COMPACTNESS ESTIMATE FOR THE $\bar{\partial}$-NEUMANN PROBLEM
ON A $q$-PSEUDOCONVEX DOMAIN IN A STEIN MANIFOLD

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ABSTRACT. We consider a smoothly bounded $q$-pseudoconvex domain $\Omega$ in an $n$-dimensional Stein manifold $X$ and suppose that the boundary $b\Omega$ of $\Omega$ satisfies $(q-P)$ property, which is the natural variant of the classical $P$ property. Then, one proves the compactness estimate for the $\bar{\partial}$-Neumann operator $N_{r,s}$ in the Sobolev $k$-space. Applications to the boundary global regularity for the $\bar{\partial}$-Neumann operator $N_{r,s}$ in the Sobolev $k$-space are given. Moreover, we prove the boundary global regularity of the $\bar{\partial}$-operator on $\Omega$.

1. INTRODUCTION AND MAIN RESULTS

The existence and regularity properties of the solutions of the system of Cauchy-Riemann equations $\bar{\partial}f = g$ on strongly pseudo-convex domains have been a central theme in the theory of several complex variables for many years. Classically many different approaches have been used: a) Vanishing of the $\bar{\partial}$-cohomology group, b) The abstract $L^2$-theory of the $\bar{\partial}$-Neumann problem, and c) The construction of rather explicit integral solution operators for $\bar{\partial}$, in analogy to the Cauchy transform in $C^1$. The first approach used by Grauert–Riemenschneider [6]. Saber [15], used this method and studied the solvability of the $\bar{\partial}$-problem with $C^\infty$ regularity up to the boundary on a strictly $q$-convex domain of an $n$-dimensional Kähler manifold $X$. The second approach was first used by Kohn [11] in studying the boundary regularity of the $\bar{\partial}$-equation when $\Omega$ is pseudoconvex with $C^\infty$ boundary. For solvability with regularity up to the boundary in a pseudoconvexity domain without corners, one refer to Kohn...
Zampieri [18] introduced a new type of notion of \( q \)-pseudoconvexity in \( \mathbb{C}^n \). Under this condition he proved local boundary regularity for any degree \( \geq q \). Other results in this direction belong to Heungju [10], Baracco-Zampieri [1] and Saber [16]. Thus the method of \( L^2 \) a priori estimates for the weighted \( \overline{\partial} \)-Neumann operator has yielded many important results on the local and global boundary regularity of the \( \overline{\partial} \)-problem. The integral formula approach was pioneered by Henkin [8] and Grauert-Lieb [7] for strictly pseudoconvex domains. They obtained uniform and Hölder estimates for the solution of \( \overline{\partial} \) on such domains. For the related results for \( \overline{\partial} \) on the pseudoconcave domains in \( \mathbb{P}^n \), see Henkin-Iordan [9].

In this paper, he compactness estimate proved in Khanh and Zampieri [17] is extended \( E \)-valued forms. Such compactness estimates immediately lead to very important qualitative properties of the \( \overline{\partial} \)-operator, such as smoothness of solutions and closed range. The main theorem generalizes Khanh and Zampieri [17] result to forms with values in a vector bundle. The proof starting with the known estimate on scalar differential forms and then obtains a similar estimate locally on bundle-valued forms using a local frame. Then, by using a partition of unity, we globalize this estimate at the cost of the constants. Consequently, we study the boundary regularity of the \( \overline{\partial} \)-equation, \( \overline{\partial}u = f \), for forms in a vector bundle on bounded \( q \)-pseudoconvex domain \( \Omega \) in a Stein manifold \( X \) of dimension \( n \). Moreover, some standard consequences of compactness are deduced.

2. \((q-P)\) PROPERTY

Let \( \Omega \) be a bounded domain of \( \mathbb{C}^n \) with \( C^1 \)-boundary \( b\Omega \) and \( \rho \) its a \( C^1 \)-defining function. An \((r,s)\)-form on \( \Omega \) is given by

\[
f = \sum_{I,J} f_{I,J} dz^I \wedge d\bar{z}^J,
\]

where \( I = (i_1, \ldots, i_r) \) and \( J = (j_1, \ldots, j_s) \) are multiindices and \( dz^I = dz_1 \wedge \cdots \wedge dz_r \), \( d\bar{z}^J = d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_s \). Here, the coefficients \( f_{I,J} \) are functions (belonging to various function classes) on \( \Omega \). Then for two \((r,s)\)-forms

\[
f = \sum_{I,J} f_{I,J} dz^I \wedge d\bar{z}^J,
\]

\[
g = \sum_{I,J} g_{I,J} dz^I \wedge d\bar{z}^J.
\]

One defines the inner product and the norm as

\[
(f,g) = \sum_{I,J} f_{I,J} \overline{g_{I,J}},
\]

\[
|f| = (f,f).
\]

The notation \( \sum' \) means the summation over strictly increasing multiindices. This definition is independent of the choice of the orthonormal basis. Denote by \( C_{r,s}^\infty(\Omega) \)
the space of complex-valued differential forms of class $C^\infty$ and of type $(r, s)$ on $\Omega$ that are smooth up to the boundary and $D_{r,s}(U)$ denotes the elements in $C^\infty(\overline{\Omega})$ that are compactly supported in $U \cap \overline{\Omega}$. $L^2_{r,s}(\Omega)$ consists of the $(r, s)$-forms $u$ satisfies
\[ \|u\|^2 = \sum_{|I|=r, |J|=s} |u_{I,J}|^2 dV < \infty. \]

Let
\[ \mathcal{D} : L^2_{r,s}(\Omega) \to L^2_{r,s+1}(\Omega) \]
be the maximal closed extension and
\[ \mathcal{D}^* : L^2_{r,s}(\Omega) \to L^2_{r,s-1}(\Omega) \]
its Hilbert space adjoint. The Laplace-Beltrami operator $\Box_{r,s}$ is defined as
\[ \Box_{r,s} = \overline{\partial} \partial + \overline{\partial}^* \overline{\partial} : \text{dom} \, \Box_{r,s} \to L^2_{r,s}(\Omega). \]

Let
\[ \mathcal{H}^{r,s} = \{ \varphi \in \text{dom} \mathcal{D} \cap \text{dom} \mathcal{D}^* : \mathcal{D} \varphi = 0 \text{ and } \text{dom} \mathcal{D}^* \varphi = 0 \}. \]

One defines the $\bar{\partial}$-Neumann operator
\[ N : L^2_{r,s}(\Omega) \to L^2_{r,s}(\Omega), \]
as the inverse of the restriction of $\Box_{r,s}$ to $(\mathcal{H}^{r,s})^\perp$. For nonnegative integer $k$, one defines the Sobolev $k$-space
\[ W^k_{r,s}(\Omega) = \{ f \in L^2_{r,s}(\Omega) : \|f\|_k < +\infty \}, \]
where the Sobolev norm of order $k$ is defined as
\[ \|f\|^2_{W^k} = \int_{\Omega} \sum_{|\alpha| \leq k} |D^\alpha f|^2 dV, \]
\[ D^\alpha = \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left( \frac{\partial}{\partial x_{2n}} \right)^{\alpha_{2n}}, \text{ for } \alpha = (\alpha_1, \ldots, \alpha_{2n}), |\alpha| = \sum \alpha_j, \]
and $x_1, \ldots, x_{2n}$ are real coordinates for $\Omega$. Detailed information on Sobolev spaces may be found for example in [4], [5]. Let $p$ be a point in the boundary of $\Omega$. Then one can choose a neighborhood $U$ of $p$ and a local coordinate system $(x_1, \ldots, x_{2n-1}, \rho) \in \mathbb{R}^{2n-1} \times \mathbb{R}$, satisfies the last coordinate is a local defining function of the boundary. Call $(U, (x, \rho))$ a special boundary chart. Denote the dual variable of $x$ by $\xi$, and define
\[ \langle x, \xi \rangle = \sum_{j=1}^{2n-1} x_j \xi_j. \]

The tangential Fourier transform for $f \in \mathcal{D}(\overline{\Omega} \cap U)$ is given in this special boundary chart by
\[ \tilde{f}(\xi, \rho) = \int_{\mathbb{R}^{2n-1}} e^{-2\pi i \langle x, \xi \rangle} f(x, \rho) dx, \]
where \( dx = dx_1 \cdots dx_{2n-1} \). For each \( k \geq 0 \), the standard tangential Bessel potential operator \( \Lambda^k \) of order \( k \) (see e.g., Chen-Shaw [4], Section 5.2) is defined as

\[
(\Lambda^k f)(x, \rho) = \int_{\mathbb{R}^{2n-1}} e^{-2\pi i (x, \xi)} (1 + |\xi|^2)^{\frac{k}{2}} \hat{f}(\xi, \rho) d\xi.
\]

The tangential \( L^2 \)-Sobolev norm of \( f \) of order \( k \) is defined as

\[
|||f|||^2_{W^k_2(\Omega)} = ||\Lambda^k f||^2 = \int_{-\infty}^{0} \int_{\mathbb{R}^{2n-1}} (1 + |\xi|^2)^k |\hat{f}(\xi, \rho)|^2 d\xi d\rho.
\]

Clearly, for \( k > 0 \), this norm is weaker than the full \( L^2 \)-Sobolev norm of order \( k \), since it just measures derivatives in the tangential directions.

Let \( T^2 b\Omega \) be the complex tangent bundle to the boundary, \( L_{b\Omega} = (\rho_{ij})_{\tau^{c}M} \) the Levi form of \( b\Omega \) and \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \) are the ordered eigenvalues of \( L_{b\Omega} \). For every positive number \( M \) and if \( \varphi^M \in C^\infty(\overline{\Omega} \cap V) \), one denote by

\[
\lambda^M_1 \leq \lambda^M_2 \leq \cdots \leq \lambda^M_{n-1}
\]

the ordered eigenvalues of the Levi form \( (\varphi^M_{ij}) \). Choose an orthonormal basis of \((1,0)\)-forms \( \omega_1, \omega_2, \ldots, \omega_n = \partial \rho \) and the dual basis of \((1,0)\)-vector fields \( L_1, L_2, \ldots, L_n \); note that \( T^{1,0} \partial \Omega = \text{Span}\{L_1, L_2, \ldots, L_{n-1}\} \). We denote by \( \rho_j \) and \( \rho_{ij} \) the coefficients of \( \partial \rho \) and \( \partial \overline{\partial} \rho \) in this basis. Following Khanh and Zampieri [17] in Section 2, we have the following definitions.

**Definition 2.1.** \( b\Omega \) is called \( q \)-pseudoconvex in a neighborhood \( V \) of \( z_0 \) if there exist a bundle \( \Xi \subset T^{1,0} b\Omega \) of rank \( q_0 < q \) with smooth coefficients in \( V \), say the bundle of the first \( q_0 \) vector fields \( L_1, \ldots, L_{q_0} \) of our basis of \( T^{1,0} b\Omega \), satisfies

\[
\sum_{j=1}^{q} \lambda_j - \sum_{j=1}^{q_0} \rho_{jj} \geq 0 \quad \text{on} \quad b\Omega \cap V.
\]

Since \( \sum_{j=1}^{q_0} \rho_{jj} \) is the trace of the restricted form \( (\rho_{jj})|_{\Xi} \), then Definition 2.1 depends only on the choice of the bundle \( \Xi \), not of its basis. Condition (2.1) is equivalent to

\[
\sum_{|I|=r}^{n-1} \sum_{i,j=1}^{n-1} \rho_{ij}u_{IiK}\overline{u}_{IjK} - \sum_{j=1}^{q_0} \rho_{jj}|u|^2 \geq 0,
\]

for any \((r, s)\) form \( u \) with \( s \geq q \). It is in this form that (2.1) will be applied. In some case, it is better to consider instead of (2.2), the variant

\[
\sum_{|I|=r}^{n-1} \sum_{i,j=1}^{n-1} \rho_{ij}u_{IiK}\overline{u}_{IjK} - \sum_{|I|=r}^{q_0} \sum_{j=1}^{q_0} \rho_{jj}|u_{IjK}|^2 \geq 0.
\]
It is obvious that if $L_{\Omega}|z$ is assumed to be diagonal, instead of less than or equal to 0, then the left side of (2.3) equals
\[
\sum_{|I|=r} \sum_{i,j=q_0+1}^{n-1} \rho_{ij} u_{i\bar{k}} \bar{u}_{i\bar{k}}.
\]
Thus, if $\overline{\nabla}$ is the Levi-orthogonal complement of $\Xi$, then (2.3) is equivalent to $L_{\Omega}|z \geq 0$. The condition in the definition below generalizes to domains which are not necessarily pseudoconvex, the celebrated $P$ property by Catlin [3].

**Definition 2.2.** $b\Omega$ is said to has the $(q-P)$ property in $V$ if for every positive number $M$ there exists a function $\varphi^M \in C^\infty(\overline{\Omega} \cap V)$ with

(i) $|\varphi^M| \leq 1$ on $\Omega$;
(ii) $\sum_{j=1}^{q} \lambda_j^M - \sum_{j=1}^{q_0} \varphi_{jj}^M \geq c \sum_{j=1}^{q_0} |\varphi_j^M|^2$ on $\overline{\Omega} \cap V$;
(iii) $\sum_{j=1}^{q} \lambda_j^M - \sum_{j=1}^{q_0} \varphi_{jj}^M \geq M$ on $b\Omega \cap V$,
where the constant $c > 0$ does not depend on $M$. (The point here is that (ii) holds in the whole $\overline{\Omega}$, (iii) only on $b\Omega$.)

There are obvious variants of (ii) and (iii) adapted to (2.3). Condition (iii) is a modification of (ii) in Definition 2 of [14]. The bigger flexibility of our condition consists in allowing subtraction of $\varphi_{jj}^M$ for $j = 1, \ldots, q_0$. We say that a compact subset $F \subset b\Omega \cap V$ satisfies $(q-P)$ if and only if (iii) holds for any $z \in F$.

**Theorem 2.1 ([13])**. Let $X$ be a complete manifold of complex dimension $n$ with a Hermitian metric $g$ and $\Omega$ be a bounded domain of $X$. Let $\Omega \Subset X$ be an submanifold with smooth boundary. Suppose the compactness estimate (3.1) holds on $\Omega$. Suppose further that the $\overline{\partial}$-closed $(r,s)$-form $\alpha$ is in $W^k(\Omega)$ and $\alpha \perp \mathcal{H}^r$, there exists a constant $C_k$ so that the canonical solution $u$ of $\overline{\partial}u = \alpha$, with $u \perp \ker \overline{\partial}$ satisfies
\[
||u||_{W^k} \leq C_k(||\alpha||_{W^k} + ||u||).
\]
Since $C^\infty(\overline{\Omega}) = \cap_{k=0}^\infty W^k(\Omega)$, it follows that if $\alpha \in C^\infty_{r,s}(\overline{\Omega})$, then $u \in C^\infty_{r,s-1}(\overline{\Omega})$.

3. Solvability of $\overline{\partial}$ in $C^a$

Following Khanh and Zampieri [17], one obtains the following theorem.

**Theorem 3.1.** Let $\Omega$ be a smoothly bounded $q$-pseudoconvex domain in $\mathbb{C}^n$, and suppose that $b\Omega$ satisfies property $(q-P)$ in a neighborhood $V$ of $z_0$. Then for every $\epsilon > 0$ there exists a function $C_\epsilon \in \mathcal{D}(\Omega)$ satisfying
\[
||u||^2 \leq \epsilon Q(u,u) + C_\epsilon ||u||^2_{W^{r-1}_{r,s}(\Omega)},
\]
for $u \in \mathcal{D}_{r,s}(\overline{\Omega} \cap V) \cap \text{dom } \overline{\partial}$ and for any $s \geq q$. Here
\[
Q(u,u) = ||\overline{\partial}u||^2 + ||\overline{\partial}^* u||^2 + ||u||^2,
\]
and $||u||_{W^{r-1}_{r,s}(\Omega)}$ is the Sobolev norm of order $-1$. 
The same statement holds if \( q \)-pseudoconvexity is understood in the sense of the variant (2.3) and if \( (q - P) \) property has the corresponding variants.

**Definition 3.1.** We will refer to (3.1) as a compactness estimate.

**Theorem 3.2.** Let \( \Omega \) be the same as in Theorem 3.1. Then, for \( f \in C^\infty_{r,s}(\overline{\Omega}) \), \( q \leq s \leq n - 2 \), satisfying \( \overline{\partial} f = 0 \), there exists \( u \in C^\infty_{r,s-1}(\overline{\Omega}) \), satisfies \( \overline{\partial} u = f \).

**Proof.** The proof follows from the estimate (3.1) and Theorem 2.1.

Property \( (q - P) \) is related to the \( \overline{\partial} \)-Neumann problem by the following theorem.

**Remark 3.1.** It is easy to observe that (3.1) implies for \( u \in \text{dom} \Box_{r,s} \):

\[
\|u\|^2 \leq \varepsilon \|\Box_{r,s} u\|^2 + C\varepsilon \|u\|_{W^{2,1}_r(\Omega)}^2.
\]

We now discuss the global regularity for \( N_{r,s} \). From the estimate (3.1) one can derive a priori estimates for \( N_{r,s} \) in the Sobolev \( k \)-space.

**Theorem 3.3.** Let \( \Omega \) be the same as in Theorem 3.1. A compactness estimate implies boundedness of the \( \overline{\partial} \)-Neumann operator \( N_{r,s} \) in \( W^k_{r,s}(\Omega) \) for any \( k > 0 \).

**Proof.** By a standard fact of elliptic regularization, one sees that the global regularity for the \( \overline{\partial} \)-Neumann operator \( N_{r,s} \) holds if

\[
\|u\|_{W^k_{r,s}(\Omega)} \lesssim \|\Box_{r,s} u\|_{W^k_{r,s}(\Omega)},
\]

for any \( u \in C^\infty_{r,s}(\overline{\Omega}) \cap \text{dom} \Box_{r,s} \). Hence,

\[
\|u\|_{W^k_{r,s}(\Omega)}^2 \lesssim \|\Box_{r,s} u\|_{W^{k-2}_{r,s}(\Omega)}^2 + \|\Lambda^{k-1} Du\|_{W^{k-2}_{r,s}(\Omega)}^2,
\]

where \( \Lambda \) is the tangential differential operator of order \( k \). By Theorem 3.1, the estimate (3.1) implies that

\[
\|\Lambda^{k-1} Du\|_{W^{k-2}_{r,s}(\Omega)}^2 \lesssim Q(u, u) + C\|u\|_{W^{2,1}_r(\Omega)}^2.
\]

In fact, it follows by the non-characteristic with respect to the boundary of \( L_n \); the operator \( D \) can be understood as \( D_r \) or \( \Lambda \).

Now we estimate the last term of (3.3), we have

\[
\|\Lambda^{k-1} Du\|^2 \lesssim \|\Lambda^{k-1} \Lambda^k u\|^2 + C\|u\|_{W^{2,1}_r(\Omega)}^2
\]

\[
\lesssim Q(\Lambda^k u, \Lambda^k u) + C\|u\|^2_{W^{2,1}_r(\Omega)}
\]

\[
\lesssim < \Lambda^k \Box_{r,s} u, \Lambda^k u > + \|\overline{\partial}, \Lambda^k u\|^2 + \|\overline{\partial}, \Lambda^k u\|^2
\]

\[
+ \|\overline{\partial}, \Lambda^k u\|^2 + \|\overline{\partial}, \Lambda^k u\|^2 + C\|u\|^2_{W^{2,1}_r(\Omega)}
\]

\[
\lesssim \|\Lambda^k \Box_{r,s} u\|^2 + \|\Lambda^{k-1} Du\|^2 + \|\Lambda^{k-2} D^2 u\|^2 + C\|u\|^2_{W^{2,1}_r(\Omega)}
\]

\[
\lesssim \|\Box_{r,s} u\|^2_{W^{2,1}_r(\Omega)} + \|\Lambda^{k-1} Du\|^2 + C\|u\|^2_{W^{2,1}_r(\Omega)}.
\]
where the second inequality follows by (3.4). Then the term $\|\Lambda^{k-1}Dv\|^2$ can be absorbed by the left-hand side term. By induction method, we obtain the estimate (3.2).

\[\Box\]

**Proposition 3.1.** Let $\Omega$ be the same as in Theorem 3.1. Then the following are equivalent.

(i) The validity of global compactness estimates.

(ii) The embedding of the space $\text{dom} \mathcal{D} \cap \text{dom} \mathcal{D}^*$, provided with the graph norm

$$\|u\| + \|\mathcal{D}u\| + \|\mathcal{D}^* u\|,$$

into $L^2_{r,s}(\Omega)$ is compact.

(iii) The $\mathcal{D}$-Neumann operators

$$N_{r,s} : L^2_{r,s}(\Omega, E) \to L^2_{r,s}(\Omega, E),$$

for $q \leq s \leq n - 1$ are compact from $L^2_{r,s}(\Omega)$ to itself.

(iv) The canonical solution operators to $\mathcal{D}$ given by

$$\mathcal{D}^* N_{r,s} : L^2_{r,s}(\Omega) \to L^2_{r,s-1}(\Omega),$$

$$N_{r,s+1} \mathcal{D}^* : L^2_{r,s+1}(\Omega) \to L^2_{r,s}(\Omega),$$

are compact.

\[\text{Proof.}\] The equivalence of (ii) and (i) is a result of Lemma 1.1 in [13]. The general $L^2$-theory and the fact that $L^2_{r,s}(\Omega)$ embeds compactly into $W^{-1}_{r,s}(\Omega)$ shows that (iii) is equivalent to (ii) and (i). Finally, the equivalence of (iii) and (iv) follows from the formula

$$N_{r,s} = (\mathcal{D}^* N_{r,s})^* \mathcal{D}^* N_{r,s} + \mathcal{D}^* N_{r,s+1} (\mathcal{D}^* N_{r,s+1})^*,$$

(see [4], page 55). We refer the reader to [14] for similar calculations.

\[\Box\]

4. **Solvability of $\mathcal{D}$ in Stein manifold**

Let $X$ be complex manifold of complex dimension $n$ with a Hermitian metric $g$ and $\Omega$ be a bounded domain of $X$. Let $\pi : E \to X$ be a vector bundle, of rank $p$, over $X$ with Hermitian metric $h$. Let $\{U_j\}$, $j \in J$, be an open covering of $X$ by charts with coordinates mappings $z_j : U_j \to \mathbb{C}^n$ satisfies $E|_{U_j}$ is trivial, namely $\pi^{-1}(U_j) = U_j \times \mathbb{C}$, and $(z_j^1, z_j^2, \ldots, z_j^n)$ be local coordinates on $U_j$. Let $\{\zeta_j\}_{j \in J}$ be a partition of unity subordinate to the holomorphic atlas $(U_j, z_j)$, of $X$. We denote by $T_z X$ the tangent bundle of $X$ at $z \in X$. An $E$-valued differential $(r, s)$-form $u$ on $X$ is given locally by a column vector $u = (u^1, u^2, \ldots, u^p)$, where $u^a$, $1 \leq a \leq p$, are $\mathbb{C}$-valued differential forms of type $(r, s)$ on $X$. The spaces $C^\infty_r(X, E)$, $\mathcal{D}_r,s(X, E)$, $C^\infty_r(\overline{\Omega}, E)$, $\mathcal{D}_r,s(\overline{\Omega}, E)$ and $W^k_{r,s}(\Omega, E)$ are defined as in Section 2 but for $E$-valued forms. Let $L^2_{r,s}(\Omega, E)$ be the Hilbert space of $E$-valued differential forms $u$ on $\Omega$, of type $(r, s)$, satisfies

$$\|u\|_\Omega = \sum_j \sum_{a=1}^p \|u^a\|_{U_j \cap \Omega} < \infty,$$
where \( \|u_0^a\|_{U_j \cap \Omega} \) is defined in (2.1). Let \( \overline{\mathcal{J}} : L^2_{r,s}(\Omega, E) \to L^2_{r,s+1}(\Omega, E) \) be the maximal closed extension of the original \( \mathcal{J} \) and \( \overline{\mathcal{J}}' : L^2_{r,s}(\Omega, E) \to L^2_{r,s-1}(\Omega, E) \) its Hilbert space adjoint. For \( k \in \mathbb{R} \), we define a \( W^k(X, E) \)-norm by the following:

\[
\|u\|_{k(X)}^2 := \sum_j \|\zeta_j u_j\|_{k(W_j)}^2,
\]

where \( W_j = z_j(U_j) \) and \( \sum_j \|\zeta_j u_j\|_{k(W_j)}^2 \) is defined as in the Euclidean case.

**Theorem 4.1.** Let \( \Omega \) be a smoothly bounded \( q \)-pseudoconvex domain in an \( n \)-dimensional Stein manifold \( X \), \( n \geq 3 \), and suppose that \( b\Omega \) satisfies property \((q-P)\) in a neighborhood \( V \) of \( z_0 \). Let \( E \) be a vector bundle, of rank \( p \), on \( X \). Then, for \( f \in C^\infty(\overline{\Omega}, E) \), \( q \leq s \leq n - 2 \), satisfying \( \mathcal{D} f = 0 \) in the distribution sense in \( X \), there exists \( u \in C^\infty_{r,s-1}(\Omega, E) \), satisfies \( \mathcal{D} u = f \) in the distribution sense in \( X \).

**Proof.** Let \( \{U_j\}_{j=1}^N \) be a finite covering of \( b\Omega \) by a local patching. Let \( e_1, e_2, \ldots, e_p \) be an orthonormal basis on \( E_z = \pi^{-1}(z) \), for every \( z \in U_j \), \( j \in J \). Thus, every \( E \)-valued differential \((r,s)\)-form \( u \) on \( X \) can be written locally, on \( U_j \), as

\[
u(z) = \sum_{a=1}^p u^a(z) e_a(z),
\]

where \( u^a \) are the components of the restriction of \( u \) on \( U_j \). Since \( b\Omega \) is compact, there exists a finite number of elements of the covering \( \{U_j\} \), say, \( U_j, j = 1, 2, \ldots, m \) satisfies \( \bigcup_{j=1}^m U_j \) cover \( b\Omega \). Let \( \{\zeta_j\}_{j=0}^m \) be a partition of the unity satisfies \( \zeta_0 \in \mathcal{D}_{r,s}(\Omega) \), \( \zeta_j \in \mathcal{D}_{r,s}(U_j) \), \( j = 1, 2, \ldots, m \), and

\[
\sum_{j=0}^m \zeta_j^2 = 1 \quad \text{on} \quad \overline{\Omega},
\]

where \( \{U_j\}_{j=1, \ldots, m} \) is a covering of \( b\Omega \). Let \( U \) be a small neighborhood of a given boundary point \( \zeta_0 \in b\Omega \) satisfies \( U \in V \subseteq U_{j_0} \), for a certain \( j_0 \in I \). If \( u \in \mathcal{D}_{r,s}(\Omega, E) \), \( 0 \leq r \leq n \), \( q \leq s \leq n - 2 \), on applying the compactness estimate of Khanh and Zampieri [17] to each \( u^a \) and adding for \( a = 1, \ldots, p \), one gets compactness estimate for \( u|_{\Omega \cap U} \)

\[
\|\zeta_0 u\|^2 \lesssim \epsilon Q(\zeta_0 u, \zeta_0 u) + C_c \|\zeta_0 u\|_{W^{r-1}_{s+1}(\Omega)}^2
\]

\[
\lesssim \epsilon Q(u, u) + C_c \|u\|_{W^{r-1}_{s+1}(\Omega)}^2.
\]

Similarly, for \( j = 1, \ldots, m \), we obtain compactness estimate for \( u|_{\Omega \cap U_j} \)

\[
\|\zeta_j u\|^2 \lesssim \epsilon Q(\zeta_j u, \zeta_j u) + C_c \|\zeta_j u\|_{W^{r-1}_{s+1}(\Omega)}^2
\]

\[
\lesssim \epsilon Q(u, u) + C_c \|u\|_{W^{r-1}_{s+1}(\Omega)}^2.
\]

Summing up over \( j \), we obtain

\[
\|u\|^2 \lesssim \epsilon Q(u, u) + C_c \|u\|_{W^{r-1}_{s+1}(\Omega)}^2.
\]

Thus the proof follows by using Theorem 2.1 and the compactness estimate (4.2). \( \square \)
**Theorem 4.2.** Denote by $\Omega$, $E$ and $X$ as in Theorem 4.1. A compactness estimate (4.2) implies boundedness of the $\overline{\mathcal{J}}$-Neumann operator $N_{r,s}$ in $W^{k}_{r,s}(\Omega, E)$ for any $k > 0$.

**Proof.** By a standard fact of elliptic regularization, one sees that the boundary global regularity for the $\partial$-Neumann operator $N_{r,s}$ holds if

$$
\|u\|_{W^{k}} \lesssim \|\Box_{r,s} u\|_{W^{k}},
$$

for any $u \in C^\infty_{r,s}(\overline{\Omega}, E) \cap \text{dom } \Box_{r,s}$ and for any positive integer $k$. As in the proof of Theorem 4.1, let $U$ be a small neighborhood of a given boundary point $\zeta_0 \in \partial \Omega$ satisfies $U \Subset V \Subset U_{j_0}$, for a certain $j_0 \in I$. If $u \in \mathcal{D}_{r,s}(\Omega, E)$, $0 \leq r \leq n$, $q \leq s \leq n - 2$, on applying the estimate (3.2) to each $u^a$ and adding for $a = 1, \ldots, p$, one gets compactness estimate for $u|_{\Omega \cap U}$

$$
\|\zeta_0 u\|_{W^{k}} \lesssim \|\Box_{r,s} \zeta_0 u\|_{W^{k}}.
$$

Similarly, for $j = 1, \ldots, m$, one gets compactness estimate for $u|_{\Omega \cap U_j}$

$$
\|\zeta_j u\|_{W^{k}} \lesssim \|\Box_{r,s} \zeta_j u\|_{W^{k}} \lesssim \|\Box_{r,s} u\|_{W^{k}}.
$$

Summing up over $j$, we obtain

$$
\|u\|_{W^{k}} \lesssim \|\Box_{r,s} u\|_{W^{k}}.
$$

Thus the proof follows. \hfill \Box

As in Proposition 3.1, one can prove the following proposition.

**Proposition 4.1.** Denote by $\Omega$, $E$ and $X$ as in Theorem 4.1. Then the following are equivalent.

(i) The compactness estimates are valid.

(ii) The embedding of $\text{dom } \overline{\mathcal{J}} \cap \text{dom } \overline{\mathcal{J}^*}$, with the graph norm

$$
\|u\| + \|\overline{\mathcal{J}} u\| + \|\overline{\mathcal{J}^*} u\|
$$

into $L^2_{r,s}(\Omega, E)$ is compact.

(iii) The $\overline{\mathcal{J}}$-Neumann operator

$$
N_{r,s} : L^2_{r,s}(\Omega, E) \longrightarrow L^2_{r,s}(\Omega, E),
$$

for $q \leq s \leq n - 1$ is compact from $L^2_{r,s}(\Omega, E)$ to itself.

(iv) The canonical solution operators to $\overline{\mathcal{J}}$ are given by

$$
\overline{\mathcal{J}^*} N_{r,s} : L^2_{r,s}(\Omega, E) \longrightarrow L^2_{r,s-1}(\Omega, E),
$$

$$
N_{r,s+1} \overline{\mathcal{J}^*} : L^2_{r,s+1}(\Omega, E) \longrightarrow L^2_{r,s}(\Omega, E),
$$

are compact.
References


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