KRAGUJEVAC JOURNAL OF MATHEMATICS
VOLUME 46(4) (2022), PAGES 525–532.

MAPS PRESERVING THE SPECTRUM OF SKEW LIE PRODUCT OF OPERATORS

EMAN ALZEDANI[1] AND MOHAMED MABROUK[2]

ABSTRACT. Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on a complex Hilbert space $\mathcal{H}$. In this paper, we show that a surjective map φ on $\mathcal{B}(\mathcal{H})$ satisfies

$$\sigma\left(\varphi(T)\varphi(S)-\varphi(S)\varphi(T)^*\right)=\sigma\left(TS-ST^*\right),\quad T,S\in\mathcal{B}(\mathcal{H}),$$

if and only if there exists a unitary operator $U\in\mathcal{B}(\mathcal{H})$ such that

$$\varphi(T)=\lambda UTU^*,\quad T\in\mathcal{B}(\mathcal{H}),$$

where $\lambda\in\{-1,1\}$.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Throughout this paper, $\mathcal{B}(\mathcal{H})$ stands for the algebra of all bounded linear operators acting on an infinite dimensional complex Hilbert space $(\mathcal{H},\langle\cdot,\cdot\rangle)$. Let $\mathcal{B}_s(\mathcal{H})$ (resp. $\mathcal{B}_a(\mathcal{H})$) be the real linear space of all self-adjoint (resp. anti-self-adjoint) operators in $\mathcal{B}(\mathcal{H})$. For every $A\in\mathcal{B}(\mathcal{H})$, the spectrum (resp. the spectral radius) of A is denoted by $\sigma(A)$ (resp. $r(A)$).

The problem of describing maps on operators and matrices that preserve certain functions, subsets and relations has been widely studied in the literature, see [3–6, 9–12, 16, 19–22] and references therein. One of the classical problems in this area of research is to characterize maps preserving the spectra of the product of operators. Molnár in [19] studied maps preserving the spectrum of operator and matrix products. His results have been extended in several directions [1, 2, 7, 8, 13–15, 17] and [18]. In [1], the problem of characterizing maps between matrix algebras preserving the spectrum

Key words and phrases. Nonlinear preservers, spectrum, Skew Lie product.
2010 *Mathematics Subject Classification.* Primary: 47B49. Secondary: 47A10, 47A11.
Received: December 29, 2019.
Accepted: February 19, 2020.

of polynomial products of matrices is considered. In particular, the results obtained therein extend and unify several results obtained in [6] and [8].

Latter in [2], the form of all maps preserving the spectrum and the local spectrum of Skew Lie product of matrices are determined. This paper is a continuation of such recent work, and examines the form of maps preserving the spectrum of skew Lie product of operators on a complex Hilbert space. Mainly, we shall give a characterization of all surjective maps $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ preserving the spectrum of the skew Lie product "$[T, S]_* = TS - ST^*$" of operators. Precisely, the following theorem is the main result of this paper.

Theorem 1.1. *A surjective map $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ satisfies*

$$\sigma\left(\varphi(T)\varphi(S) - \varphi(S)\varphi(T)^*\right) = \sigma\left(TS - ST^*\right), \quad T, S \in \mathcal{B}(\mathcal{H}), \tag{1.1}$$

if and only if there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that

$$\varphi(T) = \pm UTU^*, \tag{1.2}$$

for all $T \in \mathcal{B}(\mathcal{H})$.

Before presenting the proof of the main theorem few comments can be made. Firstly, note that the only restriction on the map φ is surjectivity; no linearity or additivity or continuity is assumed. Also, we point out that the consideration of maps φ from $\mathcal{B}(\mathcal{H})$ onto itself is for the sake of simplicity. Our result and its proof remains valid in the case where φ is a surjective map from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}(\mathcal{K})$ where $\mathcal{H}$ and $\mathcal{K}$ are two different Hilbert spaces.

The case of finite dimensional Hilbert spaces was considered in [2] where it is shown that the theorem 1.1 remains valid without the surjectivity assumption of the map φ. The proof given therein is based on a density argument and is completely different from the one presented in the current paper. This paper is divided into three sections. In Section 2, we collect some auxiliary lemmas needed in the proof of the main result. In Section 3, we present the proof of Theorem 1.1.

2. Preliminaries

Given two vectors x and y in $\mathcal{H}$, let $x \otimes y$ be the operator of at most rank one defined by

$$(x \otimes y)(z) := \langle z, y \rangle x, \quad z \in \mathcal{H},$$

and note that $(x \otimes y)^* = y \otimes x$. Let $(e_k)_{k \in I}$ be an orthonormal basis of $\mathcal{H}$. For any $A \in \mathcal{B}(\mathcal{H})$, the transpose $A^\top$ of A with respect to the basis $(e_k)_{k \in I}$ is defined as the unique operator such that

$$\langle Ae_k, e_j \rangle = \langle A^\top e_j, e_k \rangle,$$

for any $j, k \in I$.

For any $x = \sum_{k \in I} x_k e_k$, write $\bar{x} = \sum_{k \in I} \overline{x}_k \, e_k$. It is easy to see that

$$(x \otimes y)^\top = \bar{y} \otimes \bar{x},$$

for any $x, y \in \mathcal{H}$.

To prove Theorem 1.1, we need some auxiliary results that we present below. The first lemma describes the spectrum of the skew Lie product $[x \otimes y, A]_*$ for any nonzero vectors $x, y \in \mathcal{H}$ and operator $A \in \mathcal{B}(\mathcal{H})$.

Lemma 2.1. *For any nonzero vectors $x, y \in \mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$, set*

$$\Delta_A(x,y) = (\langle Ax, y\rangle + \langle Ay, x\rangle)^2 - 4\|x\|^2\langle A^2y, y\rangle$$

and

$$\Lambda_A(x,y) = (\langle x, Ay\rangle + \langle Ax, y\rangle)^2 - 4\langle x, y\rangle\langle Ax, Ay\rangle.$$

Then

(1) $\sigma\left([x \otimes y, A]_*\right) = \frac{1}{2}\left\{0, \langle Ax, y\rangle - \langle Ay, x\rangle \pm \sqrt{\Delta_A(x,y)}\right\}$;
(2) $\sigma\left([A, x \otimes y]_*\right) = \frac{1}{2}\left\{0, \langle Ax, y\rangle - \langle x, Ay\rangle \pm \sqrt{\Lambda_A(x,y)}\right\}$.

Proof. For the proof of the first item see [2]. The second statement can be proved in a similar way and we therefore omit its proof. □

Corollary 2.1. *For any $x \in \mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H})$, we have*

$$\sigma(A(x \otimes x) + (x \otimes x)A) \;=\; \left\{0, \langle Ax, x\rangle \pm \|x\|\sqrt{\langle A^2x, x\rangle}\right\}.$$

Proof. It suffices to replace x by ix and y by x in Lemma 2.1 (1). □

The second principle gives necessary and sufficient conditions for two operators to be the same.

Lemma 2.2. *For any two operators A and B in $\mathcal{B}(\mathcal{H})$, the following statements are equivalent.*

(1) $A = B$.
(2) $\sigma([X, A]_*) = \sigma([X, B]_*)$ *for every operator* $X \in \mathcal{B}(\mathcal{H})$.
(3) $\sigma([X, A]_*) = \sigma([X, B]_*)$ *for every operator* $X \in \mathcal{B}_a(\mathcal{H})$.

Proof. The proof is the same as that of [2, Corollary 3.2]. □

The next lemma characterizes real scalar operators in terms of skew Lie products.

Lemma 2.3. *For an operator $A \in \mathcal{B}(\mathcal{H})$, we have $\sigma([A, X]_*) = \{0\}$ holds for any operator $X \in \mathcal{B}(\mathcal{H})$ if and only if $A = \alpha I$ for some scalar $\alpha \in \mathbb{R}$.*

Proof. The "*if*" part is obvious. To check the "*only if*" part, assume that

$$\sigma(([A, X]_*)) = \{0\}$$

holds for any operator $X \in \mathcal{B}(\mathcal{H})$. As $A - A^*$ is anti-self-adjoint then

$$\|A - A^*\| = r(A - A^*) = r([A, I]_*) = 0,$$

it follows that $A = A^*$. If there exists a nonzero vector $x \in \mathcal{H}$ such that $\{x, Ax\}$ is a linearly independent set, then by Lemma 2.1 (2) we have

$$\sigma([A, x \otimes x]_*) = \frac{1}{2}\left\{0, \pm\sqrt{\langle Ax, x\rangle^2 - \|x\|^2\ \|Ax\|^2}\right\}.$$

This is a contradiction since $\langle Ax, x\rangle^2 - \|x\|^2\ \|Ax\|^2 \neq 0$. □

We close this section with the following lemma which gives a characterization of self-adjoint and antiself-adjoint operators in terms of the spectrum of the skew Lie product.

Lemma 2.4. *If $A \in \mathcal{B}(\mathcal{H})$ is nonzero operator, then*

(1) *$A \in \mathcal{B}_s(\mathcal{H})$ if and only if $\sigma([X, A]_*) \subset i\mathbb{R}$, for any $X \in \mathcal{B}(\mathcal{H})$;*
(2) *$A \in \mathcal{B}_a(\mathcal{H})$ if and only if $\sigma([X, A]_*) \subset \mathbb{R}$, for any $X \in \mathcal{B}(\mathcal{H})$.*

Proof. (1) If $A = A^*$, then $\sigma([X, A]_*) \subset i\mathbb{R}$, since $[X, A]_* = XA - AX^* = XA - (XA)^*$. To prove the converse, assume that $\sigma([X, A]_*) \subset i\mathbb{R}$ for any operator $X \in \mathcal{B}(\mathcal{H})$. In particular by Lemma 2.1 (1) we get

$$\sigma([x \otimes y, A]_*) = \frac{1}{2}\left\{0, \langle Ax, y\rangle - \langle Ay, x\rangle \pm \sqrt{\Delta_A(x, y)}\right\} \subset i\mathbb{R},$$

for any $x, y \in \mathcal{H}$. Which yields that

$$0 \;=\; \Re(\langle Ax, y\rangle - \langle Ay, x\rangle) = \langle (A - A)^* x, y\rangle + \langle y, (A - A^*)x\rangle.$$

Replace x by ix in the above equality, we get

$$\langle (A - A)^* x, y\rangle - \langle y, (A - A^*)x\rangle = 0.$$

Accordingly $\langle (A - A)^* x, y\rangle = 0$ for any $x, y \in \mathcal{H}$. Thus, $A = A^*$.

(2) We have

$$\begin{aligned}
A \in \mathcal{B}_a(\mathcal{H}) &\Leftrightarrow iA \in \mathcal{B}_s(\mathcal{H}) \\
&\Leftrightarrow \sigma([X, iA]_*) \subset i\mathbb{R}, \quad \text{for all } X \in \mathcal{B}(\mathcal{H}) \quad \text{(by Lemma 2.4 (1))} \\
&\Leftrightarrow i\sigma([X, A]_*) \subset i\mathbb{R}, \quad \text{for all } X \in \mathcal{B}(\mathcal{H}) \ \ (\text{since } \sigma([X, iA]_*) = i\sigma([X, A]_*)) \\
&\Leftrightarrow \sigma([X, A]_*) \subset \mathbb{R}, \quad \text{for all } X \in \mathcal{B}(\mathcal{H}).
\end{aligned}$$

□

3. Proof of Theorem 1.1

The "*if*" part is obvious. We will complete the proof of the "*only if*" part after proving several claims.

Claim 1. φ is injective.

Proof. For $A, B \in \mathcal{B}(\mathcal{H})$, assume that $\varphi(A) = \varphi(B)$. Then, for every $X \in \mathcal{B}(\mathcal{H})$, we have

$$\sigma([X, A]_*) \;=\; \sigma([\varphi(X), \varphi(A)]_*) = \sigma([\varphi(X), \varphi(B)]_* = \sigma([X, B]_*).$$

It then follows from Corollary 2.2 that $A = B$ and φ is injective. □

Claim 2. φ preserves self-adjoint and anti-self adjoint operators in both directions. In particular, we have $\varphi(0) = 0$.

Proof. Pick up an operator $A \in \mathcal{B}(\mathcal{H})$. If $A \in \mathcal{B}_s(\mathcal{H})$, then

$$\sigma\left([\varphi(X), \varphi(A)]_*\right) = \sigma\left([X, A]_*\right) \subset i\mathbb{R}.$$

As φ is surjective, then Lemma 2.4 (1) entails that $\phi(A) \in \mathcal{B}_s(\mathcal{H})$. Similarly, if $A \in \mathcal{B}_a(\mathcal{H})$, we have $\sigma([\varphi(X), \varphi(A)]_*) \subset \mathbb{R}$. By Lemma 2.4 (2), we get $\phi(A) \in \mathcal{B}_a(\mathcal{H})$.

For the converse, note that φ is bijective and φ^{-1} satisfies (1.1) A similar discussion entails that if $\varphi^{-1}(A) \in \mathcal{B}_s(\mathcal{H})$ (resp. $\varphi^{-1}(A) \in \mathcal{B}_a(\mathcal{H})$), then so is A. □

Claim 3. φ is homogenous, i.e., $\varphi(\alpha A) = \alpha\varphi(A)$ for any $\alpha \in \mathbb{C}$ and $A \in \mathcal{B}(\mathcal{H})$.

Proof. For any $\alpha \in \mathbb{C}$ and $A, X \in \mathcal{B}(\mathcal{H})$, we have

$$\begin{aligned}\sigma([\varphi(X), \varphi(\alpha A)]_*) =& \sigma([X, \alpha A]_*)\\ =& \alpha\ \sigma([X, A]_*)\\ =& \alpha\ \sigma([\varphi(X), \varphi(A)]_*)\\ =& \sigma([\varphi(X), \alpha\varphi(A)]_*).\end{aligned}$$

Hence,

$$\sigma([\varphi(X), \varphi(\alpha A)]_*) = \sigma([\varphi(X), \alpha\varphi(A)]_*),$$

for any $X \in \mathcal{B}(\mathcal{H})$. Since φ is bijective, we infer from Lemma 2.3 that $\varphi(\alpha A) = \alpha\varphi(A)$. This ends the proof of Claim 3. □

Claim 4. There exist a unitary operator $U \in \mathcal{B}(\mathcal{H})$ and a scalar $c \in \{-1, 1\}$ such that either

(i) $\varphi(A) = cUAU^*$ for every $A \in \mathcal{B}_s(\mathcal{H})$ or
(ii) $\varphi(A) = cUA^\top U^*$ for every $A \in \mathcal{B}_s(\mathcal{H})$.

Here $A^\top$ is the transpose of A with respect to an arbitrary but fixed orthonormal basis of $\mathcal{H}$.

Proof. Let $A, B \in \mathcal{B}(\mathcal{H})$. From Claim 3 and (1.1), we have

$$\begin{aligned}\sigma\left(\varphi(A)\varphi(B) + \varphi(B)\varphi(A)^*\right) =& -\sigma\left(\varphi(iA)\varphi(iB) - \varphi(iB)\varphi(iA)^*\right)\\ =& -\sigma\left(-AB - BA^*\right)\\ =& \sigma\left(AB + BA^*\right).\end{aligned}$$

Thus,

$$\sigma\left(\varphi(A)\varphi(B) + \varphi(B)\varphi(A)^*\right) = \sigma\left(AB + BA^*\right), \tag{3.1}$$

for any $A, B \in \mathcal{B}(\mathcal{H})$. Now Claim 2 implies that $\varphi(A) \in \mathcal{B}_s(\mathcal{H})$ whenever $A \in \mathcal{B}_s(\mathcal{H})$. This together with Claim 1 entail that the restriction $\varphi_{|\mathcal{B}_s(\mathcal{H})} : \mathcal{B}_s(\mathcal{H}) \to \mathcal{B}_s(\mathcal{H})$ is well defined and bijective. Moreover, (3.1) implies that

$$\sigma\left(\varphi(A)\varphi(B) + \varphi(B)\varphi(A)\right) = \sigma\left(AB + BA\right),$$

for any $A, B \in \mathcal{B}_s(\mathcal{H})$. Therefore, by [12, Theorem 3.1] (see also [23, Theorem 2], there exist a unitary operator $U \in \mathcal{B}(\mathcal{H})$ and a scalar $c \in \{-1, 1\}$ such that either

- $\varphi(A) = cUAU^*$ for every $A \in \mathcal{B}_s(\mathcal{H})$ or
- $\varphi(A) = cUA^\top U^*$ for every $A \in \mathcal{B}_s(\mathcal{H})$.

Here $A^\top$ is the transpose of A with respect to an arbitrary but fixed orthonormal basis of $\mathcal{H}$. □

In particular Claim 4 implies that $\varphi(I) = \pm I$. In the sequel we may and shall assume that $\phi(I) = I$. Define a map $\psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by putting

$$\psi(A) = U^*\varphi(A)U,$$

for every $A \in \mathcal{B}(\mathcal{H})$. Then ψ is a bijective map satisfying

$$\sigma\left(\psi(A)\psi(B) + \psi(B)\psi(A)^*\right) = \sigma\left(AB + BA^*\right), \tag{3.2}$$

for every $A, B \in \mathcal{B}(\mathcal{H})$. Moreover, we have either

$$\psi(A) = A, \quad A \in \mathcal{B}_s(\mathcal{H}), \tag{3.3}$$

or

$$\psi(A) = A^\top, \quad A \in \mathcal{B}_s(\mathcal{H}). \tag{3.4}$$

Claim 5. The form (3.4) cannot occur.

Proof. Assume for the sake of contradiction that $\psi(A) = A^\top$ for any $A \in \mathcal{B}_s(\mathcal{H})$. Let $\{e_j, j \in I\}$ be the orthonormal basis with respect to which $A^\top$ is computed, for every $A \in \mathcal{B}_s(\mathcal{H})$. To get a contradiction we shall prove that $\langle Ax, x\rangle = \langle \psi(A)x, x\rangle$ for any $x \in \mathcal{H}$ and $A \in \mathcal{B}_s(\mathcal{H})$. To do so it suffices to prove that

$$\langle Ae_k, e_l\rangle = \langle \psi(A)e_k, e_l\rangle, \tag{3.5}$$

for any k and l in I and $A \in \mathcal{B}_s(\mathcal{H})$.

Let $A \in \mathcal{B}_s(\mathcal{H})$ and pick up two elements e_k and e_l in $\{e_j, j \in I\}$. For any $\alpha, \beta \in \mathbb{R}$, set $a = \alpha e_k + \beta e_l$. Note that

$$\psi(a \otimes a) = (a \otimes a)^\top = a \otimes a.$$

Now, by (3.2) we have

$$\begin{aligned} \sigma\left((a \otimes a)A + A(a \otimes a)\right) =& \sigma\left((a \otimes a)A + A(a \otimes a)^*\right) \\ =& \sigma\left(\psi(a \otimes a)\psi(A) + \psi(A)\psi(a \otimes a)^*\right) \\ =& \sigma\left((a \otimes a)\psi(A) + \psi(A)(a \otimes a)^*\right) \\ =& \sigma\left((a \otimes a)\psi(A) + \psi(A)(a \otimes a)\right). \end{aligned}$$

Accordingly

$$\sigma\left((a \otimes a)\psi(A) + \psi(A)(a \otimes a)\right) \;=\; \sigma\left((a \otimes a)A + A(a \otimes a)\right). \tag{3.6}$$

Corollary 2.1 together with (3.6) entail that

$$\left\{0, \langle\psi(A)a, a\rangle \pm \|a\|\sqrt{\langle\psi(A)^2 a, a\rangle}\right\} = \left\{0, \langle Aa, a\rangle \pm \sqrt{\langle A^2 a, a\rangle\|a\|^2}\right\}.$$

Accordingly $\langle\psi(A)a, a\rangle = \langle Aa, a\rangle$. Since α and β are arbitrary, we infer that

$$\langle Ae_k, e_k\rangle = \langle\psi(A)e_k, e_k\rangle$$

and

$$\langle A(e_k + e_l), (e_k + e_l)\rangle = \langle\psi(A)(e_k + e_l), (e_k + e_l)\rangle,$$

for every $k, l \in I$. Since A and $\psi(A)$ are in $\mathcal{B}_s(\mathcal{H})$, we infer that

$$\langle Ae_k, e_l\rangle = \langle\psi(A)e_k, e_l\rangle.$$

This in particular implies that $\psi(A) = A$ for every for any $A \in \mathcal{B}_s(\mathcal{H})$. Which is impossible since $\psi(A) = A^\top$ for any $A \in \mathcal{B}_s(\mathcal{H})$. □

Claim 6. $\psi(A) = A$ for any $A \in \mathcal{B}(\mathcal{H})$.

Proof. We have $\psi(A) = A$ for any $A \in \mathcal{B}_s(\mathcal{H})$. For any $A \in \mathcal{B}(\mathcal{H})$, using a similar reasoning as above, one can show that $\langle Ax, x\rangle = \langle\psi(A)x, x\rangle$ for any $x \in \mathcal{H}$. Since $\mathcal{H}$ is a complex Hilbert space it yields that $\psi(A) = A$ as desired. The proof is thus complete. □

References

[1] Z. Abdelali, *Maps preserving the spectrum of polynomial products of matrices*, J. Math. Anal. Appl. **480** (2019), Paper ID 123392.

[2] Z. Abdelali, A. Bourhim and M. Mabrouk, *Preservers of spectrum and local spectrum on skew Lie products of matrices*, Contemp. Math. (to appear).

[3] L. Baribeau and T. Ransford, *Non-linear spectrum-preserving maps*, Bull. London Math. Soc. **32** (2000), 8–14.

[4] A. Bourhim and J. Mashreghi, *Maps preserving the local spectrum of triple product of operators*, Linear Multilinear Algebra (2014), 1–9.

[5] A. Bourhim and J. Mashreghi, *Maps preserving the local spectrum of product of operators*, Glasg. Math. J. **57** (2015), 709–718.

[6] A. Bourhim and J. Mashreghi, *A survey on preservers of spectra and local spectra*, in: *CRM Proceedings and Lecture Notes: Invariant Subspaces of the Shift Operator*, R. Amer. Math. Soc., Providence, 2015.

[7] A. Bourhim, J. Mashreghi and A. Stepanyan, *Maps between Banach algebras preserving the spectrum*, Arch. Math. **107** (2016), 609–621.

[8] J.-T. Chan, C.-K. Li and N.-S. Sze, *Mappings preserving spectra of products of matrices*, Proc. Amer. Math. Soc. **135** (2007), 977–986.

[9] C. Costara, *Non-surjective spectral isometries on matrix spaces*, Complex Anal. Oper. Theory **12** (2018), 859–868.

[10] C. Costara and D. Repovš, *Spectral isometries onto algebras having a separating family of finite-dimensional irreducible representations*, J. Math. Anal. Appl. **365** (2010), 605–608.

[11] J. Cui and P. Choonkil, *Maps preserving strong skew Lie product on factor von Neumann algebras*, Acta Math. Sci. **32** (2012), 531–538.

[12] J.-L. Cui and C.-K. Li, *Maps preserving peripheral spectrum of Jordan products of operators*, Oper. Matrices **6** (2012), 129–146.

[13] O. Hatori, T. Miura and H. Takagi, *Unital and multiplicatively spectrum-preserving surjections between semi-simple commutative Banach algebras are linear and multiplicative*, J. Math. Anal. Appl. **326** (2007), 281–296.

[14] J. Hou and K. He, *Non-linear maps on self-adjoint operators preserving numerical radius and numerical range of Lie product*, Linear Multilinear Algebra **64** (2016), 36–57.

[15] J. Hou, C.-K. Li and X. Qi, *Numerical range of Lie product of operators*, Integral Equations Operator Theory **83** (2015), 497–516.

[16] J. Hou, C.-W. Li and N.-C. Wong, *Maps preserving the spectrum of generalized Jordan product of operators*, Linear Algebra Appl. **432** (2010), 1049–1069.

[17] J. C. Hou, C. K. Li and N. C. Wong, *Jordan isomorphisms and maps preserving spectra of certain operator products*, Studia Math. **184** (2008), 31–47.

[18] J. C. Hou, C. K. Li and N. C. Wong, *Maps preserving the spectrum of generalized Jordan product of operators*, Linear Algebra Appl. **432** (2010), 1049–1069.

[19] L. Molnár, *Some characterizations of the automorphisms of* $\mathcal{B}(\mathcal{H})$ *and* $C(X)$, Proc. Amer. Math. Soc. **130** (2002), 111–120.

[20] P. V. R. Bhatia and A. Sourour, *Maps on matrices that preserve the spectral radius distance*, Studia Math. **134** (1999), 99–110.

[21] A. Taghavi, R. Hosseinzadeh and E. Nasrollahi, *Maps preserving a certain product of operators*, Ric. Mat. (2019), 1–12.

[22] W. Zhang and J. Hou, *Maps preserving peripheral spectrum of Jordan semi-triple products of operators*, Linear Algebra Appl. **435** (2011), 1326–1335.

[23] W. Zhang and J. Hou, *Maps preserving peripheral spectrum of generalized Jordan products of self-adjoint operators*, Abstr. Appl. Anal. **2014** (2014), Article ID 192040, 8 pages.

[1]Department of Mathematics, Faculty of Applied Sciences,
Umm Al-Qura University,
21955 Makkah, Saudi Arabia
Email address: `emanshaya2017@hotmail.com`

[2]Department of Mathematics,
Faculty of Sciences of Sfax, Tunisia
Email address: `mbsmabrouk@gmail.com`