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GENERALIZATION OF INEQUALITIES FOR DIFFERENT TYPES OF FUNCTIONS VIA ψ-CAPUTO FRACTIONAL DERIVATIVE

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Abstract. In this manuscript, we present a collection of inequalities that extend and generalize the important results of the findings from [14], with the latter serving as specific cases within our study. Our research takes many forms of fractional differential inequalities, including the comprehensive structure of the Caputo fractional derivative operator with respect to a function ψ. Furthermore, we demonstrate the practical applications of σ-convex functions in the domain of fractional calculus within this framework.

1. Introduction

The broad applicability of fractional calculus in modeling natural phenomena across various scientific and engineering fields has captured the interest of many researchers [4, 5, 12, 13, 17]. These fields include physics, finance, biology, engineering and chemistry. To facilitate a deeper understanding of these phenomena, various definitions of fractional derivatives have been introduced, including the Caputo derivative. This last derivative, proposed by Italian Caputo in his article [3], is considered as an important contribution to fractional calculus [11], consisting of the introduction of a new fractional derivation which is better adapted to to problems of partial differential equations. This derivative has been generalised in different sense so that the reader can see in [8–10] and the related references therein. In this study, our attention will be directed towards the Caputo fractional operators, aiming to establish crucial fractional inequalities. The concept of convexity is crucial in various domains in both pure and applied sciences. Therefore, the classical concepts of convex sets and convex

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functions have been extended and generalized in different research directions. For more information, we refer the reader to the references [7,14–16]. The close connection between convexity theory and the theory of inequalities has been a significant factor, drawing the attention of various researchers. The theory of mathematical inequalities has seen extensive research through numerous extensions and various generalizations, especially those dealing with Hadamard-type inequalities the reader can see in [2,6] and references therein.

In this section, we recall a few basics properties from the theory of fractional calculs. The Caputo fractional derivatives of order α are defined for the first time by the Italian mathematician Caputo which presented his fractional derivative of order $\alpha > 0$ as in the following definition σ-convex functions.

Definition 1.1. A set $\mathcal{Q} \subset \mathbb{R}$ is defined as a σ-convex set with respect to a strictly monotonic continuous function σ if

$$\mathcal{M}_{[\sigma]}(x,y) := \sigma^{-1}\left(tx + (1-t)y\right) \in \mathcal{Q}, \quad \text{for all } x, y \in \mathcal{Q},\ t \in [0,1].$$

A function $f : \mathcal{Q} \to \mathbb{R}$ is defined as a σ-convex function with respect to a strictly monotonic continuous function σ if

$$f\left(\mathcal{M}_{[\sigma]}(x,y)\right) \leq tf(x) + (1-t)f(y), \quad \text{for all } x, y \in \mathcal{Q},\ t \in [0,1].$$

The set of σ-convex functions will be denoted by $\mathcal{Q}(I)$.

Definition 1.2. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be $\mathcal{P}$-function or belongs to $\mathcal{P}(I)$, if it is nonnegative and for all $x, y \in I$ we have

$$f\left(\mathcal{M}_{[\sigma]}\left(\sigma(x), \sigma(y)\right)\right) \leq f(x) + f(y).$$

Definition 1.3. Let $h : J \subset \mathbb{R} \to \mathbb{R}$ be a positive function. We say that $f : I \subset \mathbb{R} \to \mathbb{R}$ is h-convex or that f belongs to the class of $\mathcal{S}X(h, I)$, if f is nonegative and for all $x, y \in I$ $\lambda \in [0,1]$, we have

$$f(\lambda x + (1-\lambda)y) \leq h(\lambda)f(x) + h(1-\lambda)f(y).$$

Definition 1.4. Let $\alpha > 0$, $f \in AC^n[a,b]$,

$$\mathcal{D}^{\alpha}_{a+}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, \quad t > a, \tag{1.1}$$

and

$$\mathcal{D}^{\alpha}_{b-}f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b \frac{f^{(n)}(s)}{(s-t)^{\alpha+1-n}} ds, \quad t < b. \tag{1.2}$$

If $\alpha = n \in \{1,2,3,\dots\}$ and usual derivative of order n exists, then Caputo fractional derivative ${}^c\mathcal{D}^{\alpha,\psi}_{a+}f(t)$ coincides with $f^{(n)}(t)$, while ${}^c\mathcal{D}^{\alpha,\psi}_{b-}f(t)$ coincides with $f^{(n)}(t)$ with exactness to a constant multiplier $(-1)^n$.

The Caputo derivatives given in Definition 1.4 have been generalized by the ψ-Caputo fractional derivative, as described below.

Definition 1.5 ([1])**.** Let $\alpha > 0$, $f \in AC^n[a,b]$. The ψ-Caputo fractional derivatives of order α are defined as follows:

$$ {}^{c}\mathfrak{D}^{\alpha,\psi}_{a+} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1} f^{[n]}_{\psi}(s)ds, \quad t > a, \tag{1.3}$$

and

$$ {}^{c}\mathfrak{D}^{\alpha,\psi}_{b-} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b \psi'(s)(\psi(s)-\psi(t))^{n-\alpha-1} f^{[n]}_{\psi}(s)ds, \quad t < b, \tag{1.4}$$

where $f^{[n]}_{\psi}(t) = \left(\frac{1}{\psi'(t)} \cdot \frac{d}{dt}\right)^n f(t)$.

Within this section, we formulate new inequalities for various types of functions using ψ-Caputo derivatives.

Lemma 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f^{[n+1]}_{\psi}$ is integrable over $[a,b]$, then the following generalized fractional equality*

$$\begin{aligned} &\left[f^{[n]}_{\psi}(a) + f^{[n]}_{\psi}(b)\right] - \frac{\Gamma(n-\alpha+1)}{(\psi(b)-\psi(a))^{n-\alpha}} \left[{}^{c}\mathfrak{D}^{\alpha,\psi}_{a+} f(b) + (-1)^n\, {}^{c}\mathfrak{D}^{\alpha,\psi}_{b-} f(a)\right] \\ &= \Big(\psi(b)-\psi(a)\Big) \int_0^1 \left(t^{n-\alpha} - (1-t)^{n-\alpha}\right) f^{[n+1]}_{\psi}\left(\psi^{-1}(t\psi(a) + (1-t)\psi(b))\right) dt \end{aligned} \tag{1.5}$$

holds.

Proof. Using Definition 1.5, we can write

$$\begin{aligned} {}^{c}\mathfrak{D}^{\alpha,\psi}_{a+} f(b) + (-1)^n\, {}^{c}\mathfrak{D}^{\alpha,\psi}_{b-} f(a) =& \frac{1}{\Gamma(n-\alpha)} \int_a^b \psi'(\tau) f^{[n]}_{\psi}(\tau) \\ &\times \left[\Big(\psi(b)-\psi(\tau)\Big)^{n-\alpha-1} + \Big(\psi(\tau)-\psi(a)\Big)^{n-\alpha-1}\right] d\tau. \end{aligned}$$

Using integration by parts and by change of variable, it follows by taking $u = \frac{1}{\Gamma(n-\alpha)} f^{[n]}_{\psi}(\tau)$ and $v = \psi'(\tau)\left(\psi(b)-\psi(\tau)\right)^{n-\alpha} + \psi'(\tau)\left(\psi(\tau)-\psi(a)\right)^{n-\alpha}$ that

$$\begin{aligned} &{}^{c}\mathfrak{D}^{\alpha,\psi}_{a+} f(b) + (-1)^n\, {}^{c}\mathfrak{D}^{\alpha,\psi}_{b-} f(a) \\ =& \frac{1}{\Gamma(n-\alpha)} \times \left[- f^{[n]}_{\psi}(\tau) \frac{\Big(\psi(b)-\psi(\tau)\Big)^{n-\alpha+1}}{n-\alpha+1} + f^{[n]}_{\psi}(\tau) \frac{\Big(\psi(\tau)-\psi(a)\Big)^{n-\alpha+1}}{n-\alpha+1} \right]_a^b \\ &- \frac{1}{\Gamma(n-\alpha)} \int_a^b f^{[n+1]}_{\psi}(\tau) \left[- \frac{\Big(\psi(b)-\psi(\tau)\Big)^{n-\alpha+1}}{n-\alpha+1} + \frac{\Big(\psi(\tau)-\psi(a)\Big)^{n-\alpha+1}}{n-\alpha+1} \right] d\tau. \end{aligned}$$

Now, we use the change of variable $\tau = \mathcal{M}_{[\psi]}(a,b) = \psi^{-1}(ta + (1-t)b)$, we obtain

$$\begin{aligned}{}^{c}\mathcal{D}^{\alpha,\psi}_{a+} f(b) + (-1)^n\, {}^{c}\mathcal{D}^{\alpha,\psi}_{b-} f(a) =& \frac{\Big(\psi(b)-\psi(a)\Big)^{n-\alpha}}{\Gamma(n-\alpha+1)} \Big[f^{[n]}_{\psi}(a) + f^{[n]}_{\psi}(b)\Big] \\ &+ \frac{\Big(\psi(b)-\psi(a)\Big)^{n-\alpha+1}}{\Gamma(n-\alpha+1)} \\ &\times \int_0^1 f^{[n+1]}_{\psi}\Big(\mathcal{M}_{[\psi]}(a,b)\Big) \Big[t^{n-\alpha} - (1-t)^{n-\alpha}\Big]\, dt.\end{aligned}$$

Finally, we multiply both sides of the previous equality by $\frac{(\psi(b)-\psi(a))^{n-\alpha}}{\Gamma(n-\alpha+1)}$, we obtain

$$\begin{aligned}&\frac{\Big(\psi(b)-\psi(a)\Big)^{n-\alpha}}{\Gamma(n-\alpha+1)} \Big[{}^{c}\mathcal{D}^{\alpha,\psi}_{a+} f(b) + (-1)^n\, {}^{c}\mathcal{D}^{\alpha,\psi}_{b-} f(a)\Big] \\ =&\Big[f^{[n]}_{\psi}(a) + f^{[n]}_{\psi}(b)\Big] + \Big(\psi(b)-\psi(a)\Big) \int_0^1 f^{[n+1]}_{\psi}\Big(\mathcal{M}_{[\psi]}(a,b)\Big)\Big[t^{n-\alpha} - (1-t)^{n-\alpha}\Big] dt.\end{aligned}$$

Hence,

$$\begin{aligned}&\Big[f^{[n]}_{\psi}(a) + f^{[n]}_{\psi}(b)\Big] - \frac{\Gamma(n-\alpha+1)}{(\psi(b)-\psi(a))^{n-\alpha}} \Big[{}^{c}\mathcal{D}^{\alpha,\psi}_{a+} f(b) + (-1)^n\, {}^{c}\mathcal{D}^{\alpha,\psi}_{b-} f(a)\Big] \\ =& (\psi(b)-\psi(a)) \int_0^1 \Big((1-t)^{n-\alpha} - t^{n-\alpha}\Big) f^{[n+1]}_{\psi}\Big(\mathcal{M}_{[\psi]}(a,b)\Big)\, dt.\end{aligned}$$

□

Theorem 1.1. *Let us consider a differentiable function $f : [a,b] \to \mathbb{R}$ defined over the open interval (a,b). Assume that $|f^{[n+1]}_{\psi}|$ belongs to the class $\mathcal{Q}(I)$. Then, the following inequality*

$$\begin{aligned}&\left|\Big[f^{[n]}_{\psi}(a) + f^{[n]}_{\psi}(b)\Big] - \frac{\Gamma(n-\alpha+1)}{(\psi(b)-\psi(a))^{n-\alpha}} \Big[{}^{c}\mathcal{D}^{\alpha,\psi}_{a+} f(b) + (-1)^n\, {}^{c}\mathcal{D}^{\alpha,\psi}_{b-} f(a)\Big]\right| \\ \leq&\Big(\psi(b)-\psi(a)\Big)\left\{\Big(\left|f^{[n+1]}_{\psi}(a)\right| + \left|f^{[n+1]}_{\psi}(b)\right|\Big)\left[B(0,n-\alpha+1) + \frac{1}{n-\alpha+2}\right]\right\}\end{aligned}$$

holds.

Proof. Combining the result of Lemma 1.1 and Definition 1.1, we have

$$\begin{aligned}&\left|\Big[f^{[n]}_{\psi}(a) + f^{[n]}_{\psi}(b)\Big] - \frac{\Gamma(n-\alpha+1)}{(\psi(b)-\psi(a))^{n-\alpha}} \Big[{}^{c}\mathcal{D}^{\alpha,\psi}_{a+} f(b) + (-1)^n\, {}^{c}\mathcal{D}^{\alpha,\psi}_{b-} f(a)\Big]\right| \\ =&\left|\Big(\psi(b)-\psi(a)\Big) \int_0^1 \Big((1-t)^{n-\alpha} - t^{n-\alpha}\Big) f^{[n+1]}_{\psi}\Big(\mathcal{M}_{[\psi]}(a,b)\Big)\, dt\right| \\ \leq&\Big(\psi(b)-\psi(a)\Big) \int_0^1 \left|(1-t)^{n-\alpha} - t^{n-\alpha}\right| \Big[\left|t\, f^{[n+1]}_{\psi}(a)\right| + \left|(1-t) f^{[n+1]}_{\psi}(b)\right|\Big]\, dt.\end{aligned}$$

We expand the product under the integral, and we obtain

$$\begin{aligned}&\left|\left[f_\psi^{[n]}(a)+f_\psi^{[n]}(b)\right]-\frac{\Gamma(n-\alpha+1)}{(\psi(b)-\psi(a))^{n-\alpha}}\left[{}^c\mathcal{D}_{a+}^{\alpha,\psi}f(b)+(-1)^n\,{}^c\mathcal{D}_{b-}^{\alpha,\psi}f(a)\right]\right|\\ \leq&\Big(\psi(b)-\psi(a)\Big)\left\{\left|f_\psi^{[n+1]}(a)+f_\psi^{[n+1]}(b)\right|\left[\int_0^1(1-t)^{n-\alpha}\,t^{-1}+t^{n-\alpha+1}\,dt\right]\right\}\\ =&\Big(\psi(b)-\psi(a)\Big)\left\{\left|f_\psi^{[n+1]}(a)\right|+\left|f_\psi^{[n+1]}(b)\right|\left[B(0,n-\alpha+1)+\frac{1}{n-\alpha+2}\right]\right\},\end{aligned}$$

where B is the beta function, which completes the proof of the theorem. □

Corollary 1.1. *If we take $\psi(x)=x$, the ψ-Caputo fractional operators ${}^c\mathcal{D}_{a+}^{\alpha,\psi}$ and ${}^c\mathcal{D}_{b-}^{\alpha,\psi}$ are reduced respectively to the Caputo fractional operators ${}^c\mathcal{D}_{a+}^{\alpha}$ and ${}^c\mathcal{D}_{b-}^{\alpha}$, and the function $f_\psi^{[n+1]}$ is reduced to $f_\psi^{(n+1)}$, then the inequality* (1.6) *reduces to the following inequality*

$$\begin{aligned}&\left|\left[f^{(n)}(a)+f^{(n)}(b)\right]-\frac{\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\left[{}^c\mathcal{D}_{a+}^{\alpha}f(b)+(-1)^n\,{}^c\mathcal{D}_{b-}^{\alpha}f(a)\right]\right|\\ \leq&\Big(\psi(b)-\psi(a)\Big)\left\{\left|f^{(n+1)}(a)\right|+\left|f^{(n+1)}(b)\right|\left[B(0,n-\alpha+1)+\frac{1}{n-\alpha+2}\right]\right\},\end{aligned}$$

which it exactly the inequality obtained in [14, Theorem 2.1], *when we replace the class $\mathcal{Q}(I)$ in* [14] *by that in the Definition* 1.1.

Theorem 1.2. *Let us consider a differentiable function $f:[a,b]\to\mathbb{R}$ defined over the open interval (a,b). Assume that $|f_\psi^{[n+1]}|$ belongs to the class $\mathcal{Q}(I)$. Then, the following inequality*

$$\begin{aligned}&\left|\frac{f_\psi^{[n]}(a)+f_\psi^{[n]}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2\,(\psi(b)-\psi(a))^{n-\alpha}}\left[{}^c\mathcal{D}_{a+}^{\alpha,\psi}f(b)+(-1)^n\,{}^c\mathcal{D}_{b-}^{\alpha,\psi}f(a)\right]\right|\\ \leq&\frac{\psi(b)-\psi(a)}{n-\alpha+1}\left\{\left|f_\psi^{[n+1]}\right|(a)+\left|f_\psi^{[n+1]}\right|(b)\right\}\end{aligned}\tag{1.6}$$

holds.

Proof. Since $\left|f_\psi^{[n+1]}\right|$ is a ψ-convex function, then in particular we can obtain

$$\left|f_\psi^{[n+1]}\right|\Big(\mathcal{M}_{[\psi]}(x,y)\Big)\leq t\left|f_\psi^{[n+1]}\right|(x)+(1-t)\left|f_\psi^{[n+1]}\right|(y),\quad\text{for all } x,y\in[a,b].\tag{1.7}$$

By the Lemma 1.1, we have

$$\begin{aligned}&\left|\frac{f_\psi^{[n]}(a)+f_\psi^{[n]}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2\,(\psi(b)-\psi(a))^{n-\alpha}}\left[{}^c\mathcal{D}_{a+}^{\alpha,\psi}f(b)+(-1)^n\,{}^c\mathcal{D}_{b-}^{\alpha,\psi}f(a)\right]\right|\\ =&\left|\frac{\psi(b)-\psi(a)}{2}\int_0^1\Big((1-t)^{n-\alpha}-t^{n-\alpha}\Big)\,f_\psi^{[n+1]}\Big(\mathcal{M}_{[\psi]}(a,b)\Big)\,dt\right|.\end{aligned}$$

Therefore, using inequality (1.7) to the points a and b, we obtain

$$\begin{aligned}
&\left|\frac{f_\psi^{[n]}(a)+f_\psi^{[n]}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2\left(\psi(b)-\psi(a)\right)^{n-\alpha}}\left[{}^c\mathcal{D}_{a+}^{\alpha,\psi}f(b)+(-1)^n\,{}^c\mathcal{D}_{b-}^{\alpha,\psi}f(a)\right]\right|\\
\leq&\frac{\psi(b)-\psi(a)}{2}\int_0^1\left|(1-t)^{n-\alpha}-t^{n-\alpha}\right|\left[t\left|f_\psi^{[n+1]}\right|(a)+(1-t)\left|f_\psi^{[n+1]}\right|(b)\right]dt\\
\leq&\frac{\psi(b)-\psi(a)}{2}\int_0^1\left|(1-t)^{n-\alpha}-t^{n-\alpha}\right|\left[\left|f_\psi^{[n+1]}\right|(a)+\left|f_\psi^{[n+1]}\right|(b)\right]dt\\
\leq&\frac{\psi(b)-\psi(a)}{2}\Big\{\left|f_\psi^{[n+1]}\right|(a)\int_0^1(1-t)^{n-\alpha}dt+\left|f_\psi^{[n+1]}\right|(b)\int_0^1(1-t)^{n-\alpha}dt\\
&+\left|f_\psi^{[n+1]}\right|(a)\int_0^1 t^{n-\alpha}dt+\left|f_\psi^{[n+1]}\right|(b)\int_0^1 t^{n-\alpha}dt\Big\}\\
=&\frac{\psi(b)-\psi(a)}{n-\alpha+1}\left[\left|f_\psi^{[n+1]}\right|(a)+\left|f_\psi^{[n+1]}\right|(b)\right].
\end{aligned}$$

So,

$$\begin{aligned}
&\left|\frac{f_\psi^{[n]}(a)+f_\psi^{[n]}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2\left(\psi(b)-\psi(a)\right)^{n-\alpha}}\left[{}^c\mathcal{D}_{a+}^{\alpha,\psi}f(b)+(-1)^n\,{}^c\mathcal{D}_{b-}^{\alpha,\psi}f(a)\right]\right|\\
\leq&\frac{\psi(b)-\psi(a)}{n-\alpha+1}\left[\left|f_\psi^{[n+1]}\right|(a)+\left|f_\psi^{[n+1]}\right|(b)\right].
\end{aligned}$$

□

Theorem 1.3. *Let f be a function such that $\left|f_\psi^{[n]}\right|$ belongs to the class $\mathcal{Q}(I)$, $a,b\in I$ with $0\leq a<b$ and $\left|f_\psi^{[n]}\right|\in L^1[a,b]$. Then, inequality*

$$\left|f_\psi^{[n]}\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)\right|\leq\frac{2\Gamma(n-\alpha+1)}{\left(\psi(b)-\psi(a)\right)^{n-\alpha}}\left|{}^c\mathcal{D}_{a+}^{\alpha,\psi}f(b)+(-1)^n\,{}^c\mathcal{D}_{b-}^{\alpha,\psi}f(a)\right| \tag{1.8}$$

holds for all $\alpha>0$.

Proof. Since $f_\psi^{[n]}$ is σ-convex, then we obtain

$$\left|f_\psi^{[n]}\left(\mathcal{M}_{[\sigma]}(x,y)\right)\right|\leq\lambda\left|f_\psi^{[n]}(x)\right|+(1-\lambda)\left|f_\psi^{[n]}(y)\right|,\qquad\text{for all } x,y\in[a,b],\ \lambda\in[0,1].$$

In particular for $\lambda=\frac{1}{2}$, we get

$$\left|f_\psi^{[n]}\left(\psi^{-1}\left(\frac{\psi(x)+\psi(y)}{2}\right)\right)\right|\leq 2\left(\left|f_\psi^{[n]}(x)\right|+\left|f_\psi^{[n]}(y)\right|\right),\qquad\text{for all } x,y\in[a,b].$$

Substituting $\psi(x)=t\psi(a)+(1-t)\psi(b)$ and $\psi(y)=(1-t)\psi(a)+t\psi(b)$, we obtain

$$\begin{aligned}
\left|f_\psi^{[n]}\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)\right|\leq&2\Big[f_\psi^{[n]}(\psi^{-1}(t\psi(a)+(1-t)\psi(b)))\\
&+f_\psi^{[n]}(\psi^{-1}((1-t)\psi(a)+t\psi(b)))\Big].
\end{aligned}\tag{1.9}$$

Multiplying both sides of (1.9) by $t^{n-\alpha-1}$, we can get

$$t^{n-\alpha-1}\left|f_{\psi}^{[n]}\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)\right| \leq 2t^{n-\alpha-1}\Big[f_{\psi}^{[n]}(\psi^{-1}(t\psi(a)+(1-t)\psi(b))) + f_{\psi}^{[n]}(\psi^{-1}((1-t)\psi(a)+t\psi(b)))\Big],$$

and integrating the above inequality with respect to t over the $[0,1]$, we obtain

$$\begin{aligned}
&f_{\psi}^{[n]}\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)\int_0^1 t^{n-\alpha-1}dt \leq 2(n-\alpha)\\
&\times\Big[\int_0^1 t^{n-\alpha-1}f_{\psi}^{[n]}\Big(\psi^{-1}\left(t\psi(a)+(1-t)\psi(b)\right)\Big)dt\\
&+\int_0^1 t^{n-\alpha-1}f_{\psi}^{[n]}\Big(\psi^{-1}\left((1-t)\psi(a)+t\psi(b)\right)\Big)dt\Big]\\
\leq&2(n-\alpha)\Big[\int_a^b \psi'(x)\left(\frac{\psi(b)-\psi(x)}{\psi(b)-\psi(a)}\right)^{n-\alpha-1} f_{\psi}^{[n]}(x)\frac{dx}{\psi(b)-\psi(a)}\\
&+\int_a^b \psi'(y)\left(\frac{\psi(y)-\psi(a)}{\psi(b)-\psi(a)}\right)^{n-\alpha-1} f_{\psi}^{[n]}(y)\frac{dy}{\psi(b)-\psi(a)}\Big]\\
\leq&\frac{2(n-\alpha)}{(\psi(b)-\psi(a))^{n-\alpha}}\Big[\int_a^b \psi'(x)\left(\psi(b)-\psi(x)\right)^{n-\alpha-1} f_{\psi}^{[n]}(x)dx\\
&+\int_a^b \psi'(y)\left(\psi(y)-\psi(a)\right)^{n-\alpha-1} f_{\psi}^{[n]}(y)dy\Big]\\
=&\frac{2\Gamma(n-\alpha+1)}{(\psi(b)-\psi(a))^{n-\alpha}}\Big[{}^{c}\mathcal{D}_{a+}^{\alpha,\psi}f(b)+(-1)^n\,{}^{c}\mathcal{D}_{b-}^{\alpha,\psi}f(a)\Big].
\end{aligned}$$

This ends the proof. □

Corollary 1.2. *If we take $\psi(x)=x$. Then, the inequality* (1.8) *reduces to the following inequality*

$$f^{(n)}\Big(\frac{a+b}{2}\Big) \leq \frac{2\Gamma(n-\alpha+1)}{(b-a)^{n-\alpha}}\Big|{}^{c}\mathcal{D}_{a+}^{\alpha}f(b)+(-1)^n\,{}^{c}\mathcal{D}_{b-}^{\alpha}f(a)\Big|. \tag{1.10}$$

Theorem 1.4. *Let f be a function such that $\left|f_{\psi}^{[n]}\right|$ belongs to the class $\mathcal{P}(I)$, $a,b\in I$ with $0\leq a<b$ and $\left|f_{\psi}^{[n]}\right|\in L^1[a,b]$. Then, inequality*

$$\begin{aligned}
\left|f_{\psi}^{[n]}\left(\psi^{-1}\Big(\frac{\psi(a)+\psi(b)}{2}\Big)\right)\right| \leq&\frac{2\Gamma(n-\alpha+1)}{\Big(\psi(b)-\psi(a)\Big)^{n-\alpha}}\Big|{}^{c}\mathcal{D}_{a+}^{\alpha,\psi}f(b)+(-1)^n\,{}^{c}\mathcal{D}_{b-}^{\alpha,\psi}f(a)\Big|\\
\leq&\left|f_{\psi}^{[n]}\right|(a)+\left|f_{\psi}^{[n]}\right|(b)
\end{aligned} \tag{1.11}$$

holds for all $\alpha>0$.

Proof. Using Definition 1.2 and substituting $\psi(x) = t\psi(a) + (1-t)\psi(b)$ and $\lambda = \frac{1}{2}$, we obtain

$$\left|f_\psi^{[n]}\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)\right| \leq \left|f_\psi^{[n]}\right|\Big(t\psi(a)+(1-t)\psi(b)\Big) + \left|f_\psi^{[n]}\right|\Big((1-t)\psi(a)+t\psi(b)\Big), \tag{1.12}$$

for all $t \in [0,1]$. As in the proof of the last theorem, multiplying both sides of inequality (1.12) by $t^{n-\alpha-1}$ and integrating the resulting inequality with respect to t over the interval $[0,1]$, we get

$$\left|f_\psi^{[n]}\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)\right| \int_0^1 t^{n-\alpha-1}dt \leq \int_0^1 t^{n-\alpha-1}\Big[\left|f_\psi^{[n]}\right|\Big(t\psi(a)+(1-t)\psi(b)\Big) + \left|f_\psi^{[n]}\right|\Big((1-t)\psi(a)+t\psi(b)\Big)\Big]dt.$$

Then,

$$\frac{1}{n-\alpha}\left|f_\psi^{[n]}\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)\right| \leq \frac{1}{\big(\psi(b)-\psi(a)\big)^{n-\alpha}} \int_a^b \Big[\big(\psi(b)-\psi(x)\big)^{n-\alpha-1} + \big(\psi(x)-\psi(a)\big)^{n-\alpha-1}\Big]\psi'(x)\left|f_\psi^{[n]}\right|(x)dx.$$

And thus,

$$\left|f_\psi^{[n]}\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right)\right| \leq \frac{n-\alpha}{\big(\psi(b)-\psi(a)\big)^{n-\alpha}} \int_a^b \Big[\big(\psi(b)-\psi(x)\big)^{n-\alpha-1} + \big(\psi(x)-\psi(a)\big)^{n-\alpha-1}\Big]\psi'(x)\left|f_\psi^{[n]}\right|(x)dx.$$

Therefore,

$$\left|f_\psi^{[n]}\right|\left(\psi^{-1}\left(\frac{\psi(a)+\psi(b)}{2}\right)\right) \leq \frac{\Gamma(n-\alpha+1)}{\big(\psi(b)-\psi(a)\big)^{n-\alpha}}\Big[{}^c\mathcal{D}_{a+}^{\alpha,\psi}f(b) + (-1)^n\, {}^c\mathcal{D}_{b-}^{\alpha,\psi}f(a)\Big],$$

which completes the proof of the first inequality in (1.11). Since $\left|f_\psi^{[n]}\right|$ belongs to the class $\mathcal{P}(I)$, then we can write

$$\left|f_\psi^{[n]}\left(\psi^{-1}\Big(t\psi(a)+(1-t)\psi(b)\Big)\right)\right| \leq \left|f_\psi^{[n]}(a)\right| + \left|f_\psi^{[n]}(b)\right|$$

and

$$\left|f_\psi^{[n]}\left(\psi^{-1}\Big((1-t)\psi(a)+t\psi(b)\Big)\right)\right| \leq \left|f_\psi^{[n]}(a)\right| + \left|f_\psi^{[n]}(b)\right|.$$

Combining these two inequalities, provide that the dummy variable choosing in $[0,1]$ for $\mathcal{M}_{[\psi]}$ is t, we obtain

$$\left|f_\psi^{[n]}\Big(\mathcal{M}_{[\psi]}\big(\psi(a),\psi(b)\big)\Big)\right|+\left|f_\psi^{[n]}\Big(\mathcal{M}_{[\psi]}\big(\psi(b),\psi(a)\big)\Big)\right|\leq 2\Big(\left|f_\psi^{[n]}(a)\right|+\left|f_\psi^{[n]}(b)\right|\Big). \tag{1.13}$$

If we multiplie both sides of (1.13) by $t^{n-\alpha-1}$ and integrate over the interval $[0,1]$ with respect to the variable t, we get

$$\begin{aligned}&\int_0^1 t^{n-\alpha-1}\Big[f_\psi^{[n]}\Big(\mathcal{M}_{[\psi]}\big(\psi(a),\psi(b)\big)\Big)+f_\psi^{[n]}\Big(\mathcal{M}_{[\psi]}\big(\psi(b),\psi(a)\big)\Big)\Big]dt\\ \leq&2\Big(f_\psi^{[n]}(a)+f_\psi^{[n]}(b)\Big)\int_0^1 t^{n-\alpha-1}dt,\end{aligned}$$

it follows that

$$\begin{aligned}&\frac{n-\alpha}{\big(\psi(b)-\psi(x)\big)^{n-\alpha}}\int_a^b\Big[\big(\psi(b)-\psi(x)\big)^{n-\alpha-1}+\big(\psi(x)-\psi(b)\big)^{n-\alpha-1}\Big]\psi'(x)\left|f_\psi^{[n]}(x)\right|dx\\ \leq&2\Big(\left|f_\psi^{[n]}(a)\right|+\left|f_\psi^{[n]}(b)\right|\Big),\end{aligned}$$

and therefore,

$$\begin{aligned}&\frac{\Gamma(n-\alpha+1)}{\big(\psi(b)-\psi(x)\big)^{n-\alpha}}\int_a^b\Big[{}^c\mathcal{D}_{a+}^{\alpha,\psi}f(b)+(-1)^n\,{}^c\mathcal{D}_{b-}^{\alpha,\psi}f(a)\Big]\psi'(x)\left|f_\psi^{[n]}(x)\right|dx\\ \leq&2\Big(\left|f_\psi^{[n]}(a)\right|+\left|f_\psi^{[n]}(b)\right|\Big).\end{aligned}$$

Finally, the second inequality of (1.11) is proved, which completes the proof. ☐

Theorem 1.5. *Assume that* $f_\psi^{[n]}\in\mathcal{S}X(h,I)$, $a,b\in I$ *with* $a<b$. *Then,*

$$\begin{aligned}\frac{1}{(n-\alpha)h(\frac{1}{2})}f_\psi^{[n]}\Big(\frac{a+b}{2}\Big)\leq&\frac{\Gamma(n-\alpha)}{\big(\psi(b)-\psi(a)\big)^{n-\alpha}}\Big[{}^c\mathcal{D}_{a+}^{\alpha,\psi}f(b)+(-1)^n\,{}^c\mathcal{D}_{b-}^{\alpha,\psi}f(a)\Big]\\ \leq&\Big(f_\psi^{[n]}(a)+f_\psi^{[n]}(b)\Big)\int_0^1 t^{n-\alpha-1}[h(t)+h(1-t)]dt.\end{aligned} \tag{1.14}$$

Proof. Applying the inequality in Definition 1.3, provided that $\psi(x)=t\psi(a)+(1-t)\psi(b)$, $\psi(y)=(1-t)\psi(a)+t\psi(b)$, we get

$$f_\psi^{[n]}\Big(\frac{\psi(x)+\psi(y)}{2}\Big)\leq h\left(\frac{1}{2}\right)\Big[f_\psi^{[n]}\big(\psi(x)\big)f_\psi^{[n]}\big(\psi(y)\big)\Big]. \tag{1.15}$$

Multiplying both sides of (1.15) by $t^{n-\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}\frac{1}{h\big(\frac{1}{2}\big)}\int_0^1 t^{n-\alpha-1}f_\psi^{[n]}\Big(\frac{\psi(a)+\psi(b)}{2}\Big)dt\leq&\int_0^1 t^{n-\alpha-1}f_\psi^{[n]}\Big(t\psi(a)+(1-t)\psi(b)\Big)dt\\ &+\int_0^1 t^{n-\alpha-1}f_\psi^{[n]}\Big((1-t)\psi(a)+t\psi(b)\Big)dt.\end{aligned}$$

And thus,

$$\begin{aligned}
&\frac{1}{h\left(\frac{1}{2}\right)(n-\alpha)} f_{\psi}^{[n]}\Big(\frac{\psi(a)+\psi(b)}{2}\Big)\\
\leq &\frac{1}{\Big(\psi(b)-\psi(a)\Big)^{n-\alpha}} \int_a^b \Big\{\Big(\psi(b)-\psi(x)\Big)^{n-\alpha-1} f_{\psi}^{[n]}(x)\psi'(x)\\
&+\Big(\psi(x)-\psi(a)\Big)^{n-\alpha-1} f_{\psi}^{[n]}(x)\psi'(x)\Big\}dx.
\end{aligned}$$

Consequently,

$$\frac{1}{h\left(\frac{1}{2}\right)(n-\alpha)} f_{\psi}^{[n]}\Big(\frac{\psi(a)+\psi(b)}{2}\Big) \leq \Big[{}^{c}\mathcal{D}_{a+}^{\alpha,\psi} f(b) + (-1)^n\, {}^{c}\mathcal{D}_{b-}^{\alpha,\psi} f(a)\Big],$$

and the first part of the inequality (1.14) is proved. Since $f_{\psi}^{[n]} \in \mathcal{S}X(h,I)$, we obtain

$$f_{\psi}^{[n]}(t\psi(x)+(1-t)\psi(y)) \leq h(t) f_{\psi}^{[n]}(x) + (1-t) f_{\psi}^{[n]}(y)$$

and

$$f_{\psi}^{[n]}((1-t)\psi(x)+t\psi(y)) \leq h(1-t) f_{\psi}^{[n]}(x) + h(t) f_{\psi}^{[n]}(y).$$

Combining these two inequalities, we get:

$$\begin{aligned}
&f_{\psi}^{[n]}(t\psi(x)+(1-t)\psi(y)) + f_{\psi}^{[n]}((1-t)\psi(x)+t\psi(y))\\
\leq &\Big(h(t)+h(1-t)\Big)\Big[f_{\psi}^{[n]}(x) + f_{\psi}^{[n]}(y)\Big].
\end{aligned} \tag{1.16}$$

If we take $x = a$ and $y = b$ in inequality (1.16), multiplying its both sides by $t^{n-\alpha-1}$ and integrating the resulting inequality over $[0,1]$ with respect to t we obtain

$$\begin{aligned}
&\int_0^1 t^{n-\alpha-1}\Big[f_{\psi}^{[n]}\Big(t\psi(a)+(1-t)\psi(b)\Big) + f_{\psi}^{[n]}\Big((1-t)\psi(a)+t\psi(b)\Big)\Big]dt\\
\leq &\int_0^1 t^{n-\alpha-1}\Big(h(t)+h(1-t)\Big)\Big[f_{\psi}^{[n]}(a) + f_{\psi}^{[n]}(b)\Big]dt.
\end{aligned} \tag{1.17}$$

Hence,

$$\begin{aligned}
&\frac{\Gamma(n-\alpha)}{\Big(\psi(b)-\psi(a)\Big)^{n-\alpha}} \Big[{}^{c}\mathcal{D}_{a+}^{\alpha,\psi} f(b) + (-1)^n\, {}^{c}\mathcal{D}_{b-}^{\alpha,\psi} f(a)\Big]\\
\leq &\Big[f_{\psi}^{[n]}(a) + f_{\psi}^{[n]}(b)\Big] \int_0^1 t^{n-\alpha-1}\Big(h(t)+h(1-t)\Big)dt.
\end{aligned} \tag{1.18}$$

Thus, the second inequality of (1.14) is proved, hence the proof is completed. □

2. Conclusion

We have formulated novel generalizations concerning inequalities of fractional type associated with different class of σ-convex type functions using the ψ-Caputo fractional operator knowing that ψ plays the same role as σ. Once again, we want to emphasize that our major finding, which possesses a broad and general nature, can be adapted to obtain various compelling fractional inequalities.

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