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L-FUZZY HOLLOW MODULES AND L-FUZZY MULTIPLICATION MODULES

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Abstract. In this paper, we give some characterizations of L-fuzzy hollow modules and of L-fuzzy multiplication modules.

1. Introduction

The concept of a fuzzy set, which is a generalization of a crisp set, was introduced by Zadeh [13]. Rosenfeld [12] used this concept to develop the theory of fuzzy subgroups. Naegoita and Ralescu [9] applied this concept to modules and defined a fuzzy submodule of a module.

Barnad [3] introduced the concept of a multiplication module. An R-module M is called a *multiplication module* if every submodule of M is of the form IM, for some ideal I of R. Also, Elbast and Smith [4] have studied multiplication modules.

Lee and Park [6] studied fuzzy prime submodules of a fuzzy multiplication module. Recently, Atani [2] introduced and investigated L-fuzzy multiplication modules over a commutative ring with nonzero identity. He has proved a relation between a multiplication module and an L-fuzzy multiplication module.

In this paper we introduce a notion of a hollow fuzzy module and prove some results. Our notion is different from that of Rahman [11]. We prove some results on L-fuzzy multiplication modules. We also show that an L-hollow fuzzy module is an L-fuzzy multiplication module.

Key words and phrases. L-Fuzzy hollow module, L-fuzzy multiplication module, L-fuzzy Noetherian module.

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2. Preliminaries

Throughout in this paper R denotes a commutative ring with identity, M a unitary R-module with zero element θ. We recall some definitions and results from Moderson and Malik [8] which will be used in this paper.

Definition 2.1. ([8, Definition 1.1.1]). A fuzzy subset of an R-module M is a mapping $\mu : M \to [0,1]$. We denote the set of all fuzzy subsets of M by $[0,1]^M$.

If μ is a mapping from M to L, where L is a complete Heyting algebra, then μ is called an L-subset of M. We denote the set of all L-subsets of R by L^R and the set of all L-subsets of M by L^M.

Definition 2.2. ([8, Definition 1.1.3]). If $N \subseteq M$ and $\alpha \in [0,1]^M$, then α_N is defined as

$$\alpha_N(x) = \begin{cases} \alpha, & \text{if } x \in N, \\ 0, & \text{otherwise.} \end{cases}$$

If $N = \{x\}$, then α_x is often called a fuzzy point and is denoted by χ_α. If $\alpha = 1$, then 1_N is known as the characteristic function of N and is denoted by χ_N.

If $\mu, \sigma \in [0,1]^M$, then for $x, y, z \in M$, we define

(i) $\mu \subseteq \sigma$ if and only if $\mu(x) \leq \sigma(x)$;
(ii) $(\mu \cup \sigma)(x) = \max\{\mu(x), \sigma(x)\} = \mu(x) \vee \sigma(x)$;
(iii) $(\mu \cap \sigma)(x) = \min\{\mu(x), \sigma(x)\} = \mu(x) \wedge \sigma(x)$;
(iv) $(\mu + \sigma)(x) = \vee\{\mu(y) \wedge \sigma(z) \mid y, z \in M, y + z = x\}$.

Definition 2.3. ([8, Definition 4.1.6]). Let $\zeta \in L^R$ and $\mu \in L^M$. Define $\zeta \cdot \mu$ as

$$(\zeta \cdot \mu)(x) = \vee\{\zeta(r) \wedge \mu(y) \mid r \in R, y \in M, ry = x\}, \quad \text{for all } x \in M.$$

Definition 2.4. ([8, Definition 3.1.7]). Suppose that $\mu \in L^R$ satisfies the following conditions:

(i) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$;
(ii) $\mu(xy) \geq \mu(x) \vee \mu(y)$ for all $x, y \in R$.

Then μ is called an L-ideal of R.

We denote the set of all L-ideals of R by $LI(R)$.

Definition 2.5. ([8, Definition 4.1.8]). Let M be a module over a ring R and L be a complete Heyting algebra. An L subset μ in M is called an L-submodule of M, if for every $x, y \in M$ and $r \in R$ the following conditions are satisfied:

(i) $\mu(\theta) = 1$;
(ii) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$;
(iii) $\mu(rx) \geq \mu(x)$.

Definition 2.6. ([8, Definition 4.5.1]). For $\mu, \nu \in L^M$ and $\zeta \in L^R$, define the residual quotients $\mu : \nu \in L^R$ and $\mu : \zeta \in L^M$ as follows:

$$\mu : \nu = \cup\{\eta \mid \eta \in L^R,\ \eta \cdot \nu \subseteq \mu\},$$
$$\mu : \zeta = \cup\{\xi \mid \xi \in L^M,\ \zeta \cdot \xi \subseteq \mu\}.$$

Theorem 2.1. ([8, Theorem 4.5.3]). *Let μ, $\nu \in L^M$ and $\zeta \in L^R$. Then*

(1) $(\mu : \nu)\nu \subseteq \mu$;

(2) $\zeta \cdot \nu \subseteq \mu$ *if and only if* $\zeta \subseteq (\mu : \nu)$ *if and only if* $\nu \subseteq \mu : \zeta$.

Definition 2.7 ([8])**.** Let $c \in L\backslash\{1\}$. Then

(i) c is called a prime element of L if $a \wedge b \leq c$, implies that $a \leq c$ or $b \leq c$ for all a, $b \in L$;

(ii) c is called a maximal element if there does not exist $a \in L\backslash\{1\}$ such that $c < a < 1$.

Remark 2.1 ([8]). If $\mu, \nu \in LI(R)$, then $(\mu \circ \nu)(x) = \vee\{\mu(y) \wedge \nu(z) \mid y, z \in R, yz = x\}$. We write $\mu_* = \{x \in R \mid \mu(x) = \mu(0)\}$.

Definition 2.8. ([8, Definition 3.5.1]). Let $\xi \in LI(R)$. Then ξ is called a prime L-ideal of R if ξ is non-constant and $\mu \circ \nu \subseteq \xi$, $\mu, \nu \in LI(R)$ implies either $\mu \subseteq \xi$ or $\nu \subseteq \xi$.

Definition 2.9. ([8, Definition 3.6.1]). Let $\xi \in LI(R)$ and let ρ_ξ be the family of all prime L-ideals μ of R such that $\xi \subseteq \mu$. The L-radical of ξ, denoted by $\sqrt{\xi}$, is defined by

$$\sqrt{\xi} = \begin{cases} \cap\{\mu \mid \mu \in \rho_\xi\}, & \text{if } \rho_\xi \neq \phi, \\ 1_R, & \text{if } \rho_\xi = \phi. \end{cases}$$

Definition 2.10. ([8, Definition 3.7.1]). Let $\xi \in LI(R)$. Then ξ is called a primary L-ideal of R if ξ is nonconstant and for any $\mu, \nu \in LI(R)$, $\mu \circ \nu \subseteq \xi$ implies $\mu \subseteq \xi$ or $\nu \subseteq \sqrt{\xi}$.

Theorem 2.2. ([8, Theorem 3.5.3]). *If ξ is a prime L-ideal of R, then ξ_* is a prime ideal of R.*

Theorem 2.3. ([8, Theorem 3.5.5]). *Let $\xi \in L^R$. Then ξ is a prime L-ideal of R if and only if $\xi(0) = 1$, ξ_* is a prime ideal of R, $\xi(R) = \{1, c\}$, where c is a prime element in L.*

Definition 2.11 ([5])**.** A ring R is called regular if, for each element $x \in R$, there exists $y \in R$ such that $xyx = x$.

Definition 2.12. A dense chain in a lattice L is a non-empty sublattice C such that, for all ordered pairs $x < y$ with x, $y \in C$, there exists some $z \in C$ such that $x < y < z$.

Theorem 2.4 ([8])**.** *Let R be a ring with identity, L be a dense chain and ξ be a primary L-ideal of R. Then $\sqrt{\xi}$ is a prime L-ideal of R.*

Theorem 2.5. ([7, Theorem 3.10]). *Let R be a ring with* 1 *and A be a nonconstant fuzzy left* (*right*) *ideal of R. Then there exists a fuzzy maximal left* (*right*) *ideal B of R such that $A \subseteq B$.*

Definition 2.13. ([5, Definition 4.3.2]). A fuzzy ideal μ of a ring R is called fuzzy semiprime if, for any fuzzy ideal ζ of R, the condition $\zeta^n \subseteq \mu$ implies that $\zeta \subseteq \mu$, where $n \in \mathbb{Z}_+$.

Theorem 2.6. ([5, Theorem 4.4.3]). *A commutative ring with unity is regular if and only if each of its fuzzy ideal is fuzzy semiprime.*

Definition 2.14 ([2])**.** Let M be a module over a commutative ring R. M is called an L-fuzzy multiplication module provided for each L-fuzzy submodule μ of M, there exists $\zeta \in LI(R)$ with $\zeta(0_R) = 1$ such that $\mu = \zeta\chi_M$.

One can easily show that if $\mu = \zeta\chi_M$ for some $\zeta \in LI(R)$ with $\zeta(0_R) = 1$, then $\mu = (\mu : \chi_M)\chi_M$.

Theorem 2.7. ([2, Theorem 10]). *Let M be an R-module. Then M is a multiplication module if and only if M is an L-fuzzy multiplication module.*

Theorem 2.8. ([1, Theorem 2]). *Let P be a primary ideal of R and M a faithful multiplication R-module. Let $a \in R$, $x \in M$ satisfy $ax \in PM$. Then $a \in \sqrt{P}$ or $x \in PM$.*

Definition 2.15. ([10, Definition 4.1]). Let M be a module over a ring R and $\mu \in L(M)$. Then μ is said to be a small L-submodule of M, if for any $\nu \in L(M)$ satisfying $\nu \neq \chi_M$ implies $\mu + \nu \neq \chi_M$.

Definition 2.16. ([11, Definition 2.10]). A fuzzy submodule $\mu(\neq \chi_\theta)$ of a module M is said to be fuzzy indecomposable if there do not exist fuzzy submodules σ, γ of M with $\sigma \neq \chi_\theta$, $\gamma \neq \chi_\theta$ and $\sigma \neq \mu$, $\gamma \neq \mu$ such that $\mu = \sigma \oplus \gamma$.

Theorem 2.9. ([10, Theorem 5.2]). *Let $\mu \in L^M$. Then μ is a maximal L-submodule of M if and only if μ can be expressed as $\mu = \chi_{\mu_*} \cup \alpha_M$, where μ_* is a maximal submodule of M and α is a maximal element of $L - \{1\}$.*

Definition 2.17. ([11, Definition 3.1]). A fuzzy submodule ν with $\nu_* \neq \{\theta\}$ of M is said to be a fuzzy hollow submodule if for every fuzzy submodule μ of ν with $\mu_* \neq \nu_*$, μ is a fuzzy small submodule of ν. We say that an R-module $M \neq \{\theta\}$ is fuzzy hollow module if for every $\sigma \in F(M)$ with $\sigma_* \neq M$ implies $\sigma \ll_f M$.

Theorem 2.10. ([11, Theorem 3.6]). *Every fuzzy hollow submodule is indecomposable.*

Theorem 2.11. ([2, Theorem 14]). *Let M be a non-zero L-fuzzy multiplication R-module. Then every L-fuzzy submodule $\mu \neq \chi_M$ of M is contained in a generalized maximal L-fuzzy submodule of M.*

Proposition 2.1. ([2, Proposition 18]). *Suppose that M is a faithful L-fuzzy multiplication R-module. Let ζ be an L-fuzzy prime ideal of R. If η is an L-fuzzy ideal of R such that $\eta\chi_M \subseteq \zeta\chi_M$ and $\zeta\chi_M \neq \chi_M$, then $\eta \subseteq \zeta$. In particular, $(\zeta\chi_M : \chi_M) = \zeta$.*

Notations:

$fspec(R)$: the set of all prime L-submodules of R;

$Max_L(M)$: the set of all maximal L-submodules of M;

$JLR(M)$: the intersection of all maximal L-submodules of M is known as Jacobson L-radical of M.

Definition 2.18 ([2]). An R-module M is called an L-fuzzy Noetherian module, if every ascending chain of L-fuzzy submodules is stationary.

Definition 2.19. A module M is called L-local if M has exactly one maximal L-submodule.

Definition 2.20. A module M is called L-serial if any two L-submodules of M are comparable with respect to inclusion.

3. L-Fuzzy Hollow Modules and L-Fuzzy Multiplication Modules

In this section we introduce a slightly different notion of L-fuzzy hollow modules. Also, we obtain some properties of the same and L-fuzzy multiplication module.

Definition 3.1. Let M be a module over a commutative ring R. M is called an L-fuzzy hollow module if either $Max_L(M) = \chi_\theta$ or for each maximal L-fuzzy submodule μ of M and for each L-fuzzy submodule σ of M, the equality $\mu + \sigma = \chi_M$ implies that $\sigma = \chi_M$.

Theorem 3.1. *Let M be a non-zero module. Then the following statements are equivalent.*

(1) *M is an L-fuzzy hollow module and $Max_L(M) \neq \chi_\theta$.*

(2) *M is a cyclic and an L-local module.*

(3) *M is a finitely generated L-local module.*

Proof. (1) $\Rightarrow$ (2) Let μ be a maximal L-submodule of M and for $m \in M$, $\chi_{\{m\}}$ be an L-submodule of M such that $\chi_{\{m\}} \nsubseteq \mu$. Since, $\mu + \chi_{\{m\}} = \chi_M$, and as M is a L-fuzzy hollow module we have $\chi_M = \chi_{\{m\}}$. Hence, M has only one maximal L-submodule.

Also, as $\chi_M = \langle \chi_{\{m\}} \rangle = \chi_{Rm}$ implies that, $M = Rm$. Hence, M is cyclic.

(2) $\Rightarrow$ (3) It is obivous.

(3) $\Rightarrow$ (1) Let μ be a maximal L-submodule of M and σ be an L-fuzzy submodule of M. If $\mu + \sigma = \chi_M$ and $\sigma \neq \chi_M$, then by Zorn's lemma there exists a maximal L-submodule δ of M containing σ. Since, M is an L-local module, $\delta = \mu$ and so $\chi_M = \mu + \sigma = \mu$, a contradiction. Thus, $\sigma = \chi_M$. □

Theorem 3.2. *Let M be an R-module and μ be an L-fuzzy submodule of M. Then the following statements are equivalent.*

(1) *μ is a serial submodule.*

(2) *μ is an L-fuzzy hollow submodule.*

(3) *μ is fuzzy indecomposable.*

Proof. (1) $\Rightarrow$ (2) Suppose that $Max_L(\mu) \neq \chi_\theta$ and $\mu_1, \mu_2 \in L(M)$ be such that $\mu_1 + \mu_2 = \mu$, where μ_1 is a maximal L-submodule of μ and μ_2 is an L-submodule of μ. Since, μ_1, μ_2 are L-submodules of μ and μ is a serial submodule either $\mu_1 \subseteq \mu_2$ or $\mu_2 \subseteq \mu_1$.

If $\mu_1 \subseteq \mu_2$, then $\mu = \mu_1 + \mu_2 = \mu_2$. If $\mu_2 \subseteq \mu_1$, then $\mu = \mu_1 + \mu_2 = \mu_1$, which is not possible as μ_1 is a maximal L-submodule of μ. Thus, μ is an L-fuzzy hollow submodule of M.

(2) $\Rightarrow$ (3) Follows from Theorem 2.10.

(3) $\Rightarrow$ (1) Let μ_1, μ_2 be L-fuzzy submodules of μ with $\mu_1 \neq \chi_\theta$, $\mu_2 \neq \chi_\theta$, $\mu_1 \neq \mu$, $\mu_2 \neq \mu$ and $\mu_1 \nsubseteq \mu_2$. As μ is fuzzy indecomposable, μ_1, μ_2 does not satisfy $\mu_1 + \mu_2 = \mu$ and $\mu_1 \cap \mu_2 = \chi_\theta$. Then, $\mu_2 \subseteq \mu_1$, thus μ is a serial submodule. ☐

Lemma 3.1. *Let M be an L-fuzzy multiplication module and μ be an L-fuzzy submodule of M. Then the following are equivalent.*

(1) $\mu \subseteq JLR(M)$.

(2) μ *is an L-small submodule in M.*

Proof. (1) $\Rightarrow$ (2) Let σ be an L-fuzzy submodule of M such that $\chi_M = \mu + \sigma$. If $\sigma \neq \chi_M$, then by Theorem 2.11, there exists a maximal L-submodule δ of M such that $\sigma \subseteq \delta$. But, $\mu \subseteq JLR(M) \subseteq \delta$ implies that $\mu + \sigma \subseteq \delta \neq \chi_M$. Thus, $\sigma = \chi_M$ implies that μ is an L-small submodule in M.

(2) $\Rightarrow$ (1) Assume that μ is an L-small submodule of M. Suppose that $\mu \nsubseteq JLR(M)$. Then there exists a maximal L-submodule β of M such that $\mu \nsubseteq \beta$. Thus, $\mu + \beta = \chi_M$. But $\beta \neq \chi_M$, a contradiction. Hence, $\mu \subseteq \beta$. ☐

Theorem 3.3. *If M is an L-fuzzy hollow module, then M is an L-fuzzy multiplication module.*

Proof. As M is an L-fuzzy hollow module, by Theorem 3.1, M is cyclic. But, we know that every cyclic module is a multiplication module. Thus, by Theorem 2.7, M is an L-fuzzy multiplication module. ☐

We give an example of an L-fuzzy multiplication module by using Theorem 3.3.

Example 3.1. Let $L = \{0, 0.25, 0.5, 0.75, 1\}$. Then L is a complete Heyting algebra together with the operations minimium (meet), maximium (join) and $\leq$ (partial ordering), then 0.75 is a maximal element of $L - \{1\}$.

Consider, $M = \mathbb{Z}_{27} = \{0, 1, 2, \ldots, 26\}$ under addition modulo 27, then M is a module over the ring $\mathbb{Z}$. Let $A = \{0, 3, 6, \ldots, 24\}$.

Define, $\mu \in [0,1]^M$ as follows:

$$\mu(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0.75, & \text{otherwise.} \end{cases}$$

Then $\mu_* = \{0, 3, 6, \ldots, 24\} = A$, which is a maximal submodule of $\mathbb{Z}_{27}$. Also, $\mu = \chi_{\mu_*} \cup 0.75_M$, where 0.75 is a maximal element of $L - \{1\}$. So, by Theorem 2.9, μ is a maximal L-submodule of $\mathbb{Z}_{27}$. Infact, μ is the only maximal L-submodule of $\mathbb{Z}_{27}$.

Let $B = \{0, 9, 18\}$ and define $\nu \in [0,1]^M$ as follows,

$$\nu(x) = \begin{cases} 1, & \text{if } x \in B, \\ \alpha, & \text{otherwise,} \end{cases}$$

where $\alpha < 0.75$. Then clearly μ, ν are the only fuzzy submodules of M. Also, here $\nu \neq \chi_M$ implies that $\mu + \nu \neq \chi_M$. This shows that M is an L-fuzzy hollow module and by Theorem 3.3, M is an L-fuzzy multiplication module.

Corollary 3.1. *For $\xi_1, \xi_2 \in L^R$ with $\xi_1 \subseteq \xi_2$, then $\xi_1 \cdot \chi_M \subseteq \xi_2 \cdot \chi_M$ and thus $(\xi_1\chi_M : \chi_M) \subseteq (\xi_2\chi_M : \chi_M)$.*

Proof. We have

$$\begin{aligned}(\xi_1 \cdot \chi_M)(x) &= \bigvee\{\xi_1(r) \wedge \chi_M(y) \mid r \in R, y \in M \ \wedge \ ry = x\} \\ &= \bigvee\{\xi_1(r) \mid r \in R, x \in rM\} \\ &\leq \bigvee\{\xi_2(r) \mid r \in R, x \in rM\} \\ &\leq \bigvee\{\xi_2(r) \wedge \chi_M(y) \mid r \in R, y \in M \ \wedge \ ry = x\} \\ &= (\xi_2 \cdot \chi_M)(x).\end{aligned}$$

Hence, $\xi_1 \cdot \chi_M \subseteq \xi_2 \cdot \chi_M$, for all $x \in M$.

Again we have

$$\begin{aligned}(\xi_1\chi_M : \chi_M) &= \bigvee\{\eta \mid \eta \in L^R, \eta \cdot \chi_M \subseteq \xi_1 \cdot \chi_M\} \\ &\leq \bigvee\{\eta \mid \eta \in L^R, \eta \cdot \chi_M \subseteq \xi_2 \cdot \chi_M\} \\ &\leq (\xi_2\chi_M : \chi_M).\end{aligned}$$

Hence, $(\xi_1\chi_M : \chi_M) \subseteq (\xi_2\chi_M : \chi_M)$. □

Theorem 3.4. *Let M be an L-fuzzy multilpication module. Then μ is a maximal L-fuzzy submodule of M if and only if there exists a maximal ideal ξ of $LI(R)$ such that $\mu = \xi\chi_M \neq \chi_M$.*

Proof. By Theorem 2.11, if ξ is a maximal L-fuzzy ideal of R and $\chi_M \neq \xi\chi_M$, then $\xi\chi_M$ is a maximal L-submodule of M.

Conversely, assume that μ is a maximal L-submodule of M. Then there exists an L-ideal ν of $LI(R)$ such that $\mu = \nu\chi_M$. Suppose that ν is not a maximal L-ideal of R. Then $\nu \subseteq \beta$ for some $\beta \in LI(R)$ and so $\nu\chi_M \subseteq \beta\chi_M$ implies that $\mu \subseteq \beta\chi_M$. This implies μ is not a maximal L-submodule of M, a contradiction. Thus, ν is a maximal L-fuzzy ideal of R. □

Theorem 3.5. *Let M be a faithful L-fuzzy Noetherian R-module. Then R satisfies the ascending chain condition on L-prime ideals.*

Proof. Let $\xi_1 \subseteq \xi_2 \subseteq \xi_3 \subseteq \cdots$ be an ascending chain of L-prime ideals of R. Then by Corollary 3.1, $\xi_1\chi_M \subseteq \xi_2\chi_M \subseteq \xi_3\chi_M \subseteq \cdots$. But as M is an L-fuzzy Noetherian R-module, there exists some $n \in \mathbb{N}$ such that $\xi_n\chi_M = \xi_{n+1}\chi_M = \cdots$. Hence, by Proposition 2.1, $\xi_1 \subseteq \xi_2 \subseteq \xi_3 \subseteq \cdots \subseteq \xi_n$. □

Theorem 3.6. *Let R be regular ring with unity which satisfies ascending chain condition on fuzzy semiprime ideals and M be an L-fuzzy multiplication module. Then M is an L-fuzzy Noetherian module.*

Proof. Let $\mu_1 \subseteq \mu_2 \subseteq \mu_3 \subseteq \cdots$ be an ascending chain of L-fuzzy submodules of M. Then by Corollary 3.1, $(\mu_1 : \chi_M) \subseteq (\mu_2 : \chi_M) \subseteq (\mu_3 : \chi_M) \subseteq \cdots$ is an ascending chain of ideals of R. By Theorem 2.6, $(\mu_1 : \chi_M) \subseteq (\mu_2 : \chi_M) \subseteq (\mu_3 : \chi_M) \subseteq \cdots$ is an ascending chain of fuzzy semiprime ideals of R. By assumption there exists positive integer t such that $(\mu_t : \chi_M) = (\mu_{t+s} : \chi_M)$, for every positive integer s. Hence, $\mu_t = (\mu_t : \chi_M)\chi_M = (\mu_{t+s} : \chi_M)\chi_M = \mu_{t+s}$ gives $\mu_t = \mu_{t+s}$ for every s and so the chain is stationary. Hence, M is an L-fuzzy Noetherian module. □

Theorem 3.7. *Let M be an faithful L-fuzzy multiplication module. Then for every L-fuzzy submodule μ of M, if $\mu\chi_M \subseteq \xi\chi_M$, where $\xi \in fspec(R)$, then $\mu \subseteq \xi$.*

Proof. Given, $\mu\chi_M \subseteq \xi\chi_M$. As, $\mu \subseteq (\mu\chi_M : \chi_M) \subseteq (\xi\chi_M : \chi_M) = \xi$ by Proposition 2.1. Hence, $\mu \subseteq \xi$. □

Theorem 3.8. *Let R be a ring and M be an L-fuzzy multiplication R-module. Then $\xi\chi_M \neq \chi_M$ for any proper fuzzy ideal ξ of R.*

Proof. As ξ is a proper fuzzy ideal of R, by Theorem 2.5, there exists a maximal fuzzy ideal η of R such that $\xi \subseteq \eta$. Let μ be a proper L-fuzzy submodule of M. As M is an L-fuzzy multiplication module, by Theorem 2.11, μ is contained in a generalized maximal L-fuzzy submodule of M say ν. Then, ν is a maximal L-fuzzy submodule of M. Hence, by Theorem 3.4, $\nu = \eta\chi_M \neq \chi_M$. But as $\xi \subseteq \eta$ implies that $\xi\chi_M \subseteq \eta\chi_M \neq \chi_M$ and so $\xi\chi_M \neq \chi_M$. □

Theorem 3.9. *Let L be a dense chain and M be a faithful L-fuzzy multiplication R-module. Let μ be a primary L-fuzzy ideal of R, $a, b \in L$ and $r_a x_b \in \mu\chi_M$ for some $r \in R$ and $x \in M$. Then $r_a \in \mu$ or $x_b \in \mu\chi_M$.*

Proof. As μ is a primary L-fuzzy ideal of R and L is a dense chain, then by Theorem 2.4 $\sqrt{\mu}$ is prime L-fuzzy ideal of R. Now, by Theorem 2.3, for each $r \in R$, there exist a prime ideal P of R and a prime element $c \in L$ such that

$$\sqrt{\mu(r)} = \begin{cases} 1, & \text{if } r \in P, \\ c, & \text{otherwise.} \end{cases}$$

(I) As $r_a x_b \in \mu\chi_M$, it follows that $\mu\chi_M(rx) \geq a \wedge b$.

But, (II)

$$\begin{aligned}\mu\chi_M(rx) &= \vee\{\mu(s) \wedge \chi_M(y) \mid s \in R, y \in M, rx = sy\}\\ &= \vee\{\mu(s) \mid s \in R, rx \in sM\}.\end{aligned}$$

Let $A = \{s \in P \mid rx \in sM\}$.

Case(I). If $A = \emptyset$, then there does not exist $s \in P$ such that $rx \in sM$. Hence, from (I) $\mu\chi_M(rx) = c \geq a \wedge b$. As c is a prime element of L, either $c \geq a$ or $c \geq b$.

(i) Suppose that $c \geq a$. As $\mu(r) \in \{1, c\}$, we have $\sqrt{\mu(r)} \geq a$ and so $r_a \in \sqrt{\mu}$.

(ii) If $c \geq b$, then similarly from (II) $\mu\chi_M(x) = \vee\{\mu(s') : s' \in R, x \in s'M\}$. So, $\mu\chi_M(x) \in \{1, c\}$. Therefore, $\mu\chi_M(x) \geq b$ and so $x_b \in \mu\chi_M$.

Case (II). If $A \neq \emptyset$, then there exists $s' \in P$ such that $rx \in s'M$. Therefore, using (I) we have $\mu\chi_M(rx) = \vee\{\mu(s) \mid s \in R, rx \in sM\} = 1$ and $rx \in s'M \subseteq PM$. Now, by using Theorem 2.7 and Theorem 2.8, we get either $r \in P$ or $x \in PM$.

(i) If $r \in P$, then $\sqrt{\mu(r)} = 1 \geq a$ implies that $ra \in \sqrt{\mu}$.

(ii) If $x \in PM$, then $x = r_1x_1 + \cdots + r_nx_n$ for some $r_i \in P$ and $x_i \in M$ such that $i = 1, 2, \ldots, n$. Hence, $\mu\chi_M(x) = \mu\chi_M(\Sigma r_i x_i) \geq \mu\chi_M(r_1x_1) \wedge \cdots \wedge \mu\chi_M(r_nx_n) = 1 \geq b$ and so, $x_b \in \mu\chi_M$. □

Corollary 3.2. *Assume that M is a faithful L-fuzzy multiplication R-module and μ is a primary L-fuzzy ideal of R such that $\chi_M \neq \mu\chi_M$. Then $\mu\chi_M$ is a primary L-fuzzy submodule of M.*

Proof. Let μ be a primary L-fuzzy ideal of R and M be a faithful L-fuzzy multiplication R-module. If $r_a x_b \in \mu\chi_M$, for $r \in R$ and $x \in M$, then by Theorem 3.9, $r_a \in \sqrt{\mu} \subseteq \sqrt{(\mu\chi_M : \chi_M)}$ or $x_b \in \mu\chi_M$. Thus, $\mu\chi_M$ is a primary L-fuzzy submodules of M. □

4. Conclusion

In this article, we have defined an L-fuzzy hollow submodule in a different way and some of its properties are investigated. Also, some theorems on L-fuzzy multiplication modules are proved. Thus, this concept of an L-fuzzy multiplication module can be extended to an L-fuzzy fully invariant multiplication modules.

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References

[1] S. E. Atani, F. Çallialp and Ü. Tekri, *A short note on the primary submodules of multiplication modules*, Int. J. Algebra **1**(8) (2007), 381–384. http://dx.doi.org/10.12988/ija.2007.07042

[2] S. E. Atani and F. E. K. Sarai, *On L-fuzzy multiplication modules*, Discuss. Math. Gen. Algebra Appl. **37** (2017), 209–221. https://doi.org/10.7151/dmgaa.1268

[3] A. Barnad, *Multiplication modules*, J. Algebra **71** (1981), 174–178. https://doi.org/10.1016/0021-8693(81)90112-5
[4] Z. Elbast and P. F. Smith, *Multiplication modules*, Comm. Algebra **16** (1988), 755–779. https://doi.org/10.1080/00927878808823601
[5] R. Kumar, *Fuzzy Algebra*, Publication Division, University of Delhi, Delhi, 1993.
[6] D. S. Lee and C. H. Park, *On fuzzy prime submodules of fuzzy multiplication modules*, East Asian Math. J. **27**(1) (2011), 75–82. https://doi.org/10.7858/eamj.2011.27.1.075
[7] D. S. Malik and J. N. Morderson, *Fuzzy maximal, radical, and primary ideals of a ring*, Inform. Sci. **53** (1991), 237–250. https://doi.org/10.1016/0020-0255(91)90038-V
[8] J. N. Moderson and D. S. Malik, *Fuzzy Commutative Algebra*, World Scientific, River Edge, NJ, USA, 1998.
[9] C. V. Nagoita and D. A. Ralesca, *Applications of Fuzzy Sets in System Analysis*, Birkhauser, Basel, Switzerland, 1975.
[10] S. Rahman and H. K. Saikia, *Fuzzy small submodule and Jacobson L-radical*, Int. J. Math. Math. Sci. Act. **2011** (2011) Article ID 980320, 12 pages. https://doi.org/10.1155/2011/980320
[11] S. Rahman, *Fuzzy hollow submodules*, Ann. Fuzzy Math. Inform. **12**(5) (2016), 601–608.
[12] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512–517. https://doi.org/10.1016/0022-247X(71)90199-5
[13] L. A. Zadeh, *Fuzzy sets*, Information and Control **8** (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X

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