

CONVERGENCE ANALYSIS OF LEAST SQUARES-EPSILON-RITZ ALGORITHM FOR SOLVING A GENERAL CLASS OF PANTOGRAPH EQUATIONS

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ABSTRACT. In this paper, we propose an approximate method for solving a general class of pantograph differential equations. The proposed method is based on a combination of least squares, epsilon and Ritz methods. The convergence properties of the method are analyzed and discussed. Finally, several numerical examples are given to illustrate the applicability and efficiency of the method.

1. INTRODUCTION

Pantograph equations are a kind of functional differential equations which have wide applications in various fields such as analytic number theory, nonlinear dynamical systems and probability theory on algebraic structures [3, 13, 14]. The name pantograph originated from the Ockendon and Tayler's work on the collection of current by the pantograph head of an electric locomotive [16]. In recent years there has been the growing interest in the pantograph equations [1, 3, 11, 15]. Pantograph equations are usually difficult to solve analytically, so numerical methods are required. There has been number of some numerical methods for solving pantograph equations are presented by researchers of this field. So far various numerical methods have been proposed to solve pantograph equations such as the variational and modified variational iteration method [2, 11, 19, 27], the homotopy method [28], Taylor polynomial method [21–23], the Bessel collocation method [24, 29, 30], the reproducing kernel space method [4], the discontinuous Galerkin methods [1], the Runge-Kutta methods [9] and [10], the θ -methods [25] and [26], the Chebyshev polynomials method [20], the Hermite

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method [31], the rational approximation method [7], the linear multistep methods [6], the shifted Legendre approximation method [32], the generalized fractional-order Bernoulli wavelet method [17] and the shifted orthonormal Bernstein polynomials method [8].

In this work, by combining least square and epsilon methods [5, 12] we present an efficient method for solving the multi-pantograph equation. The remainder of the paper is structured as follows. After explaining the problem, we obtain the least squares-Epsilon method for solving the multi-pantograph equation in Section 3 and Section 4. In Section 5, the convergence results of the least squares-epsilon method are given. We use numerical example to confirm the efficiency of the method in Section 6.

2. STATEMENT OF THE PROBLEM

Our goal in this article is to find approximate solution for multi-pantograph equation of the following form

$$(2.1) \quad \sum_{i=1}^{\rho} \frac{f_i(x)}{x^l(x-\lambda)^d} u^i(x) + \sum_{i=1}^{\bar{\rho}} \frac{g_i(x)}{x^l(x-\lambda)^d} u^{(i)}(x) + \sum_{i=1}^{\delta} \sum_{j=1}^{\gamma} \frac{\alpha_{i,j}(x)}{x^l(x-\lambda)^d} u^i(p_j x) \\ + \sum_{i=1}^{\bar{\delta}} \sum_{j=1}^{\bar{\gamma}} \frac{\beta_{i,j}(x)}{x^l(x-\lambda)^d} u^{(i)}(q_j x) = h(x), \quad 0 < x < \lambda,$$

with boundary conditions

$$(2.2) \quad u^{(i)}(0) = \eta_{0,i}, \quad u^{(j)}(\lambda) = \eta_{1,j},$$

$i = 0, 1, \dots, n_1 - 1, j = 0, 1, \dots, n_2 - 1, n_1 + n_2 = n = \max\{\bar{\rho}, \bar{\delta}\}$, where $0 < p_i, q_j < 1$ and $l, d \in \mathbb{Z}^+$. Here we suppose that $f_i(x), g_i(x), \alpha_{i,j}(x), \beta_{i,j}(x), h(x) \in C(0, \lambda)$. In this problem unknown function $u(x)$ belongs to the space $C^n(0, \lambda)$.

The problem (2.1) with boundary conditions (2.2) is a singular equation defined on the open interval $(0, \lambda)$. In the subsequent development, we proceed solving the following problem

$$(2.3) \quad \sum_{i=1}^{\rho} f_i(x) u^i(x) + \sum_{i=1}^{\bar{\rho}} g_i(x) u^{(i)}(x) + \sum_{i=1}^{\delta} \sum_{j=1}^{\gamma} \alpha_{i,j}(x) u^i(p_j x) \\ + \sum_{i=1}^{\bar{\delta}} \sum_{j=1}^{\bar{\gamma}} \beta_{i,j}(x) u^{(i)}(q_j x) = x^l(x-\lambda)^d h(x), \quad 0 \leq x \leq \lambda,$$

$$(2.4) \quad u^{(i)}(0) = \eta_{0,i}, \quad u^{(j)}(\lambda) = \eta_{1,j},$$

$i = 0, 1, \dots, n_1 - 1, j = 0, 1, \dots, n_2 - 1, n_1 + n_2 = n = \max\{\bar{\rho}, \bar{\delta}\}$, where $u(x) \in C^n[0, \lambda]$ and $f_i(x), g_i(x), \alpha_{i,j}(x), \beta_{i,j}(x), h(x) \in C[0, \lambda]$. It is obvious that the solution of the problem (2.3) with conditions (2.4) also satisfies the problem (2.1) with conditions (2.2).

3. LEAST SQUARES-EPSILON METHOD

In this section, we develop an approximate method for solving problem (2.3) with conditions (2.4) by combining least square and epsilon methods. Without loss of generality, we consider $\lambda = 1$ in (2.3) and (2.4). Consider Equation (2.3) in the following form

$$(3.1) \quad F(x, u(x), u^{(1)}(x), \dots, u^{(\bar{p})}(x), u(p_1x), \dots, u(p_\gamma x), u^{(1)}(q_1x), \dots, u^{(1)}(q_{\bar{\gamma}}x), \dots, u^{(\bar{\delta})}(q_1x), \dots, u^{(\bar{\delta})}(q_{\bar{\gamma}}x)) = 0, \quad 0 \leq x \leq 1.$$

Define functional $J : M \rightarrow \mathbb{R}$ as follows

$$(3.2) \quad J[u] := \int_0^1 F^2 dx = \|F\|_{L^2}^2,$$

where

$$(3.3) \quad M := \{u \in C^n[0, 1] : u^{(i)}(0) = \eta_{0,i}, u^{(j)}(1) = \eta_{1,j}, i = 0, 1, \dots, n_1 - 1,$$

$$(3.4) \quad j = 0, 1, \dots, n_2 - 1\}.$$

Now we assume that Equation (2.3) has a solution u_{ex} in M . So, the functional J has a minimizing solution $u_{ex} \in M$ with minimum value $J[u_{ex}] = 0$. Our purpose is to find an approximate minimizing solution for the functional J given by (3.2) on M . The minimization problem $\min J|_M$ is a constrained optimization problem. By applying the epsilon method, we transform the constrained optimization problem into the following unconstrained problem

$$(3.5) \quad \tilde{J}[u] = \int_0^1 F^2 dx + \frac{1}{\epsilon} \left(\sum_{i=0}^{n_1-1} (u^{(i)}(0) - \eta_{0,i})^2 + \sum_{j=0}^{n_2-1} (u^{(j)}(1) - \eta_{1,j})^2 \right),$$

where $\epsilon > 0$ is given.

We minimize the functional \tilde{J} on $C^n[0, 1]$. It is obvious that u_{ex} is also a minimizing solution for \tilde{J} for any considered value of ϵ and $\tilde{J}[u_{ex}] = 0$. Theorem 5.2 ensures that solving the problem $\min \tilde{J}|_{C^n[0,1]}$ by utilizing the Ritz method leads to an approximate minimizing solution for the functional J on M .

4. APPROXIMATE SOLUTION OF PANTOGRAPH EQUATION

Consider expansion $u_{k,\epsilon}(t)$, in the following form

$$(4.1) \quad u_k(t) = C_k^T \cdot \Psi_k(t), \quad \Psi_k(t) = \begin{pmatrix} \phi_0(t) \\ \phi_1(t) \\ \vdots \\ \phi_k(t) \end{pmatrix}, \quad C_k = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_k \end{pmatrix}.$$

Here, $\phi_j(t)$, $j \in \{0\} \cup \mathbb{N}$ are shifted Legendre orthonormal polynomials

$$(4.2) \quad \phi_j(t) = \sqrt{2j+1} \sum_{k=0}^j (-1)^{j+k} \frac{(j+k)! t^k}{(j-k)!(k!)^2}, \quad j = 0, 1, 2, \dots, t \in [0, 1].$$

Substituting u_k into (3.5) gives us

$$(4.3) \quad \tilde{J}[c_0, \dots, c_k] = \int_0^1 F_k^2 dx + \frac{1}{\epsilon} \left(\sum_{i=0}^{n_1-1} (u_k^{(i)}(0) - \eta_{0,i})^2 + \sum_{j=0}^{n_2-1} (u_k^{(j)}(1) - \eta_{1,j})^2 \right),$$

where

$$F_k := F(x, u_k(x), u_k^{(1)}(x), \dots, u_k^{(\bar{\rho})}(x), u_k(p_1x), \dots, u_k(p_\gamma x), u_k^{(1)}(q_1x), \dots, u_k^{(1)}(q_{\bar{\gamma}}x), \dots, u_k^{(\bar{\delta})}(q_1x), \dots, u_k^{(\bar{\delta})}(q_{\bar{\gamma}}x)),$$

which is an algebraic function of unknowns $c_j, j = 0, 1, \dots, k$. If c_j s be determined by minimizing function \tilde{J} , then by (4.1) we achieve function that approximate minimum value of \tilde{J} in (4.3). According to differential calculus, the following system of equations is the necessary condition of optimization for the multidimensional function (4.3)

$$(4.4) \quad \frac{\partial \tilde{J}}{\partial c_j} = 0, \quad 0 \leq j \leq k.$$

By solving system (4.4), we can determine minimizing values of $c_j, j = 0, 1, \dots, k$, for function (4.3). Hence, we achieve functions u_j , by (4.1), which approximate minimum value of J by

$$J[c_0, \dots, c_k] = \int_0^1 F_k^2 dx,$$

while

$$\begin{aligned} u_k^{(i)}(0) &\simeq \eta_{0,i}, & i = 0, 1, \dots, n_1 - 1, \\ u_k^{(j)}(1) &\simeq \eta_{1,j}, & j = 0, 1, \dots, n_2 - 1. \end{aligned}$$

5. ON THE CONVERGENCE OF THE METHOD

In this section we discuss the convergence of the method presented in Section 4. Lemma 5.1 shows that the functional \tilde{J} is continuous on $C^n[0, 1]$ with respect to the norm, $\|\cdot\|_n$, defined as follows

$$\|u\|_n = \|u\|_\infty + \dots + \|u^{(n)}\|_\infty.$$

We use this important property later in Theorem 5.2. The following theorem from real analysis plays key roll in the proof of Lemma 5.1.

Theorem 5.1. *Let f be continuous mapping of a compact metric space X into a metric space Y , then f is uniformly continuous.*

Proof. See [18]. □

Lemma 5.1. *The functional J is continuous on $C^n[0, 1]$.*

Proof. Let $u^* \in C^n[0, 1]$. Notice that $\Delta > 0$ is given. Consider $d > 0$ and

$$I = [0, 1] \times \prod_{i=1}^{1+\bar{\rho}+\gamma+\bar{\gamma}\bar{\delta}} [-L - d, L + d],$$

where $L = \max\{\|u\|_\infty, \|u'\|_\infty, \dots, \|u^{(n)}\|_\infty\}$. Obviously

$$(5.1) \quad U^*(x) := (x, u^*(x), u^{*(1)}(x), \dots, u^{*(\bar{\rho})}(x), u^*(p_1x), \dots, u^*(p_\gamma x), u^{*(1)}(q_1x), \dots, u^{*(1)}(q_{\bar{\gamma}}x), \dots, u^{*(\bar{\delta})}(q_1x), \dots, u^{*(\bar{\delta})}(q_{\bar{\gamma}}x)) \in I, \quad x \in [0, 1].$$

$\Delta_1 > 0$ is given. Let $\delta_1 > 0$ and $\|u - u^*\|_n < \delta_1$, then we have

$$(5.2) \quad \|u^{(j)} - u^{*(j)}\|_\infty < \delta_1, \quad 1 \leq j \leq \bar{\rho},$$

$$(5.3) \quad \max\{|u(p_jx) - u^*(p_jx)| : x \in [0, 1]\} < \delta_1, \quad 1 \leq j \leq \gamma,$$

$$(5.4) \quad \max\{|u^{(i)}(q_jx) - u^{*(i)}(q_jx)| : x \in [0, 1]\} < \delta_1, \quad 1 \leq j \leq \bar{\gamma}, 1 \leq i \leq \bar{\delta}.$$

According to (5.2)-(5.4) it is easy to see that for small enough value of δ_1 we have

$$(5.5) \quad U(x) := (x, u(x), u^{(1)}(x), \dots, u^{(\bar{\rho})}(x), u(p_1x), \dots, u(p_\gamma x), u^{(1)}(q_1x), \dots, u^{(1)}(q_{\bar{\gamma}}x), \dots, u^{(\bar{\delta})}(q_1x), \dots, u^{(\bar{\delta})}(q_{\bar{\gamma}}x)) \in I, \quad x \in [0, 1],$$

$$|U(x) - U^*(x)| < \Delta_1, \quad x \in [0, 1].$$

Since function F is continuous on I and I is a compact set, according to Theorem 5.1, F is uniformly continuous on I . So if $\delta_1 > 0$ be sufficiently small, then $|U(x) - U^*(x)| < \Delta_1$ implies that $|F(U(x)) - F(U^*(x))| < \Delta, x \in [0, 1]$, and $|J[u] - J[u^*]| < \Delta$. \square

Define $L_\epsilon : C^n[0, 1] \rightarrow \mathbb{R}$ as follows

$$(5.6) \quad L_\epsilon[u] := \frac{1}{\epsilon} \left(\sum_{i=0}^{n_1-1} (u^{(i)}(0) - \eta_{0,i})^2 + \sum_{j=0}^{n_2-1} (u^{(j)}(1) - \eta_{1,j})^2 \right).$$

It is easy to observe that the functional L_ϵ is continuous on $C^n[0, 1]$ with respect to $\|\cdot\|_n$. So by Lemma 5.1, for any selection of $\epsilon \in \mathbb{R}$, the functional \tilde{J} is continuous on $C^n[0, 1]$.

Theorem 5.2. *Let $\hat{\mu}_k = \tilde{J}[u_k]$ be the minimum of the functional \tilde{J} on $P_k[0, 1]$ and $\mu_k := J[u_k]$, then*

$$\lim_{k \rightarrow \infty} \mu_k = 0,$$

and

$$\begin{aligned} \lim_{j \rightarrow \infty} |u_k^{(i)}(0) - \eta_{0,i}| &= 0, \quad i = 0, 1, \dots, n_1 - 1, \\ \lim_{j \rightarrow \infty} |u_k^{(i)}(1) - \eta_{1,i}| &= 0, \quad i = 0, 1, \dots, n_2 - 1. \end{aligned}$$

TABLE 1. The approximate values of μ_k for $k = 2, 3, 4$ for Example 6.1

k	μ_k
2	0.000813977
3	9.440282×10^{-6}
4	1.3411×10^{-7}

Proof. For any given $\Delta > 0$, we consider $u^* \in C^n[0, 1]$ such that

$$(5.7) \quad \tilde{J}[u^*] < \Delta.$$

Such u^* exists by the properties of minimum. On the other hand, \tilde{J} is continuous on $C^n[0, 1]$ so we have

$$(5.8) \quad |\tilde{J}[u] - \tilde{J}[u^*]| < \Delta,$$

provided that $\|u - u^*\|_n < \eta$. According to Weierstrass theorem given in [18], for large enough value of k , there exist $p_k(x) \in P_k[0, 1]$ such that $\|p_k - u^*\| < \eta$. Moreover let u_k be the element of $P_k[0, 1]$ such that $\tilde{J}[u_k] = \hat{\mu}_k$, then using (5.7) and (5.8) we have

$$0 \leq \tilde{J}[u_k] \leq \tilde{J}[p_k] < 2\Delta.$$

Since $\Delta > 0$ is arbitrary, it follows that $\lim_{k \rightarrow \infty} \hat{\mu}_k = 0$ and the result can be easily obtained. \square

6. TEST PROBLEMS

Example 6.1. As the first example, we study the following singular multi-pantograph delay differential equation

$$u^{(2)}(x) + \frac{1}{x}u^{(1)}\left(\frac{x}{2}\right) + \frac{1}{x^2}u^{(1)}\left(\frac{x}{4}\right) - \frac{1}{x-1}u(x) = h(x), \quad 0 < x \leq 1,$$

$$u(0) = 1, \quad u(1) = e,$$

$$h(x) = e^x + \frac{1}{x}e^{\frac{x}{2}} + \frac{1}{x^2}e^{\frac{x}{4}} - \frac{1}{x-1}e^x.$$

For above problem, we can see that exact solution is $u(x) = e^x$. Applying the method presented in Section 4, we achieve the approximate solution for the problem. The approximate values of μ_k , for different number of basis functions k , are demonstrated in Table 1. Also Table 2 shows absolute error $|u_k(x) - u_{ex}(x)|$.

Example 6.2. In this example, we consider the following nonlinear equation

$$u^{(1)}(x) + 2u^2\left(\frac{x}{2}\right) = 1, \quad 0 < x \leq 1,$$

$$u(0) = 0.$$

Noting to this problem, we can verify that the exact solution is $u(x) \sin(x)$. Table 3 shows the values of μ_k for different values of k and Table 4 demonstrates the absolute error $|u_k(x) - u_{ex}(x)|$.

TABLE 2. Absolute error for $k = 2, 3, 4$ for Example 6.1

x	$k = 2$	$k = 3$	$k = 4$
0	1.71305×10^{-7}	2.2301×10^{-8}	5.19587×10^{-9}
0.1	0.000846204	2.18047×10^{-6}	2.23933×10^{-6}
0.2	0.0022476	0.00099343	0.0000524334
0.3	0.00811796	0.00236922	0.0000851558
0.4	0.0154792	0.00362989	0.0000788594
0.5	0.0229106	0.00441537	0.0000430139
0.6	0.0288417	0.004515	3.88571×10^{-6}
0.7	0.0315373	0.00388327	0.0000111775
0.8	0.0290793	0.00265722	8.8842×10^{-6}
0.9	0.0193481	0.00117555	0.0000396475
1	1.08041×10^{-6}	9.01796×10^{-8}	7.90223×10^{-9}

TABLE 3. The approximate values of μ_k for $k = 1, 3, 5$ for Example 6.2

k	μ_k
1	0.0124095
3	2.19831×10^{-6}
5	2.27916×10^{-11}

TABLE 4. Absolute error for $k = 1, 3, 5$ for Example 6.2

x	$k=1$	$k=3$	$k=5$
0	3.37176×10^{-6}	1.27022×10^{-9}	1.89865×10^{-14}
0.1	0.0142226	0.000169393	1.15815×10^{-7}
0.2	0.0274443	0.000108566	4.02153×10^{-7}
0.3	0.038681	0.0000550637	3.08125×10^{-7}
0.4	0.0469649	0.000213911	2.1473×10^{-7}
0.5	0.0513579	0.000289893	5.49804×10^{-7}
0.6	0.0509607	0.000243804	3.47773×10^{-7}
0.7	0.0449217	0.0000843014	1.61122×10^{-7}
0.8	0.0324459	0.000123587	3.33028×10^{-7}
0.9	0.0128025	0.00025052	1.60343×10^{-7}
1	0.0146676	0.0000955398	4.00845×10^{-8}

From the above numerical results, we can see that the proposed method is quite efficient.

7. CONCLUSION

This paper have developed an approximate method based on least squares, epsilon and Ritz methods for solving a general class of singular and nonsingular multi-pantograph equations. The convergence of the method has been extensively discussed

and illustrative test examples have been included to demonstrate validity and applicability of the new method.

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