the second order elliptic equation, in which the utilization of stabilization for the flux variable has an important role in the mixed formulation. In this paper, we are interested with the development of weak Galerkin mixed formulation for the Maxwell problem which consists in finding two unknown functions u and p such that

$$\nabla \times (\mu^{-1}\nabla \times u) - \varepsilon\nabla p = J, \quad \text{in } \Omega \subset \mathbb{R}^d, \tag{1.1}$$

$$\nabla \cdot (\varepsilon u) = 0, \quad \text{in } \Omega \subset \mathbb{R}^d, \tag{1.2}$$

$$n \times u = 0, \quad \text{on } \partial\Omega, \tag{1.3}$$

$$p = 0, \quad \text{on } \partial\Omega. \tag{1.4}$$

Here Ω is a bounded convex polygonal domain in $\mathbb{R}^2$ or a bounded polyhedral domain in $\mathbb{R}^3$ with boundary $\partial\Omega$. μ and ε denote the magnetic permeability and the electric permittivity of the medium and are assumed sufficiently smooth and in $L^\infty(\Omega)$. p is a Lagrange multiplier and u is related to the electric field E by the relation $E(x,t) = \text{Re}(u(x)\exp(i\omega t))$ with a given non zero frequency ω. Our goal in this paper is to introduce and study a mixed weak Galerkin finite element method for (1.1)–(1.4) that is potent and sturdy by allowing the use of discontinuous functions on finite element partitions consisting of arbitrary elements with certain shape regularity.

The organization of this paper is as follows. In the next section, we recall some notations in Sobolev spaces and we describe in detail our mixed weak Galerkin discrete scheme. In Section 3, we study the properties of the bilinear forms given in the formulation while in Section 4, we analyse the convergence of the proposed numerical formulation and prove some optimal error estimations. Section 5 is done for studying some numerical examples for confirming the proven theoretical results.

2. Preliminaries and Notations

2.1. **Meshes.** In this work, we use the standard notations for Sobolev spaces and their norms [5], such as, $H^s(\mathcal{O})^d$, $d = 1, 2, 3$, $\|\cdot\|_{s,\mathcal{O}} = \|\cdot\|_{H^s(\mathcal{O})^d}$ for a domain $\mathcal{O}$ and a positive integer or fractional regularity exponent s. The space $H_0(\nabla\times,\Omega)$ is the space of vector-valued functions $u \in L^2(\Omega)^d$ such that $\nabla \times u \in L^2(\Omega)^d$ and $n \times u = 0$ on the boundary of Ω. The space $H(\nabla_\varepsilon\cdot,\Omega)$ is the space of vector-valued functions in $L^2(\Omega)^d$ where $\nabla \cdot \varepsilon u \in L^2(\Omega)$.

Consider $\mathcal{T}_h$ be a shape-regular partition of Ω which consists of tetrahedra in $\mathbb{R}^3$ or triangles in $\mathbb{R}^2$. We denote by $\mathcal{E}_h^I$ the set of all interior faces or edges, $\mathcal{E}_h^D$ the set of all exterior faces or edges of the triangulation and we set $\mathcal{E}_h = \mathcal{E}_h^I \cup \mathcal{E}_h^D$. For any T in $\mathcal{T}_h$, we denote by h_T its diameter and $h = \max_{T\in\mathcal{T}_h} h_T$ the mesh size of the partition

$\mathcal{T}_h$. For $d = 1, 2, 3$, we introduce the piecewise Sobolev spaces as

$$H^s(\mathcal{T}_h)^d := \Big\{ v \in L^2(\Omega)^d : v_{|K} \in H^s(K)^d \text{ for all } K \in \mathcal{T}_h \Big\}.$$

2.2. **Weak Galerkin formulation.** First, we introduce the following two finite element spaces $\mathcal{V}_h$ and $\mathcal{W}_h$ for $k \geq 0$:

$$\mathcal{V}_h := \Big\{ v = \{v^0, v^b\} : v^0 \in [P_k(T)]^d,\ v^b \in [P_k(e)]^d,\ e \in \mathcal{E}_h,\ T \in \mathcal{T}_h \Big\}$$

and

$$\mathcal{W}_h := \Big\{ \psi \in L^2(\Omega) : \psi_{|T} \in P_{k+1}(T),\ \ T \in \mathcal{T}_h \Big\}.$$

Next, introduce the subspace of $\mathcal{V}_h$ as

$$\mathcal{V}_h^0 := \Big\{ v \in \mathcal{V}_h : v^b \times n = 0 \text{ on } \partial\Omega \Big\}$$

and the subspace of $\mathcal{W}_h$ as

$$\mathcal{W}_h^0 := \Big\{ \psi \in \mathcal{W}_h : \psi = 0 \text{ on } \partial\Omega \Big\}.$$

2.2.1. *Weak differential operators.* Let T be any element in $\mathcal{T}_h$ and v any function in $\mathcal{V}_h$, a weak divergence $\nabla_w \cdot v \in P_k(T)$ is defined as the unique polynomial satisfying

$$(\nabla_w \cdot v, \psi)_T = -(v^0, \nabla\psi)_T + \langle v^b \cdot n, \psi\rangle_{\partial T}, \quad \text{for all } \psi \in P_k(T) \tag{2.1}$$

and a weak curl $\nabla_w \times v \in [P_k(T)]^d$ is defined as the only polynomial satisfying

$$(\nabla_w \times v, w)_T = (v^0, \nabla \times w)_T - \langle v^b \times n, w\rangle_{\partial T}, \quad \text{for all } w \in [P_k(T)]^d. \tag{2.2}$$

With these two definitions, one can naively formulate a finite element discretization of the problem (1.1)-(1.4) as: Find $(u_h, p_h) \in \mathcal{V}_h^0 \times \mathcal{W}_h^0$ such that

$$\begin{aligned}
&\sum_{T\in\mathcal{T}_h} (\mu^{-1}\nabla_w \times u_h, \nabla_w \times v_h)_T + (\nabla_w \cdot \varepsilon u_h, \nabla_w \cdot \varepsilon v_h)_T + \sum_{T\in\mathcal{T}_h} (p_h, \nabla_w \cdot \varepsilon v_h)_T \\
&= \sum_{T\in\mathcal{T}_h} (J, v_h^0)_T, \quad \text{for all } v_h \in \mathcal{V}_h^0, \\
&\sum_{T\in\mathcal{T}_h} (\nabla_w \cdot \varepsilon u_h, \psi_h)_T = 0, \quad \text{for all } \psi_h \in \mathcal{W}_h^0.
\end{aligned}$$

This system does not have only one solution due to an insufficient enforcement of the components u_h^0 and u_h^b and we stabilize the bilinear form

$$\sum_{T\in\mathcal{T}_h} (\mu^{-1}\nabla_w \times u_h, \nabla_w \times v_h)_T + (\nabla_w \cdot \varepsilon u_h, \nabla_w \cdot \varepsilon v_h)_T,$$

by requiring some communications between u_h^0 and u_h^b. Hence, for $(u_h, v_h) \in \mathcal{V}_h^0 \times \mathcal{V}_h^0$ and $(v_h, p_h) \in \mathcal{V}_h^0 \times \mathcal{W}_h^0$, we define the following bilinear forms

$$\begin{aligned} a(u_h, v_h) &:= \sum_{T\in\mathcal{T}_h} (\mu^{-1}\nabla_w \times u_h, \nabla_w \times v_h)_T + (\nabla_w \cdot \varepsilon u_h, \nabla_w \cdot \varepsilon v_h)_T, \\ B(v_h, p_h) &:= \sum_{T\in\mathcal{T}_h} (p_h, \nabla_w \cdot \varepsilon v_h)_T, \\ s_T(u_h, v_h) &:= r \sum_{\partial T\in\mathcal{E}_h^I} h_T^{-1}\langle(\varepsilon u_h^0 - \varepsilon u_h^b)\cdot n, (\varepsilon v_h^0 - \varepsilon v_h^b)\cdot n\rangle_{\partial T} \\ &\quad + r \sum_{\partial T\in\mathcal{E}_h} h_T^{-1}\langle(u_h^0 - u_h^b)\times n, (v_h^0 - v_h^b)\times n\rangle_{\partial T}, \end{aligned}$$

where r is an arbitrary real parameter and assumed greater than zero. Next, for an approximate solution of (1.1)–(1.4), we find $u_h = \{u_h^0, u_h^b\} \in \mathcal{V}_h^0$, $p_h \in \mathcal{W}_h^0$ satisfying

$$A_s(u_h, v_h) + B(v_h, p_h) = \sum_{T\in\mathcal{T}_h} (J, v_h^0), \quad \text{for all } v_h \in \mathcal{V}_h^0, \tag{2.3}$$

$$B(u_h, \psi_h) = 0, \quad \text{for all } \psi_h \in \mathcal{W}_h^0, \tag{2.4}$$

where we have denoted by

$$A_s(u_h, v_h) := a(u_h, v_h) + s_T(u_h, v_h), \quad \text{for } (u_h, v_h) \in \mathcal{V}_h^0 \times \mathcal{V}_h^0.$$

Since our numerical scheme was given in (2.3)–(2.4), we first analyse its well posedness in the following theorem.

Theorem 2.1. *The mixed weak Galerkin scheme* (2.3)–(2.4) *is well posed and it has a unique solution* $(u_h, p_h) \in \mathcal{V}_h^0 \times \mathcal{W}_h^0$.

Proof. Take $J = 0$ in (2.3)–(2.4), then we have to prove that $u_h = 0$ and $p_h = 0$. Substituting $v = u_h$ and $\psi = p_h$ in (2.3)–(2.4) and substracting the second equation from the first and obtain $A_s(u_h, u_h) = 0$. It follows from the definition of $A_s(\cdot,\cdot)$ that $\nabla_w \times u_h = \nabla_w \cdot \varepsilon u_h = 0$ on each element $T \in \mathcal{T}_h$ and $u^0 \times n = u^b \times n$, $\varepsilon u^0 \cdot n = \varepsilon u^b \cdot n$ on each edge $e \in \mathcal{E}_h$. Therefore, from the definition of the weak curl operator and $\nabla_w \times u_h = 0$, one can obtain for any $w \in P_k(T)^d$,

$$\begin{aligned} 0 = (\nabla_w \times u, w)_T &= (u^0, \nabla \times w)_T - \langle u^b \times n, w\rangle_{\partial T} \\ &= (\nabla \times u^0, w)_T - \langle (u^0 - u^b) \times n, w\rangle_{\partial T} \\ &= (\nabla \times u^0, w)_T, \end{aligned}$$

which gives $\nabla \times u^0 = 0$ on each $T \in \mathcal{T}_h$. From the fact that $u^0 \times n = u^b \times n$ on each edge $e \in \mathcal{E}_h$ and $v^b \times n = 0$ on $\partial\Omega$ we deduce that $u^0 \in H_0(\nabla\times, \Omega)$ with $\nabla \times u^0 = 0$ in Ω. Similarly, since $\nabla_w \cdot \varepsilon u = 0$ on each $T \in \mathcal{T}_h$ and $\varepsilon u^0 \cdot n = \varepsilon u^b \cdot n$ on each edge $e \in \mathcal{E}_h$ we conclude that $u^0 \in H(\nabla_\varepsilon \cdot, \Omega)$ with $\nabla \cdot \varepsilon u^0 = 0$ and it follows that $u^0 = 0$ in Ω. Then, $u^b \times n = \varepsilon u^b \cdot n = 0$ and therefore $u^b = 0$ in $\mathcal{T}_h$. Next, using the definition of the bilinear form B, the weak divergence operator and the first equation in (2.3)–(2.4) we deduce also that $p_h = 0$ and this end the proof. □

3. Error Estimations

Let us start by introducing the local projection operators. Define $\mathbb{Q}_0$ the projection from $(L^2(T))^d$ to $(P_k(T))^d$, $\mathbb{Q}_b$ the projection from $(L^2(e))^d$ to $(P_k(e))^d$ on each elements $T \in \mathcal{T}_h$ and $e \in \mathcal{E}_h$, respectively. We denote by $\mathbb{Q}_h$ the L^2-projection of $v = \{v^0, v^b\} \in \mathcal{V}_h^0$ defined as $\mathbb{Q}_h := \{\mathbb{Q}_0(v^0), \mathbb{Q}_b(v^b)\}$ and for $p \in \mathcal{W}_h^0$, we denote by $Q_h(p)$ the local projection from $L^2(T)$ onto $P_{k+1}(T)$. In the following lemma, we introduce and prove some essential equations and results which we need for proving some error equations that are essential for the study of error estimations.

Lemma 3.1. *Let* (u, p) *be the solution of* (1.1)–(1.4)*, then*

$$\begin{aligned}
\nabla_w \cdot (\mathbb{Q}_h(u))_T =& Q_h(\nabla \cdot u)_T,\\
\nabla_w \times (\mathbb{Q}_h(u))_T =& \mathbf{Q}_h(\nabla \times u)_T,\\
(\nabla_w \times v, \mathbb{Q}_h(\varphi))_T =& (v^0, \nabla \times \varphi)_T + \langle (v^b - v^0) \times n, \varphi - \mathbb{Q}_h(\varphi)\rangle_{\partial T} - \langle v^b \times n, \varphi\rangle_{\partial T}.
\end{aligned}$$

Proof. From the definition of weak-divergence (2.1), $\mathbb{Q}_h$, Q_h, we have for $\psi \in P_k(T)$

$$\begin{aligned}
(\nabla_w \cdot \mathbb{Q}_h(u), \psi)_T =& -(\mathbb{Q}_0(u), \nabla\psi)_T + \langle \mathbb{Q}_b(u) \cdot n, \psi\rangle_{\partial T}\\
=& -(u, \nabla\psi)_T + \langle u \cdot n, \psi\rangle_{\partial T}\\
=& (\nabla \cdot u, \psi)_T - \langle u \cdot n, \psi\rangle_{\partial T} + \langle u \cdot n, \psi\rangle_{\partial T}\\
=& (\nabla \cdot u, \psi)_T = (Q_h(\nabla \cdot u), \psi)_T,
\end{aligned}$$

which means the first equation in the lemma and similarly we can prove the second equation. For the proof of the third assertion of the lemma, fix $v \in \mathcal{V}_h$ and φ sufficiently regular function, then from the definition of the weak curl operator (2.2) one can have

$$\begin{aligned}
(\nabla_w \times v, \mathbb{Q}_h(\varphi))_T =& (v^0, \nabla \times \mathbb{Q}_h(\varphi))_T - \langle v^b \times n, \mathbb{Q}_h(\varphi)\rangle_{\partial T}\\
=& (\nabla \times v^0, \mathbb{Q}_h(\varphi))_T + \langle v^0 \times n, \mathbb{Q}_h(\varphi)\rangle_{\partial T} - \langle v^b \times n, \mathbb{Q}_h(\varphi)\rangle_{\partial T}\\
=& (\nabla \times v^0, \mathbb{Q}_h(\varphi))_T + \langle (v^0 - v^b) \times n, \mathbb{Q}_h(\varphi)\rangle_{\partial T}\\
=& (\nabla \times v^0, \varphi)_T + \langle (v^0 - v^b) \times n, \mathbb{Q}_h(\varphi)\rangle_{\partial T}\\
=& (v^0, \nabla \times \varphi)_T - \langle v^0 \times n, \varphi\rangle_{\partial T} + \langle (v^0 - v^b) \times n, \mathbb{Q}_h(\varphi)\rangle_{\partial T}\\
=& (v^0, \nabla \times \varphi)_T + \langle (v^b - v^0) \times n, \varphi - \mathbb{Q}_h(\varphi)\rangle_{\partial T} - \langle v^b \times n, \varphi\rangle_{\partial T}. \square
\end{aligned}$$

In the following section, we derive some error equations which we need to establish optimal error estimates for the weak Galerkin mixed finite element scheme (2.3)–(2.4).

3.1. Error equations.

Let (u, p) be a sufficiently smooth solution of (1.1)–(1.4) and for the sake of simplicity, assume that the coefficients μ, ε are constants and to be equal to the identity. The use of Lemma 3.1, the definition of weak curl operator (2.2) and the usual integration by parts, implies

$$\begin{aligned}(\nabla_w \times (\mathbb{Q}_h(u)), \nabla_w \times v)_T =&(\mathbf{Q}_h(\nabla \times u), \nabla_w \times v)_T\\ =&(v^0, \nabla \times \mathbf{Q}_h(\nabla \times u))_T - \langle v^b \times n, \mathbf{Q}_h(\nabla \times u)\rangle_{\partial T}\\ =&(\nabla \times v^0, \mathbf{Q}_h(\nabla \times u))_T + \langle (v^0 - v^b) \times n, \mathbf{Q}_h(\nabla \times u)\rangle_{\partial T}.\end{aligned}$$

Therefore,

$$(\nabla_w \times (\mathbb{Q}_h(u)), \nabla_w \times v)_T =(\nabla \times v^0, \nabla \times u)_T + \langle (v^0 - v^b) \times n, \mathbf{Q}_h(\nabla \times u)\rangle_{\partial T}. \tag{3.1}$$

Also, the use of the definition of weak divergence operator (2.1), the usual integration by parts and the fact that $\sum_{T\in\mathfrak{T}_h}\langle v^b \cdot n, p\rangle_{\partial T} = 0$, gives

$$\begin{aligned}(\nabla_w \cdot v, Q_h(p))_\Omega =& -\sum_{T\in\mathfrak{T}_h}(v^0, \nabla(Q_h(p)))_T + \langle v^b \cdot n, Q_h(p)\rangle_{\partial T}\\ =& \sum_{T\in\mathfrak{T}_h}(\nabla \cdot v^0, Q_h(p))_T - \langle (v^0 - v^b)\cdot n, Q_h(p)\rangle_{\partial T}\\ =& \sum_{T\in\mathfrak{T}_h}(\nabla \cdot v^0, p)_T - \langle (v^0 - v^b)\cdot n, Q_h(p)\rangle_{\partial T}\\ =& -\sum_{T\in\mathfrak{T}_h}(v^0, \nabla p)_T + \langle v^0 \cdot n, p\rangle_{\partial T} - \sum_{T\in\mathfrak{T}_h}\langle (v^0 - v^b)\cdot n, Q_h(p)\rangle_{\partial T}\\ =& -\sum_{T\in\mathfrak{T}_h}(v^0, \nabla p)_T + \langle (v^0 - v^b)\cdot n, p\rangle_{\partial T} - \sum_{T\in\mathfrak{T}_h}\langle (v^0 - v^b)\cdot n, Q_h(p)\rangle_{\partial T}\\ =& -(v^0, \nabla p)_\Omega + \sum_{T\in\mathfrak{T}_h}\langle (v^0 - v^b)\cdot n, p - Q_h(p)\rangle_{\partial T},\end{aligned}$$

which implies that

$$(v^0, \nabla p)_\Omega = -(\nabla_w \cdot v, Q_h(p))_\Omega + \sum_{T\in\mathfrak{T}_h}\langle (v^0 - v^b)\cdot n, p - Q_h(p)\rangle_{\partial T}. \tag{3.2}$$

Now, testing the first equation in (1.1)–(1.4) by using v^0 in $v = \{v^0, v^b\} \in V_h^0$ and get

$$(\nabla \times \nabla \times u, v^0)_\Omega - (\nabla p, v^0)_\Omega = (J, v^0)_\Omega. \tag{3.3}$$

After an integration by parts and using the fact that $\sum_{T\in\mathfrak{T}_h}\langle v^b \times n, (\nabla \times u)\rangle_{\partial T} = 0$, one can arrive to

$$(\nabla \times \nabla \times u, v^0)_\Omega = \sum_{T\in\mathfrak{T}_h}(\nabla \times u, \nabla \times v^0)_T + \langle (v^0 - v^b) \times n, (\nabla \times u)\rangle_{\partial T}.$$

The use of this last equation together with (3.1) implies that

$$\begin{aligned}(\nabla \times \nabla \times u, v^0)_\Omega =&(\nabla_w \times (\mathbb{Q}_h(u)), \nabla_w \times v)_\Omega - \sum_{T\in\mathfrak{T}_h} \langle (v^0 - v^b) \times n, \mathbf{Q}_h(\nabla \times u)\rangle_{\partial T}\\ &+ \sum_{T\in\mathfrak{T}_h} \langle (v^0 - v^b) \times n, (\nabla \times u)\rangle_{\partial T}\\ =&(\nabla_w \times (\mathbb{Q}_h(u)), \nabla_w \times v)_\Omega\\ &- \sum_{T\in\mathfrak{T}_h} \langle (v^0 - v^b) \times n, \mathbf{Q}_h(\nabla \times u) - \nabla \times u\rangle_{\partial T}.\end{aligned}$$

Substituting the previous equation and (3.2) into (3.3) and get

$$\begin{aligned}(\nabla_w \times (\mathbb{Q}_h(u))_\Omega,&\nabla_w \times v)_\Omega + (\nabla_w \cdot v, Q_h(p))_\Omega\\ &=(J, v^0)_\Omega + \sum_{T\in\mathfrak{T}_h} \langle (v^0 - v^b) \cdot n, p - Q_h(p)\rangle_{\partial T}\\ &\quad + \sum_{T\in\mathfrak{T}_h} \langle (v^0 - v^b) \times n, \mathbf{Q}_h(\nabla \times u) - \nabla \times u\rangle_{\partial T}.\end{aligned}$$

As to the second equation in (1.1)–(1.4), we test it by a function $\nabla_w \cdot v$ and write

$$0 = (\nabla \cdot u, \nabla \cdot v)_\Omega =(Q_h(\nabla \cdot u), \nabla_w \cdot v)_\Omega = (\nabla_w \cdot (\mathbb{Q}_h(u)), \nabla_w \cdot v)_\Omega.$$

The addition of these two last equations gives

$$\begin{aligned}(\nabla_w \times (\mathbb{Q}_h(u)),&\nabla_w \times v)_\Omega + (\nabla_w \cdot (\mathbb{Q}_h(u)), \nabla_w \cdot v)_\Omega + (\nabla_w \cdot v, Q_h(p))_\Omega\\ &=(J, v^0)_\Omega + \sum_{T\in\mathfrak{T}_h} \langle (v^0 - v^b) \cdot n, p - Q_h(p)\rangle_{\partial T}\\ &\quad + \sum_{T\in\mathfrak{T}_h} \langle (v^0 - v^b) \times n, \mathbf{Q}_h(\nabla \times u) - \nabla \times u\rangle_{\partial T}.\end{aligned} \tag{3.4}$$

Now, it is the moment to introduce and prove the error equations, we have the following.

Lemma 3.2. *Let $e_h := u_h - \mathbb{Q}_h(u)$ and $\epsilon_h := p_h - Q_h(p)$ be the errors, then*

$$\begin{aligned}A_s(e_h, v) + B(v, \epsilon_h) =& \sum_{T\in\mathfrak{T}_h} \langle (v^b - v^0) \times n, \mathbf{Q}_h(\nabla \times u) - \nabla \times u\rangle_{\partial T}\\ &+ \sum_{T\in\mathfrak{T}_h} \langle (v^b - v^0) \cdot n, p - Q_h(p)\rangle_{\partial T} - s_T(\mathbb{Q}_h(u), v),\\ B(e_h, \psi) =&0.\end{aligned}$$

Proof. By adding $s_T(\mathbb{Q}_h(u), v)$ to the two sides of (3.4), one can obtain

$$\begin{aligned} &A_s(\mathbb{Q}_h(u),v) + B(v, Q_h(p)) \\ &\quad=(J, v^0)_\Omega + \sum_{T\in\mathfrak{T}_h} \langle (v^0 - v^b) \times n, \mathbf{Q}_h(\nabla\times u) - \nabla \times u\rangle_{\partial T} \\ &\qquad + s_T(\mathbb{Q}_h(u), v) + \sum_{T\in\mathfrak{T}_h} \langle (v^0 - v^b) \cdot n, p - Q_h(p)\rangle_{\partial T}. \end{aligned}$$

Substract this equation from the first equation in (2.3)–(2.4), one can get

$$\begin{aligned} &A_s(e_h, v) + B(v, \epsilon_h) \\ &= \sum_{T\in\mathfrak{T}_h} \langle (v^b - v^0) \times n, \mathbf{Q}_h(\nabla\times u) - \nabla \times u\rangle_{\partial T} - s_T(\mathbb{Q}_h(u), v) \\ &\quad + \sum_{T\in\mathfrak{T}_h} \langle (v^b - v^0) \cdot n, p - Q_h(p)\rangle_{\partial T}. \end{aligned}$$

Testing the second equation in (1.1)–(1.4) by a function ψ, then

$$0 = (\nabla \cdot u, \psi)_\Omega = (Q_h(\nabla \cdot u), \psi)_\Omega = (\nabla_w \cdot (\mathbb{Q}_h(u)), \psi)_\Omega,$$

which means that $B(\mathbb{Q}_h(u), \psi) = 0$. Substract the previous equation from the second equation in (2.3)–(2.4), we obtain $B(e_h, \psi) = 0$. □

The weak Galerkin mixed finite element formulation (2.3)–(2.4) is a typical saddle-point scheme which can be studied with the well known Babuška-Brezzi theory [1,2]. Thus, we have a great interest for studying the properties of the bilinear forms introduced in (2.3)–(2.4).

3.2. **Study of the bilinear forms.** First, we define the norms on the space $\mathcal{V}_h^0$ and $\mathcal{W}_h^0$. For $\psi \in \mathcal{W}_h^0$, we use $\|\psi\|$ the usual L^2-norm of ψ and we introduce a norm in $\mathcal{V}_h^0$ as

$$|||u|||^2 := A_s(u, u).$$

Note that from the proof of Theorem 2.1, we immediately deduce that $u = 0$ if $A_s(u, u) = 0$ and hence the triple-bar norm just introduced above define norm on the space $\mathcal{V}_h^0$. Also from this definition of norm, we remark that the coercivity of A_s follows directly. While, the continuity of the bilinear forms A_s and B can be demonstrated from classical techniques due to the Cauchy-Schwarz inequalities. Therefore, for an application of the Babuška-Brezzi theory, it remains to demonstrate an inf-sup condition for B. This is the objective of the following lemma.

Lemma 3.3. *There exists a constant β independent of h such that*

$$\inf_{\psi\in\mathcal{W}_h^0\setminus\{0\}} \sup_{v\in\mathcal{V}_h^0\setminus\{0\}} \frac{B(\varphi, v)}{|||v||| \cdot \|\varphi\|} \geq \beta > 0.$$

Proof. Let $\psi \in \mathcal{W}_h^0$, then ψ is in $L_0^2(\Omega)$ and it is well known that there exists $v \in H_0^1(\Omega)^d$ such that $(\nabla \cdot v, \psi) \geq C\|\psi\| \cdot \|v\|_1$. Choose $\tilde{v} = \mathbb{Q}_h(v)$ and let us prove that $|||\tilde{v}||| \leq C\|v\|_1$, we have

$$\begin{aligned}\sum_{T\in\mathfrak{T}_h} \|\nabla_w \times \tilde{v}\|^2 &= \sum_{T\in\mathfrak{T}_h} \|\nabla_w \times \mathbb{Q}_h(v)\|^2 = \sum_{T\in\mathfrak{T}_h} \|\mathbf{Q}_h(\nabla \times v)\|^2 \\ &\leq \sum_{T\in\mathfrak{T}_h} \|\nabla \times v\|^2 \leq \|v\|_1^2 \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\sum_{T\in\mathfrak{T}_h} \|\nabla_w \cdot \tilde{v}\|^2 &= \sum_{T\in\mathfrak{T}_h} \|\nabla_w \cdot \mathbb{Q}_h(v)\|^2 = \sum_{T\in\mathfrak{T}_h} \|Q_h(\nabla \cdot v)\|^2 \\ &\leq \sum_{T\in\mathfrak{T}_h} \|\nabla \cdot v\|^2 \leq \|v\|_1^2. \end{aligned} \tag{3.6}$$

By selecting $\tilde{v} = \mathbb{Q}_h(v)$ in the definition of $s_T(\tilde{v}, \tilde{v})$, we need to estimate the following two terms

$$s_{NT}(\tilde{v}, \tilde{v}) := r \sum_{\partial T\in\mathcal{E}_h} h_T^{-1}\|(\mathbb{Q}_0(v - \mathbb{Q}_b(v)) \cdot n\|_{\partial T}^2$$

and

$$s_{TT}(\tilde{v}, \tilde{v}) := r \sum_{\partial T\in\mathcal{E}_h} h_T^{-1}\|(\mathbb{Q}_0(v - \mathbb{Q}_b(v)) \times n\|_{\partial T}^2.$$

For an estimation of the term $s_{NT}(\tilde{v}, \tilde{v})$, one can get

$$\begin{aligned} s_{NT}(\tilde{v}, \tilde{v}) \leq& 2r \sum_{\partial T\in\mathcal{E}_h} h_T^{-1}\|(\mathbb{Q}_0(v) - v) \cdot n\|_{\partial T}^2 + 2r \sum_{\partial T\in\mathcal{E}_h} h_T^{-1}\|(\mathbb{Q}_b(v) - v) \cdot n\|_{\partial T}^2 \\ \leq& 2r \sum_{\partial T\in\mathcal{E}_h} h_T^{-1}\|\mathbb{Q}_0(v) - v\|_{\partial T}^2 + 2r \sum_{\partial T\in\mathcal{E}_h} h_T^{-1}\|\mathbb{Q}_0(v) - v\|_{\partial T}^2 \\ \leq& Cr \sum_{\partial T\in\mathcal{E}_h} h_T^{-1}\|\mathbb{Q}_0(v) - v\|_{\partial T}^2 \\ \leq& Cr \sum_{T\in\mathfrak{T}_h} h_T^{-2}\|\mathbb{Q}_0(v) - v\|_T^2 + \|\nabla(\mathbb{Q}_0(v) - v)\|_T^2 \\ \leq& Cr \sum_{T\in\mathfrak{T}_h} h_T^{-2}\|\mathbb{Q}_0(v) - v\|_T^2 + \|\nabla(\mathbb{Q}_0(v) - v)\|_T^2 \\ \leq& C\|\nabla v\|^2, \end{aligned} \tag{3.7}$$

and similarly, one can obtain

$$s_{TT}(\tilde{v}, \tilde{v}) \leq C\|\nabla v\|^2. \tag{3.8}$$

It follows from (3.5), (3.6), (3.7) and (3.8) that

$$|||\tilde{v}||| \leq C\|v\|_1,$$

and the use of Lemma 3.1, the definition of Q_h, means

$$B(\tilde{v}, \psi) = (\nabla_w \cdot \mathbb{Q}_h(v), \psi) = (Q_h(\nabla \cdot v), \psi) = (\nabla \cdot v, \psi) \geq C\|v\|_1\|\psi\| \geq \beta|||\tilde{v}||| \cdot \|\psi\|,$$

which ends the proof. □

In the next subsection, we shall demonstrate optimal order error estimates for the electrostatic field u_h in a norm which is equivalent to the standard $H_0(\nabla\times,\Omega)\cap H(\nabla\cdot,\Omega)$ norm, and for the Lagrange multiplier p_h in the usual L^2 norm. Moreover, we give an error estimate result for the electrostatic field u_h in the L^2 norm.

3.3. **Error estimations.** Let us start by introducing the following lemma which we need for a rigourous proof of error convergence results.

Lemma 3.4. (a) *Given* $u \in H^{k+2}(\Omega)$ *and* $s \in [0, k+1]$, *then*

$$\sum_{T\in\mathcal{T}_h} \|u-\mathbb{Q}_h(u)\|_T^2 + h_T^2\|\nabla(u-\mathbb{Q}_h(u))\|_T^2 \leq h^{2(s+1)}\|u\|_{s+1}^2, \tag{3.9}$$

$$\sum_{T\in\mathcal{T}_h} \|\nabla(u-\mathbb{Q}_h(u))\|_T^2 \leq h^{2s}\|u\|_{s+1}^2. \tag{3.10}$$

(b) *For any* $\theta \in H^1(T)$, $T \in \mathcal{T}_h$ *and* $e \in \mathcal{E}_h$,

$$\|\theta\|_e^2 \leq C\Big(h_T^{-1}\|\varphi\|_T^2 + h_T\|\nabla\theta|_T^2\Big). \tag{3.11}$$

Proof. See the equalities (4.3), (4.2) and the inequality (A.1) in [14]. □

One of our main result in this paper, which demonstrate clearly the optimal convergence of the mixed weak Galerkin formulation (2.3)–(2.4) is given and proven in the following theorem.

Theorem 3.1. *Let* $(u_h, p_h) \in \mathcal{V}_h^0 \times \mathcal{W}_h^0$ *be the approximate solution of* (2.3)–(2.4), (u, p) *the exact solution of* (1.1)–(1.4) *and suppose that* $(u, p) \in H^{k+2}(\Omega) \times H^{k+1}(\Omega)$ *with* $k \geq 0$, *then, we have the following two convergence results*

$$|||\mathbb{Q}_h(u)-u_h||| + \|Q_h(p)-p_h\| \leq Ch^{s+1}\Big(\|u\|_{s+2} + \|p\|_{s+1}\Big) \tag{3.12}$$

and

$$\|\mathbb{Q}_0(u)-u_0\| \leq Ch^{k+2}\Big(\|u\|_{k+2} + \|p\|_{k+1}\Big). \tag{3.13}$$

Proof. Define

$$\begin{aligned}
T_1(u,v) :=& \sum_{T\in\mathcal{T}_h} \langle (v^b - v^0)\times n, \mathbf{Q}_h(\nabla\times u) - \nabla\times u\rangle_{\partial T},\\
T_2(u,v) :=& s_T(\mathbb{Q}_h(u), v),\\
=& r\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle (\mathbb{Q}_0(u)-\mathbb{Q}_b(u))\times n, (v^0-v^b)\times n\rangle_{\partial T}\\
&+ r\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle (\mathbb{Q}_0(u)-\mathbb{Q}_b(u))\cdot n, (v^0-v^b)\cdot n\rangle_{\partial T},\\
T_3(p,v) :=& \sum_{T\in\mathcal{T}_h} \langle (v^b - v^0)\cdot n, p - Q_h(p)\rangle_{\partial T}
\end{aligned}$$

and

$$\ell(v) := T_1(u,v) - T_2(u,v) + T_3(p,v).$$

Then the error equations in Lemma 3.2 can be written as

$$A_s(e_h, v) + B(v, \varepsilon_h) = \ell(v), \quad B(e_h, \psi) = 0,$$

and we deduce from the general theory of Babuška and Brezzi that

$$|||e_h||| + \|\varepsilon_h\| \leq C\|\ell\|_{\mathcal{V}_h^{0\prime}}.$$

Then, it is sufficient to find a bound of $\|\ell\|_{\mathcal{V}_h^{0\prime}}$. Let us start by estimating the term $\|u - \mathbb{Q}_b(u)\|_{\partial T}$ which we need for estimating $T_2(u,v)$. We have

$$\begin{aligned}\|u - \mathbb{Q}_b(u)\|_{\partial T}^2 =& \langle u - \mathbb{Q}_b(u), u - \mathbb{Q}_b(u)\rangle_{\partial T} \\ =& \langle u - \mathbb{Q}_b(u), u - \mathbb{Q}_0(u)\rangle_{\partial T} \\ \leq& \|u - \mathbb{Q}_b(u)\|_{\partial T}\|u - \mathbb{Q}_0(u)\|_{\partial T}\end{aligned}$$

and then,

$$\|u - \mathbb{Q}_b(u)\|_{\partial T} \leq \|u - \mathbb{Q}_0(u)\|_{\partial T}. \tag{3.14}$$

Now, the use of the definition of $\mathbb{Q}_0$, $\mathbb{Q}_b$, Cauchy Schwarz inequality, (3.14), (3.11) and (3.9), (3.10) imply that

$$\begin{aligned}& r\sum_{T\in\mathfrak{T}_h} |h_T^{-1}\langle(\mathbb{Q}_b(u) - \mathbb{Q}_0(u)) \times n, (v^0 - v^b) \times n\rangle_{\partial T}| \\ =& r\sum_{T\in\mathfrak{T}_h} |h_T^{-1}\langle(u - \mathbb{Q}_0(u)) \times n - (u - \mathbb{Q}_b(u)) \times n, (v^0 - v^b) \times n\rangle_{\partial T}| \\ \leq& \left(r\sum_{T\in\mathfrak{T}_h} h_T^{-1}\|(v^0 - v^b) \times n\|_{\partial T}^2\right)^{\frac{1}{2}} \\ & \times \left(r\sum_{T\in\mathfrak{T}_h} h_T^{-1}\|(u - \mathbb{Q}_0(u) - (u - \mathbb{Q}_b(u))) \times n\|_{\partial T}^2\right)^{\frac{1}{2}} \\ \leq& C|||v|||\left(\sum_{T\in\mathfrak{T}_h} h_T^{-1}\|u - \mathbb{Q}_0(u)\|_{\partial T}^2 + h_T^{-1}\|u - \mathbb{Q}_b(u)\|_{\partial T}^2\right)^{\frac{1}{2}} \\ \leq& C|||v|||\left(\sum_{T\in\mathfrak{T}_h} h_T^{-1}\|u - \mathbb{Q}_0(u)\|_{\partial T}^2\right)^{\frac{1}{2}} \\ \leq& C|||v|||\left(\sum_{T\in\mathfrak{T}_h} \left(h_T^{-2}\|u - \mathbb{Q}_0(u)\|_T^2 + \|\nabla(u - \mathbb{Q}_0(u)\|_T^2\right)\right)^{\frac{1}{2}} \\ \leq& Ch^{s+1}|||v||| \cdot \|u\|_{s+2}.\end{aligned}$$

With a similar, one can obtain

$$r\sum_{T\in\mathfrak{T}_h}|h_T^{-1}\langle(\mathbb{Q}_b(u)-\mathbb{Q}_0(u))\cdot n,(v^0-v^b)\cdot n\rangle_{\partial T}|\le Ch^{s+1}|||v|||\cdot\|u\|_{s+2}.$$

and deduce that

$$|T_2(u,v)|\le Ch^{s+1}\|u\|_{s+2}|||v|||.$$

For finding an estimation of $T_1(u,v)$, we use the Cauchy Schwarz inequality, the definition of $|||\cdot|||$ and the trace inequality (3.11) and write

$$\begin{aligned}|T_1(u,v)|\le&\sum_{T\in\mathfrak{T}_h}\left|\langle(v^b-v^0)\times n,\mathbf{Q}_h(\nabla\times u)-\nabla\times u\rangle_{\partial T}\right|\\ \le&\sum_{T\in\mathfrak{T}_h}h_T^{-\frac12}\|(v^b-v^0)\times n\|_{0,\partial T}h_T^{\frac12}\|\mathbf{Q}_h(\nabla\times u)-\nabla\times u\|_{\partial T}\\ \le&\left(\sum_{T\in\mathfrak{T}_h}h_T^{-1}\|(v^b-v^0)\times n\|_{0,\partial T}^2\right)^{\frac12}\\ &\times\left(\sum_{T\in\mathfrak{T}_h}h_T\|\mathbf{Q}_h(\nabla\times u)-\nabla\times u\|_{\partial T}^2\right)^{\frac12}\\ \le&\left(\sum_{T\in\mathfrak{T}_h}h_T\|\mathbf{Q}_h(\nabla\times u)-\nabla\times u\|_{\partial T}^2\right)^{\frac12}|||v|||\\ \le&\left(\sum_{T\in\mathfrak{T}_h}\|\mathbf{Q}_h(\nabla\times u)-\nabla\times u\|_T^2\right.\\ &\left.+\sum_{T\in\mathfrak{T}_h}h_T^2\|\nabla(\mathbf{Q}_h(\nabla\times u)-\nabla\times u)\|_T^2\right)^{\frac12}|||v|||.\end{aligned}$$

Using (3.9) for $\nabla\times u$, we get

$$|T_1(u,v)|\le h^{s+1}\|\nabla\times u\|_{s+1}|||v|||\le h^{s+1}\|u\|_{s+2}|||v|||.$$

The same technique applied for $T_1(u,v)$ can also be applied for estimating $T_3(p,v)$ and we obtain $|T_3(p,v)|\le Ch^{s+1}\|p\|_{s+1}|||v|||$. The inequality (3.12) follows immediately from the previous inequalities and for the proof of (3.13), we can use a similar technique to the given in [6, 14] for the second order Laplacien operator. □

4. Numerical Tests

In this paragraph, two numerical examples are tested for the two dimensional Maxwell equations (1.1)–(1.4) with constant coefficients $\mu(x)=1$ and $\varepsilon(x)=1$ on a domain $\Omega=(0,1)^2$. The parameter r which appears in (2.3)–(2.4) is chosen as $r=1$ and can be taken as any strictly positive real number. The approximate solution (u_h,p_h) is discritised with the lowest order (i.e., $k=0$) on the space $\mathcal{V}_h^0\times\mathcal{W}_h^0$. The numerical experiments indicate that the weak Galerkin methods are accurate and easy to implement and the numerical convergence results obtained on the two examples confirm perfectly the estimations proven in theorem 3.1.

Example 4.1. In this example, we consider the Maxwell equations and Lagrange multiplier together with boundary conditions (1.1)–(1.4) on the unit square $\Omega = (0,1)^2$. We assume that the true solutions are given by $u(x,y) = \begin{pmatrix} y(y-1)\cos(y) \\ x(x-1)\cos(x) \end{pmatrix}$ and $p(x,y) = x(x-1)y(y-1)\cos(x+y)$. The numerical experiments of the algorithm are presented in Table 1. We see that these results show the $O(h)$ error for the electrostatic field in the $|||\cdot|||$-norm and $O(h^2)$ error of the Lagrange multiplier in the L^2-norm. The convergence rate with respect $O(h^2)$ for the electric field u in the L^2-norm is also observed, which confirms the proven estimations (3.12) and (3.13).

Table 1. Numerical results for Example 1.

h	$\lvert\lvert\lvert e_h \rvert\rvert\rvert$	rate	$\lVert \varepsilon_h \rVert_{1,h}$	rate	$\lVert u_h^0 - \mathbb{Q}_h(u^0) \rVert$	rate
$\frac{1}{2}$	6.2592e-01	-	1.7009e-02	-	1.7249e-01	-
$\frac{1}{4}$	3.5102e-01	8.3443e-01	2.6726e-03	2.6700	3.5422e-02	2.2838e
$\frac{1}{8}$	1.8526e-01	9.2201e-01	6.0702e-04	2.1384	8.9524e-03	1.9843
$\frac{1}{16}$	9.3879e-02	9.8066e-01	1.4837e-04	2.0325	2.2470e-03	1.9943
$\frac{1}{32}$	4.7096e-02	9.9518e-01	3.6887e-05	2.0080	5.6233e-04	1.9985
$\frac{1}{64}$	2.3568e-02	9.9880e-01	9.2090e-06	2.0020	1.4062e-04	1.9996
$\frac{1}{128}$	1.1786e-02	9.9970e-01	2.3014e-06	2.0005	3.5157e-05	1.9999

Example 4.2. In this numerical example, we shall consider the 2-dimensional Maxwell problem with Lagrange multiplier (1.1)–(1.4). Consider $\Omega = (0,1) \times (0,1)$ and the right-hand side function J be chosen such that the functions $u(x,y) = (e^y \sin(y^2 - y), e^x \sin(x^2 - x))^T$ and $p(x,y) = e^{x+y}\sin((x^2-x)(y^2-y))$ are the true solutions of the problem (1.1)–(1.4). The convergence results and error profiles are presented in Table 2. It can be observed $|||\cdot|||$-error, L^2-error for the electric field u, and L^2-error for the Lagrange multiplier p converge, respectively, with respect to $O(h)$, $O(h^2)$, and $O(h^2)$, which confirms the theoretical estimations (3.12) and (3.13).

5. Conclusions and Remarks

In this paper, we analysed the new formulation of weak Galerkin mixed finite element method for solving numerically the Maxwell equations with Lagrange multiplier. The well posedness as well as the optimal convergence of the numerical scheme was shown, established and tested numerically. The results obtained in this paper are powerful and encourage applications to other systems of partial differential equations.

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Table 2. Numerical results for Example 2.

h	$\|\|\|e_h\|\|\|$	rate	$\|\varepsilon_h\|_{1,h}$	rate	$\|u_h^0 - \mathbb{Q}_h(u^0)\|$	rate
$\frac{1}{2}$	1.3330e+00	-	4.1335e-02	-	3.9249e-01	-
$\frac{1}{4}$	6.7970e-01	9.7174e-01	1.1172e-02	1.8874	7.2042e-02	2.4457
$\frac{1}{8}$	3.5995e-01	9.1709e-01	2.9750e-03	1.9090	1.8123e-02	1.9910
$\frac{1}{16}$	1.8272e-01	9.7819e-01	7.5563e-04	1.9771	4.5461e-03	1.9951
$\frac{1}{32}$	9.1709e-02	9.9449e-01	1.8965e-04	1.9943	1.1376e-03	1.9986
$\frac{1}{64}$	4.5898e-02	9.9862e-01	4.7460e-05	1.9986	2.8447e-04	1.9996
$\frac{1}{128}$	2.2955e-02	9.9966e-01	1.1868e-05	1.9996	7.1123e-05	1.9999

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