Kragujevac Journal of Mathematics
Volume 51(3) (2027), Pages 381–399.

# ON A THEOREM OF LEGENDRE ON DIOPHANTINE APPROXIMATION

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Abstract. Legendre's theorem states that every irreducible fraction $\frac{p}{q}$ which satisfies the inequality $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$ is convergent to $\alpha$. Later Barbolosi and Jager improved this theorem. In this paper we refine these results.

## 1. Introduction

The theory of simple continued fractions plays the most important tools in mathematical analysis, probability theory, physics, approximation theory and other branches of natural sciences. During the last three centuries many famous mathematicians came with interesting results in Diophantine approximations and continued fractions. Among them let us mention for example [5, 7, 15, 27, 31].

In 1830, Legendre [27, page 23], proved the sufficient condition for a fraction $p/q$ to be convergent of real number $\alpha$.

**Theorem 1.1** (Legendre)**.** *Let $p$ and $q$ be relatively prime integers with $q > 0$ and such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}.$$

*Then, $\frac{p}{q}$ is a convergent to $\alpha$.*

In Theorem 3.1 we refine this theorem, replacing 2 by $2-(q-1)/q^2$. The proof is elementary in character but lengthy, as it involves a detailed case analysis. Theorems 3.2 and 3.3 give other alternative to Theorem 3.1.

*Key words and phrases.* Continued fraction, approximation, theorem of Legendre.
2020 *Mathematics Subject Classification.* Primary: 11J82. Secondary: 11A55.
DOI
*Received*: January 23, 2025.
*Accepted*: September 04, 2025.

Legendre's theorem has some history. In 1965, Billingsley [4] made use of Legendre's method in the ergodic theory. In 1988, Ito [16] tried to extend Legendre's constant $\frac{1}{2}$ to the other kind of continued fractions. To do this he used a special algorithm how to find the best constant. Such kind of methods are used for example in the theory of dynamical systems and ergodic theory. For other results see [17, 18] and [28].

There is a nice result of Koksma [23] stated in the following theorem.

**Theorem 1.2** (Koksma)**.** *If $p/q$ is a rational and $\alpha$ an irrational number and if $q|q\alpha - p| < 2/3$, then $p/q$ is either a convergent or a first mediant of $\alpha$. The constant $2/3$ is best possible.*

Barbolosi and Jager [2] refined Legendre's theorem by considering two special cases from which the constant 2/3 appears. They proved the following theorem.

**Theorem 1.3** (Barbolosi and Jager)**.** *Let $p/q = [b_0; b_1, \dots, b_n]$, where $b_n \geq 2$, $(p, q) = 1$, $q > 0$ be the simple continued fraction expansion and let $\alpha$ be an irrational number. If $(-1)^n \operatorname{sgn}(\alpha - p/q) = 1$ and $q|q\alpha - p| < 2/3$, then $p/q$ is a convergent of $\alpha$.*

This is an interesting result because it shows the possibility of improving the constant 1/2 in Legendre's theorem. However, in some cases, determining the length of the continued fraction representing $p/q$ is not easy.

We improve Theorem 1.2 and give some alternative to Theorem 1.3 in Theorem 3.4, when we replace 2/3 by $(3/2 - ((q-1)(q-2))/(2q^3))^{-1}$. The similar method was used in [3] and [10–12] but for completely different results.

Interesting results can be found in [1, 9, 19, 24–26]. An excellent source of basic background is given in [13]. The books [6, 14, 20–22, 29, 32] and [8] are also very useful.

## 2. Notation

Throughout the paper, $\mathbb{Z}^+$, $\mathbb{N}_0$ and $\mathbb{R}$ will denote the sets of positive, non-negative integers and real numbers, respectively. Let $\alpha$ be a real number and suppose $n \in \mathbb{N}_0$. Let $\alpha = [a_0; a_1, a_2, \dots]$ be its simple continued fraction expansion. Also, let $p_n/q_n = [a_0; a_1, a_2, \dots, a_n]$ be its $n$-th convergent. The following recurrence relations for convergents are known

$$\begin{aligned} p_{-2} &= 0, \quad p_{-1} = 1, \quad p_0 = a_0, \quad p_1 = a_1 a_0 + 1, \\ p_{n+2} &= a_{n+2} p_{n+1} + p_n, \\ q_{-2} &= 1, \quad q_{-1} = 0, \quad q_0 = 1, \quad q_1 = a_1, \\ q_{n+2} &= a_{n+2} q_{n+1} + q_n. \end{aligned}$$

For a simple continued fraction expansion we have that

$$\alpha = [a_0; a_1, a_2, \dots] = \Big[a_0; a_1, a_2, \dots, a_n, [a_{n+1}; a_{n+2}, a_{n+3}, \dots]\Big].$$

Therefore, we can write

$$\alpha = [a_0; a_1, a_2, \dots, a_n, r_{n+1}],$$

where $r_{n+1} = [a_{n+1}; a_{n+2}, a_{n+3}, \dots]$. From this we obtain that

$$\alpha = \frac{r_{n+1}p_n + p_{n-1}}{r_{n+1}q_n + q_{n-1}}. \tag{2.1}$$

In this article, we will write $[a_0]$ or $[a_0 + t]$, where $t \in \mathbb{Z}$ to define continued fractions with only the first integer part.

Taking a difference of two consecutive convergents we obtain that

$$q_{n+1}p_n - p_{n+1}q_n = (-1)^{n+1}. \tag{2.2}$$

For the finite simple continued fraction expansion, if we have $\alpha = [a_0; a_1, a_2, \dots, a_k]$ with $k \geq 1$, then we suppose that $a_k \geq 2$.

The sequence of mediants of real irrational number $\alpha$ is the sequence of irreducible fractions of the form

$$\frac{bp_n + p_{n-1}}{bq_n + q_{n-1}},$$

with $n \geq 0$, $b = 1, 2, \dots, a_{n+1} - 1$, ordered in such a way that the denominators form an increasing sequence. The mediant with $b = 1$ or $b = a_{n+1} - 1$ is called nearest mediant.

More details on the discussion in this section can be found in [2, page 7] or [30, page 10].

## 3. New Results

Our main result is the following theorem which concerns with the Legendre's theorem.

**Theorem 3.1.** *Let $p, q$ be relatively prime integers with $q \geq 1$ and*

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{\left(2 - \frac{q-1}{q^2}\right) q^2}. \tag{3.1}$$

*Then, $p/q$ is a convergent to $\alpha$ excluding the case when $\alpha = [a_0; 2]$ and $p/q = [a_0 + 1]$. For this special case, we have equality in (3.1) and $p/q$ is a nearest mediant of $\alpha$.*

We can improve this theorem in the following way.

**Theorem 3.2.** *Let $p, q$ be relatively prime integers with $q \geq 1$ and*

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{\left(2 - \frac{1}{q}\right) q^2}. \tag{3.2}$$

*Then, $p/q$ is a convergent to $\alpha$ excluding the following cases:*

(a) $\alpha = [a_0]$ *and* $p/q = [a_0 - 1]$;
(b) $\alpha = [a_0]$ *and* $p/q = [a_0 + 1]$;
(c) $\alpha = [a_0; a_1, a_2, \dots]$, $a_1 \geq 2$, *and* $p/q = [a_0 + 1]$;
(d) $\alpha = [a_0; 3]$ *and* $p/q = [a_0; 2]$;
(e) $\alpha = [a_0; a_1, 2]$ *and* $p/q = [a_0; a_1 + 1]$.

*For all special cases* (a)-(e), *we have that* $p/q$ *is the nearest mediant of* $\alpha$. *For cases* (a), (b), (d) *and* (e) *we have equality in* (3.2), *case* (c) *satisfies sharp inequality in* (3.2).

For every large $n$ we have the following result.

**Theorem 3.3.** *Let* $\alpha$ *be a real number,* $p, q$ *be relatively prime integers with* $q \geq 1$, $n$ *be a positive integer and* $p_{n-1}/q_{n-1}$ *be a convergent of* $\alpha$ *and also* $p/q$. *Assume that*

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{\left(2 - \frac{q_{n-1}}{q}\right) q^2}. \tag{3.3}$$

*Then,* $p/q$ *is a convergent to* $\alpha$, *excluding the following cases:*

(a) $\alpha = [a_0; 2]$ *and* $p/q = [a_0 + 1]$;
(b) $\alpha = [a_0; 3]$ *and* $p/q = [a_0; 2]$;
(c) $\alpha = [a_0; a_1, \ldots, a_{n-1}, a_n, 2]$ *and* $p/q = [a_0; a_1, \ldots, a_{n-1}, a_n + 1]$.

*For all these cases* $p/q$ *is the nearest mediant of* $\alpha$ *and we have equality in* (3.3).

The next theorem improves the result of Barbolosi and Jager.

**Theorem 3.4.** *Let* $p, q$ *be relatively prime integers with* $q \geq 1$. *If*

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{\left(1 - \frac{1}{2q}\right) q^2},$$

*then* $p/q$ *is either a convergent or nearest mediant of* $\alpha$, *excluding the following cases:*

(a) $\alpha = [a_0]$ *and* $p/q = [a_0 - 2]$;
(b) $\alpha = [a_0; a_1, a_2, \ldots]$ *and* $p/q = [a_0 + 2]$;
(c) $\alpha = [a_0; 6]$ *and* $p/q = [a_0; 2]$;
(d) $\alpha = [a_0; 5, a_2, \ldots]$ *and* $p/q = [a_0; 2]$;
(e) $\alpha = [a_0; a_1, 4]$ *and* $p/q = [a_0; a_1, 2]$;
(f) $\alpha = [a_0; 5]$ *and* $p/q = [a_0; 3]$;
(g) $\alpha = [a_0; 4, a_2, \ldots]$ *and* $p/q = [a_0; 2]$.

*Example* 3.1. Let $\alpha = \sum_{n=1}^{+\infty} \frac{1}{2^{2^n-1} A^{2^n}}$, where $A \in \mathbb{Z}^+$. Set

$$\frac{p}{q} = \frac{\left(2^{2^N-1} A^{2^N}\right) \sum_{n=1}^{N} \frac{1}{2^{2^n-1} A^{2^n}}}{2^{2^N-1} A^{2^N}} = \sum_{n=1}^{N} \frac{1}{2^{2^n-1} A^{2^n}},$$

where $N \in \mathbb{Z}^+$. From Theorem 3.1 we obtain that $p/q$ is a convergent of $\alpha$ which is not an immediate consequence of the Legendre's theorem.

*Example* 3.2. From Example 3.1 we obtain that $\sum_{n=1}^{N} \frac{1}{2^{2^n-1}}$ is a convergent of $\sum_{n=1}^{+\infty} \frac{1}{2^{2^n-1}}$. This is not an immediate consequence of the Legendre's theorem.

*Example* 3.3. Let $\alpha = \sum_{n=1}^{+\infty} \frac{1}{2^{2^n} A^{2^n}}$, where $A \in \mathbb{Z}^+$. Set

$$\frac{p}{q} = \frac{\left(2^{2^N} A^{2^N}\right) \sum_{n=1}^{N} \frac{1}{2^{2^n} A^{2^n}}}{2^{2^N} A^{2^N}} = \sum_{n=1}^{N} \frac{1}{2^{2^n} A^{2^n}},$$

where $N \in \mathbb{Z}^+$. From Theorem 3.4 we obtain that $p/q$ is a convergent or nearest mediant of $\alpha$ which is not an immediate consequence of Barbolosi and Jager's theorem.

## 4. Proofs

*Proof of Theorem* 3.1. Let $\alpha = [a_0; a_1, a_2, \dots, a_{n-1}, a_n, \dots]$ be a simple continued fractional expansion of the number $\alpha$. For any irreducible fraction $p/q$, which is not a convergent of $\alpha$ and $m, n \in \mathbb{N}_0$ we can write as

$$\frac{p}{q} = [a_0; a_1, a_2, \dots, a_{n-1}, b_n, b_{n+1}, \dots, b_{n+m}],$$

where $b_n, b_{n+1}, \dots, b_{n+m} \in \mathbb{Z}^+$ and $b_{n+m} \geq 2$.

1. Suppose that $q = 1$. Then, $p/q = [b_0] = b_0/1$. Now we prove that

$$\left|\alpha - \frac{p}{q}\right| > \frac{1}{\left(2 - \frac{q-1}{q^2}\right) q^2} = \frac{1}{2}.$$

i. Assume that $a_0 = b_0$. Then, $p/q$ is a convergent of $\alpha$.

ii. Let $a_0 > b_0$. Then, we obtain that

$$\left|\alpha - \frac{p}{q}\right| = a_0 - b_0 + [0; a_1, \dots] \geq 1 + [0; a_1, \dots] > \frac{1}{2}.$$

iii. Assume that $b_0 \geq a_0 + 2$. Then, we have

$$\left|\alpha - \frac{p}{q}\right| = b_0 - a_0 - [0; a_1, \dots] \geq 2 - [0; a_1, \dots] > \frac{1}{2}.$$

iv. Suppose that $b_0 = a_0 + 1$. It yields

$$\left|\alpha - \frac{p}{q}\right| = b_0 - a_0 - [0; a_1, \dots] = 1 - [0; a_1, \dots].$$

(i) If $a_1$ does not exist, then we obtain that

$$\left|\alpha - \frac{p}{q}\right| = b_0 - a_0 = 1 > \frac{1}{2}.$$

(ii) If $a_1 = 1$ and $a_2$ exists, then $p/q$ is a convergent of $\alpha$.

(iii) If $a_1 \geq 2$, then we have

$$\left|\alpha - \frac{p}{q}\right| = 1 - \frac{1}{a_1 + [0; a_2, \dots]} \geq \frac{1}{2}.$$

The equality occurs only in the case when $a_1 = 2$ and $a_2$ does not exist. This is the exception mentioned in Theorem 3.1.

2. Now we suppose that $q \geq 2$. Then, we have $2-(q-1)/q^2 > 2-1/q$ for all $q \in \mathbb{Z}^+$. Therefore, instead of (3.1) it is enough to prove (3.2). The proof falls into two main cases. Here is the plan of our proof:

| | | |
|---|---|---|
| a. $n=0$, | i. $a_0 > b_0$, | |
| | ii. $b_0 > a_0$, | |
| b. $n \geq 1$, | i. $a_n > b_n$, | A. $d \geq 1, c \geq 2$, |
| | | B. $d=0, c=1$, |
| | ii. $b_n > a_n$, | A. $d \geq 1, c \geq 2$, |
| | | B. $d=0, c=1$. |

a. Assume that $n=0$. Then, we have

$$\frac{p}{q} = [b_0; b_1, b_2, \ldots, b_m] = \frac{cb_0+d}{c},$$

where $\frac{d}{c} = [0; b_1, b_2, \ldots, b_m]$ and $c > d > 0$. Note that $c \neq 1$, otherwise $c = q = 1$. Set $A = [0; a_1, \ldots]$. Then, we have

$$\left|\alpha - \frac{p}{q}\right| = \left|a_0 - b_0 + A - \frac{d}{c}\right| = \frac{1}{c^2}c^2\left|a_0 - b_0 + A - \frac{d}{c}\right|. \tag{4.1}$$

Now we prove that

$$\left|\alpha - \frac{p}{q}\right| > \frac{1}{\left(2-\frac{1}{q}\right)q^2}.$$

From this, the fact that $q = c \geq 2$ and (4.1) we obtain that it is enough to prove that

$$\frac{1}{c^2}c^2\left|a_0 - b_0 + A - \frac{d}{c}\right| > \frac{1}{\left(2-\frac{1}{c}\right)c^2},$$

which is equivalent to

$$c(2c-1)\left|a_0 - b_0 + A - \frac{d}{c}\right| > 1. \tag{4.2}$$

Let us suppose two cases.

i. Assume that $a_0 > b_0$. This yields that inequality (4.2) has the form

$$c(2c-1)\left(a_0 - b_0 + A - \frac{d}{c}\right) > 1,$$

which is obviously true since $c \geq 2$ and

$$\begin{aligned} c(2c-1)\left(a_0 - b_0 + A - \frac{d}{c}\right) &\geq c(2c-1)\left(1+0-\frac{d}{c}\right) \\ &= (2c-1)(c-d) \geq 3 > 1. \end{aligned}$$

Hence, inequality (4.2) follows.

ii. Let $b_0 > a_0$. Then, inequality (4.2) is equivalent to

$$c(2c-1)\left(b_0 - a_0 - A + \frac{d}{c}\right) > 1,$$

which is also obviously true since $c \geq 2$, $d \geq 1$ and

$$c(2c-1)\left(b_0 - a_0 - A + \frac{d}{c}\right) > c(2c-1)\left(1 - 1 + \frac{d}{c}\right) = (2c-1)d > 1.$$

Therefore, inequality (4.2) follows.

b. Suppose that $n \geq 1$. Then, we have

$$\alpha = [a_0; a_1, a_2, \dots, a_{n-1}, a_n, a_{n+1}, \dots] = [a_0; a_1, \dots, a_{n-1}, a_n + r],$$

where $r = [0; a_{n+1}, a_{n+2}, \dots]$ satisfies $0 \leq r < 1$. If $r = 0$, then $a_{n+1}$ does not exist. This and equality (2.1) yield

$$\alpha = \frac{(a_n + r)p_{n-1} + p_{n-2}}{(a_n + r)q_{n-1} + q_{n-2}} \tag{4.3}$$

and

$$\frac{p}{q} = [a_0; a_1, a_2, \dots, a_{n-1}, b_n, b_{n+1}, \dots, b_{n+m}].$$

Set $d/c = [0; b_{n+1} \dots, b_{n+m}]$, where $c > d$. If $d = 0$, then $c = 1$. Otherwise, $p$ and $q$ are not coprime. This and equality (2.1) imply

$$\frac{p}{q} = \frac{p_{n-1}\left(b_n + \frac{d}{c}\right) + p_{n-2}}{q_{n-1}\left(b_n + \frac{d}{c}\right) + q_{n-2}} = \frac{p_{n-1}(cb_n + d) + cp_{n-2}}{q_{n-1}(cb_n + d) + cq_{n-2}}.$$

From this, (2.2) and (4.3) we obtain that

$$\begin{aligned}
\left|\alpha - \frac{p}{q}\right| &= \left|\frac{(a_n + r)p_{n-1} + p_{n-2}}{(a_n + r)q_{n-1} + q_{n-2}} - \frac{p_{n-1}(cb_n + d) + cp_{n-2}}{q_{n-1}(cb_n + d) + cq_{n-2}}\right| \\
&= \frac{|cb_n + d - ca_n - cr|}{((a_n + r)q_{n-1} + q_{n-2})(q_{n-1}(cb_n + d) + cq_{n-2})} \\
&= \frac{1}{q^2} \cdot \frac{q^2|cb_n + d - ca_n - cr|}{((a_n + r)q_{n-1} + q_{n-2})(q_{n-1}(cb_n + d) + cq_{n-2})} \\
&= \frac{1}{q^2} \cdot \frac{q|cb_n + d - ca_n - cr|}{(a_n + r)q_{n-1} + q_{n-2}} \\
&= \frac{1}{q^2} \cdot \frac{1}{\frac{(a_n + r)q_{n-1} + q_{n-2}}{q|cb_n + d - ca_n - cr|}} \\
&= \frac{1}{q^2} \cdot \frac{c}{\frac{c(a_n + r)q_{n-1} + cq_{n-2} - (cb_n + d)q_{n-1} + (cb_n + d)q_{n-1}}{q|cb_n + d - ca_n - cr|}} \\
&= \frac{1}{q^2} \cdot \frac{c}{\frac{q_{n-1}(c(a_n + r) - cb_n - d) + q}{q|cb_n + d - c(a_n + r)|}}
\end{aligned}$$

$$= \frac{1}{q^2} \frac{c}{\frac{1}{|c(b_n-a_n-r)+d|} + \frac{q_{n-1}\operatorname{sgn}(a_n-b_n)}{q}}. \tag{4.4}$$

i. If $a_n > b_n$ then (4.4) has the form

$$\left|\alpha - \frac{p}{q}\right| = \frac{1}{q^2} \cdot \frac{c}{\frac{1}{c(a_n-b_n+r)-d} + \frac{q_{n-1}}{q}}. \tag{4.5}$$

A. Assume that $d \geq 1$. Then, $c \geq 2$. From this, (4.5) and the fact that $q = q_{n-1}(cb_n + d) + cq_{n-2}$ we obtain that

$$\begin{aligned}\left|\alpha - \frac{p}{q}\right| &= \frac{1}{q^2} \cdot \frac{c}{\frac{1}{c(a_n-b_n+r)-d} + \frac{q_{n-1}}{q}} \geq \frac{1}{q^2} \cdot \frac{2}{1 + \frac{q_{n-1}}{q}} \\ &= \frac{1}{q^2} \cdot \frac{2}{1 + \frac{q_{n-1}}{q_{n-1}(cb_n+d)+cq_{n-2}}} \\ &= \frac{1}{q^2} \cdot \frac{2}{1 + \frac{1}{cb_n+d+c\frac{q_{n-2}}{q_{n-1}}}} \geq \frac{1}{q^2} \cdot \frac{3}{2} > \frac{1}{q^2} \cdot \frac{1}{2 - \frac{1}{q}}.\end{aligned}$$

B. Suppose that $d = 0$. Then, $c = 1$ and $b_{n+1}$ does not exist. Therefore, $q = b_n q_{n-1} + q_{n-2}$, where $b_n \geq 2$. From this and (4.5) we obtain that

$$\left|\alpha - \frac{p}{q}\right| = \frac{1}{q^2} \cdot \frac{c}{\frac{1}{c(a_n-b_n+r)-d} + \frac{q_{n-1}}{q}} \geq \frac{1}{q^2} \cdot \frac{1}{1 + \frac{q_{n-1}}{q}}.$$

Hence,

$$\begin{aligned}\left|\alpha - \frac{p}{q}\right| &\geq \frac{1}{q^2} \cdot \frac{1}{1 + \frac{q_{n-1}}{q}} = \frac{1}{q^2} \cdot \frac{1}{1 + \frac{q_{n-1}}{b_n q_{n-1}+q_{n-2}}} \\ &= \frac{1}{q^2} \cdot \frac{1}{1 + \frac{1}{b_n+\frac{q_{n-2}}{q_{n-1}}}} \geq \frac{1}{q^2} \cdot \frac{1}{1 + \frac{1}{b_n}} \geq \frac{1}{q^2} \cdot \frac{2}{3} \geq \frac{1}{q^2} \cdot \frac{1}{2 - \frac{1}{q}}.\end{aligned}$$

ii. Let $b_n > a_n$. Then, (4.4) has the form

$$\left|\alpha - \frac{p}{q}\right| = \frac{1}{q^2} \cdot \frac{c}{\frac{1}{c(b_n-a_n-r)+d} - \frac{q_{n-1}}{q}}. \tag{4.6}$$

A. Assume that $d \geq 1$. Then, $c \geq 2$. From this, (4.6) and the facts $0 \leq r < 1$, $q_{n-1} \geq 1$ we obtain that

$$\begin{aligned}\left|\alpha - \frac{p}{q}\right| &= \frac{1}{q^2} \cdot \frac{c}{\frac{1}{c(b_n-a_n-r)+d} - \frac{q_{n-1}}{q}} \\ &> \frac{1}{q^2} \cdot \frac{2}{\frac{1}{2(1-1)+1} - \frac{q_{n-1}}{q}} \\ &= \frac{1}{q^2} \cdot \frac{2}{1 - \frac{q_{n-1}}{q}} \geq \frac{1}{q^2} \cdot \frac{2}{1 - \frac{1}{q}} > \frac{1}{q^2} \cdot \frac{1}{2 - \frac{1}{q}}.\end{aligned}$$

B. Suppose that $d = 0$. Then, $c = 1$ and $b_{n+1}$ does not exist. Therefore, $q = b_n q_{n-1} + q_{n-2}$, where $b_n \geq 2$. Now we prove that

$$b_n - a_n - r \geq \frac{1}{2}. \tag{4.7}$$

Let $b_n = a_n + 1$.

- If $a_{n+1}$ does not exist, then $b_n - a_n - r = 1 > \frac{1}{2}$.
- If $a_{n+1} = 1$, then $\frac{p}{q}$ is a convergent of $\alpha$.
- If $a_{n+1} = 2$ and $a_{n+2}$ does not exist, then $r = \frac{1}{2}$ and we have

$$b_n - a_n - r = 1 - \frac{1}{2} = \frac{1}{2}.$$

- If $a_{n+1} = 2$ and $a_{n+2} \geq 1$, then

$$b_n - a_n - r = 1 - [0; 2, a_{n+2}, \dots] > \frac{1}{2}.$$

- If $a_{n+1} \geq 3$, then we have $b_n - a_n - r \geq 1 - [0; 3] = 2/3 > 1/2$.

Let $b_n \neq a_n + 1$. Then, $b_n - a_n - r \geq 2 - r > 1/2$. Hence, (4.7) follows.

From (4.6), (4.7), the facts that $c = 1$, $d = 0$ and $q_{n-1} \geq 1$ we obtain that

$$\left|\alpha - \frac{p}{q}\right| = \frac{1}{q^2} \cdot \frac{c}{\frac{1}{c(b_n - a_n - r) + d} - \frac{q_{n-1}}{q}} \geq \frac{1}{q^2} \cdot \frac{1}{2 - \frac{q_{n-1}}{q}} \geq \frac{1}{q^2} \cdot \frac{1}{2 - \frac{1}{q}}.$$

The proof of Theorem 3.1 is complete. □

*Proof of Theorem* 3.2. We only follow Case 2. of Theorem 3.1 with following discussions.

(a) In case 2.a.i we proved that

$$\left|\alpha - \frac{p}{q}\right| > \frac{1}{\left(2 - \frac{1}{q}\right) q^2},$$

for all cases when $q = c \geq 2$. Suppose that $q = c = 1$. Then, $d = 0$ and we have

$$c(2c-1)\left(a_0 - b_0 + A - \frac{d}{c}\right) = a_0 - b_0 + A \geq 1.$$

The equality occurs when $b_0 = a_0 - 1$ and $a_1$ does not exist, so $A = 0$. This is the first exception.

(b) Suppose that $q = c = 1$, $d = 0$ in case 2.a.ii. Then, we have

$$c(2c-1)\left(b_0 - a_0 - A + \frac{d}{c}\right) = b_0 - a_0 - A.$$

i. Let $b_0 \geq a_0 + 2$. Then, we obtain that

$$b_0 - a_0 - A \geq 2 - A > 1.$$

ii. Assume that $b_0 = a_0 + 1$. It implies that

$$b_0 - a_0 - A = 1 - A.$$

• If $a_1$ does not exist, then $A = 0$ and we have $b_0 - a_0 - A = 1 - A = 1$. This is the second exception.

• Let $a_1 = 1$ and $a_2$ exists. So, $\alpha = [a_0; 1; a_2, \dots]$ and $\frac{p}{q} = [a_0 + 1]$ is convergent of $\alpha$.

• If $a_1 \geq 2$, then we obtain that

$$b_0 - a_0 - A = 1 - \frac{1}{a_1 + [0; a_2, \dots]} < 1,$$

which is the third exception.

(c) In case 2.b.i.B we proved that

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{\left(2 - \frac{1}{q}\right) q^2}.$$

The equality occurs only when $n = 1$, $b_1 = 2$, $a_1 = 3$ and $a_2$ does not exist. Hence, $r = 0$. This is the fourth exception.

(d) In case 2.b.ii.B, we also proved that $\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{\left(2 - \frac{1}{q}\right) q^2}$. The equality occurs when $n = 1, b_1 = a_1 + 1, a_{n+1} = 2$ and $a_{n+2}$ does not exist. This is the fifth exception.

Proofs of other cases are the same like in the proof of Theorem 3.1.

The proof of Theorem 3.2 is complete. □

*Proof of Theorem* 3.3. The proof of this theorem follows Case 2. in the proof of Theorem 3.1 with some following discussions.

(a) For case 2.a if $n = 0$, then $q_{n-1} = q_{-1} = 0$. So, it is enough to prove that

$$\left|\alpha - \frac{p}{q}\right| = \frac{1}{c^2} c^2 \left|a_0 - b_0 + A - \frac{d}{c}\right| > \frac{1}{\left(2 - \frac{q_{n-1}}{q}\right) q^2} = \frac{1}{2q^2} = \frac{1}{2c^2},$$

which is equivalent to

$$2c^2 \left|a_0 - b_0 + A - \frac{d}{c}\right| > 1. \tag{4.8}$$

i. Suppose that $a_0 > b_0$. Then, inequality (4.8) has the form

$$2c^2 \left(a_0 - b_0 + A - \frac{d}{c}\right) > 1,$$

which is obviously true since

$$2c^2 \left(a_0 - b_0 + A - \frac{d}{c}\right) \geq 2c(c - d) \geq 2.$$

This is also true for $q = c \geq 1$ and $d \geq 0$.

ii. Assume that $b_0 \geq a_0 + 2$. Then, inequality (4.8) is equivalent to

$$2c^2 \left(b_0 - a_0 - A + \frac{d}{c}\right) > 1,$$

which is obviously true since

$$2c^2\left(b_0 - a_0 - A + \frac{d}{c}\right) > 2c^2\left(2 - 1 + \frac{d}{c}\right) \geq 2 > 1,$$

for all values of $c \geq 1$, $d \geq 0$.

iii. Let $b_0 = a_0 + 1$. Then, inequality (4.8) is equivalent to

$$2c^2\left(b_0 - a_0 - A + \frac{d}{c}\right) > 1,$$

which is obviously true since

$$2c^2\left(b_0 - a_0 - A + \frac{d}{c}\right) > 2c^2\left(1 - 1 + \frac{d}{c}\right) \geq 4 > 1,$$

for all values of $c \geq 2$, $d \geq 1$.

Now we suppose that $c = 1$, $d = 0$. Then, we obtain

$$2c^2\left(b_0 - a_0 - A + \frac{d}{c}\right) = 2(1 - A).$$

• If $a_1$ does not exist, then $A = 0$ and we have $2(1 - A) = 2 > 1$.
• If $a_1 = 1$ and $a_2$ exists, then $\frac{p}{q}$ is a convergent of $\alpha$.
• If $a_1 \geq 2$, then we have $2(1 - A) \geq 2\,(1 - 1/a_1) \geq 1$. The equality occurs when $\alpha = [a_0; 2]$ and $p/q = [a_0 + 1]$, which is the first exception.

(b) Suppose that $n \geq 1$. Then, we follow case 2.b in the proof of Theorem 3.1 with these exceptions.

i. In case 2.b.i.A we have

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{q^2} \cdot \frac{2}{1 + \frac{q_{n-1}}{q}} > \frac{1}{q^2} \cdot \frac{1}{2 - \frac{q_{n-1}}{q}}.$$

ii. In case 2.b.i.B we have $b_n \geq 2$, $q = b_n q_{n-1} + q_{n-2} \geq 2q_{n-1}$ and

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{q^2} \cdot \frac{1}{1 + \frac{q_{n-1}}{q}} \geq \frac{1}{q^2} \cdot \frac{1}{2 - \frac{q_{n-1}}{q}}.$$

The equality occurs when $\alpha = [a_0; 3]$ and $p/q = [a_0; 2]$, this is the second exception.

iii. In case 2.b.ii.A we have

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{q^2} \cdot \frac{2}{1 - \frac{q_{n-1}}{q}} > \frac{1}{q^2} \cdot \frac{1}{2 - \frac{q_{n-1}}{q}}.$$

iv. In case 2.b.ii.B we have

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{q^2} \frac{1}{2 - \frac{q_{n-1}}{q}}.$$

The equality occurs when $\alpha = [a_0; a_1, \ldots, a_{n-1}, a_n, 2]$ and $p/q = [a_0; a_1, \ldots, a_{n-1}, a_n + 1]$, this is the third exception.

The proof of Theorem 3.3 is complete. □

*Proof of Theorem* 3.4. Let $\alpha = [a_0; a_1, a_2, \dots, a_{n-1}, a_n, a_{n+1}, \dots]$ be a simple continued fraction expansion of number $\alpha$. For any irreducible fraction $p/q$ which is neither convergent nor nearest mediant of $\alpha$ and $m, n \in \mathbb{N}_0$ we can write as

$$\frac{p}{q} = [a_0; a_1, a_2, \dots, a_{n-1}, b_n, b_{n+1}, \dots, b_{n+m}],$$

where $b_n, b_{n+1}, \dots, b_{n+m} \in \mathbb{Z}^+$, $b_n \neq a_n$ and $b_{n+m} \geq 2$. If $b_{n+1}$ does not exist, then $b_n \notin \{a_n + 1, a_n - 1\}$.

The proof falls into two main cases. Here is the plan of our proof.

1. $n = 0$.
   a. $c \geq 2$, $d \geq 1$ and $a_0 \geq b_0 + 1$.
   b. $c = 1$, $d = 0$ and $a_0 \geq b_0 + 2$.
   c. $c \geq 2$, $d \geq 1$ and $b_0 \geq a_0 + 1$.
   d. $c = 1$, $d = 0$ and $b_0 \geq a_0 + 2$.
2. $n \geq 1$.
   a. $a_n > b_n$.
      i. $c \geq 2$, $d \geq 1$ and $a_n \geq b_n + 1$.
      ii. $a_n > b_n$, $c = 1$ and $d = 0$.
         A. $a_n \geq b_n + 4$.
         B. $a_n = b_n + 3$, $n \geq 2$.
         C. $a_n = b_n + 3$, $n = 1$.
         D. $a_n = b_n + 2$, $n \geq 2$.
         E. $a_n = b_n + 2$, $n = 1$.
   b. $b_n > a_n$
      i. $c \geq 2$, $d \geq 1$ and $b_n \geq a_n + 1$.
      ii. $c = 1$, $d = 0$ and $b_n \geq a_n + 2$.

1. Suppose that $n = 0$. Then,

$$\alpha = [a_0; a_1, a_2, \dots, a_{n-1}, a_n, a_{n+1}, \dots] = [a_0 + A]$$

and

$$\frac{p}{q} = [b_0; b_1, \dots, b_m] = b_0 + \frac{d}{c} = \frac{cb_0 + d}{c},$$

where $A = [0; a_1, a_2, \dots] \in [0; 1)$ and $d/c = [0; b_1, b_2, \dots, b_m] \in [0; 1)$. Note that $b_0 \neq a_0$, $b_m \geq 2$ and if $d = 0$, $c = 1$ which mean $b_1$ does not exist, then $b_0 \notin \{a_0 + 1, a_0 - 1\}$.

From this we have

$$\left|\alpha - \frac{p}{q}\right| = \left|a_0 - b_0 + A - \frac{d}{c}\right| = \frac{1}{c^2} c^2 \left|a_0 - b_0 + A - \frac{d}{c}\right|.$$

Now we prove that

$$\left|\alpha - \frac{p}{q}\right| > \frac{1}{\left(1 - \frac{1}{2q}\right) q^2},$$

which is equivalent to

$$\left(c^2-\frac{c}{2}\right)\left|a_0-b_0+A-\frac{d}{c}\right|>1. \tag{4.9}$$

a. Let $c\geq 2$, $d\geq 1$ and $a_0\geq b_0+1$. It yields that inequality (4.9) can be written as

$$\left(c^2-\frac{c}{2}\right)\left(a_0-b_0+A-\frac{d}{c}\right)>1,$$

which is obviously true since

$$\begin{aligned}\left(c^2-\frac{c}{2}\right)\left(a_0-b_0+A-\frac{d}{c}\right)&\geq\left(c^2-\frac{c}{2}\right)\left(1-\frac{d}{c}\right)\\&=\left(c-\frac{1}{2}\right)(c-d)>1.\end{aligned}$$

b. Assume that $c=1$, $d=0$ and $a_0\geq b_0+2$. It yields that inequality (4.9) can be written as

$$\left(c^2-\frac{c}{2}\right)\left(a_0-b_0+A-\frac{d}{c}\right)>1,$$

which is obviously true since

$$\left(c^2-\frac{c}{2}\right)\left(a_0-b_0+A-\frac{d}{c}\right)\geq\frac{1}{2}(a_0-b_0+A)\geq\frac{1}{2}(2+A)\geq 1.$$

The equality occurs when $A=0$ and $a_0=b_0+2$. It implies that $\alpha=[a_0]$ and $p/q=[a_0-2]$. This is the first exception.

c. Suppose that $c\geq 2$, $d\geq 1$ and $b_0\geq a_0+1$. Then, inequality (4.9) has the form

$$\left(c^2-\frac{c}{2}\right)\left(b_0-a_0-A+\frac{d}{c}\right)>1,$$

which is obviously true since

$$\begin{aligned}\left(c^2-\frac{c}{2}\right)\left(b_0-a_0-A+\frac{d}{c}\right)&>\left(c^2-\frac{c}{2}\right)\left(1-1+\frac{d}{c}\right)\\&=\left(c^2-\frac{c}{2}\right)\frac{d}{c}=\left(c-\frac{1}{2}\right)d>1.\end{aligned}$$

d. Assume that $b_0\geq a_0+2$, $c=1$ and $d=0$. It implies that inequality (4.9) is equivalent to

$$\frac{1}{2}(b_0-a_0-A)>1,$$

which is obviously true since the following hold.

- If $b_0\geq a_0+3$, then we obtain that

$$\frac{1}{2}(b_0-a_0-A)\geq\frac{1}{2}(3-A)>1.$$

• Suppose that $b_0 = a_0 + 2$. Hence,

$$\frac{1}{2}(b_0 - a_0 - A) = \frac{1}{2}(2 - A) \leq 1.$$

The equality occurs when $A = 0$ which mean $a_1$ does not exist. On the other side, when $a_1$ exists the sharp inequality satisfied. These are the cases included in exception 2.

2. Let $n \geq 1$. Then,

$$\alpha = [a_0; a_1, \dots, a_{n-1}, a_n, a_{n+1}, \dots] = \frac{(a_n + r)p_{n-1} + p_{n-2}}{(a_n + r)q_{n-1} + q_{n-2}}$$

and

$$\frac{p}{q} = [a_0; a_1, \dots, a_{n-1}, b_n, b_{n+1}, \dots, b_{n+m}] = \frac{(cb_n + d)p_{n-1} + cp_{n-2}}{(cb_n + d)q_{n-1} + cq_{n-2}},$$

where $r = [0; a_{n+1}, \dots] \in [0; 1)$ and $d/c = [0; b_{n+1}, \dots, b_{n+m}] \in [0, 1)$, $c > d$, $b_n \neq a_n$. If $d = 0$, then $c = 1$, otherwise $p, q$ are not coprime and then $b_n \notin \{a_n + 1, a_n - 1\}$. From this and (4.4) we obtain that

$$\left|\alpha - \frac{p}{q}\right| = \frac{1}{q^2} \cdot \frac{c}{\frac{1}{|c(b_n - a_n - r) + d|} + \frac{q_{n-1}\operatorname{sgn}(a_n - b_n)}{q}}.$$

Now we prove that

$$\left|\alpha - \frac{p}{q}\right| > \frac{1}{\left(1 - \frac{1}{2q}\right) q^2},$$

which is equivalent to

$$\frac{c}{\frac{1}{|c(b_n - a_n - r) + d|} + \frac{q_{n-1}\operatorname{sgn}(a_n - b_n)}{q}} > \frac{1}{1 - \frac{1}{2q}}. \tag{4.10}$$

a. Assume that $a_n > b_n$. Then, (4.10) has the form

$$\frac{c}{\frac{1}{c(a_n - b_n + r) - d} + \frac{q_{n-1}}{q}} > \frac{1}{1 - \frac{1}{2q}}. \tag{4.11}$$

Now we consider the following cases.

(i) Let $a_n \geq b_n + 1, c \geq 2$ and $d \geq 1$. Then, $q = (cb_n + d)q_{n-1} + cq_{n-2} \geq cb_n + d \geq 3$. From this we obtain that

$$\frac{c}{\frac{1}{c(a_n - b_n + r) - d} + \frac{q_{n-1}}{q}} \geq \frac{c}{\frac{1}{c - d} + \frac{q_{n-1}}{q}} \geq \frac{2}{1 + \frac{q_{n-1}}{q}}.$$

So to prove (4.11), it is enough to prove

$$\frac{2}{1 + \frac{q_{n-1}}{q}} > \frac{1}{1 - \frac{1}{2q}},$$

which can be written as

$$1 > \frac{1 + q_{n-1}}{q}.$$

This is obviously true since

$$q = (cb_n + d)q_{n-1} + cq_{n-2} \geq 3q_{n-1} + 2q_{n-2} > 1 + q_{n-1}.$$

(ii) Suppose that $a_n > b_n$, $c = 1$ and $d = 0$. Hence, $b_{n+1}$ does not exist and $b_n \geq 2$. Then, $q = b_n q_{n-1} + q_{n-2}$. Therefore, (4.11) has the form

$$\frac{1}{\frac{1}{a_n - b_n + r} + \frac{q_{n-1}}{q}} > \frac{1}{1 - \frac{1}{2q}}. \tag{4.12}$$

(A) Let $a_n \geq b_n + 4$. From $q = b_n q_{n-1} + q_{n-2} \geq b_n \geq 2$ we obtain that $\frac{1}{1-\frac{1}{2q}} \leq \frac{4}{3}$ and we have

$$\frac{1}{\frac{1}{a_n - b_n + r} + \frac{q_{n-1}}{q}} \geq \frac{1}{\frac{1}{4} + \frac{1}{b_n + \frac{q_{n-2}}{q_{n-1}}}} \geq \frac{1}{\frac{1}{4} + \frac{1}{2}} = \frac{4}{3} \geq \frac{1}{1 - \frac{1}{2q}}.$$

The equality occurs in the third exception.

(B) Assume that $n \geq 2$ and $a_n = b_n + 3$. Therefore, we have

$$\frac{1}{\frac{1}{a_n - b_n + r} + \frac{q_{n-1}}{q}} = \frac{1}{\frac{1}{3+r} + \frac{q_{n-1}}{q}} \geq \frac{1}{\frac{1}{3} + \frac{q_{n-1}}{q}}.$$

So to prove (4.12), it is enough to prove

$$\frac{1}{\frac{1}{3} + \frac{q_{n-1}}{q}} > \frac{1}{1 - \frac{1}{2q}},$$

which is equivalent to

$$\frac{4}{3} > \frac{1 + 2q_{n-1}}{q}.$$

This inequality is true obviously since

$$q = b_n q_{n-1} + q_{n-2} \geq 2q_{n-1} + 1.$$

(C) Suppose that $n = 1$ and $a_1 = b_1 + 3$. Then, we have $q = b_n q_{n-1} + q_{n-2} = b_1$. From this we obtain that

$$\frac{1}{\frac{1}{3+r} + \frac{q_{n-1}}{q}} = \frac{1}{\frac{1}{3+r} + \frac{1}{b_1}}.$$

So, to prove (4.12) it is enough to prove

$$\frac{1}{\frac{1}{3+r} + \frac{1}{b_1}} > \frac{1}{1 - \frac{1}{2q}}.$$

• Let $b_1 \geq 3$. Then, $q \geq 3$ and $\frac{1}{1-\frac{1}{2q}} \leq \frac{6}{5}$. Hence,

$$\frac{1}{\frac{1}{3+r} + \frac{1}{b_1}} \geq \frac{1}{\frac{1}{3} + \frac{1}{3}} = \frac{3}{2} > \frac{6}{5} \geq \frac{1}{1 - \frac{1}{2q}}.$$

• Suppose that $b_1 = 2$. Then, $q = 2$ and $\frac{1}{1-\frac{1}{2q}} = \frac{4}{3}$. It yields

$$\frac{1}{\frac{1}{3+r} + \frac{1}{b_1}} = \frac{1}{\frac{1}{3+r} + \frac{1}{2}}.$$

★ Assume $a_2$ does not exist. Then, $r = 0$. Hence,

$$\frac{1}{\frac{1}{3+r}+\frac{1}{2}} = \frac{1}{\frac{1}{3}+\frac{1}{2}} = \frac{6}{5} < \frac{4}{3} = \frac{1}{1-\frac{1}{2q}}.$$

In this case we have $\alpha = [a_0; 5]$ and $p/q = [a_0; 2]$. This is the fourth exception when $a_2$ does not exist.

★ Suppose that $a_2 \geq 1$. Then, we have

$$\frac{1}{\frac{1}{3+r}+\frac{1}{2}} < \frac{1}{\frac{1}{3+1}+\frac{1}{2}} = \frac{4}{3} = \frac{1}{1-\frac{1}{2q}}.$$

This is the fourth exception.

(D) Let $n \geq 2$ and $a_n = b_n + 2$. From this we obtain that

$$\frac{1}{\frac{1}{a_n-b_n+r}+\frac{q_{n-1}}{q}} \geq \frac{1}{\frac{1}{2}+\frac{q_{n-1}}{q}}.$$

So to prove (4.12) it is enough to prove that

$$\frac{1}{\frac{1}{2}+\frac{q_{n-1}}{q}} \geq \frac{1}{1-\frac{1}{2q}}$$

which can be written as

$$1 \geq \frac{1+2q_{n-1}}{q}.$$

This is obviously true since $q = b_n q_{n-1} + q_{n-2} \geq 2q_{n-1} + 1$.

The equality occurs in the fifth exception.

(E) Assume that $n = 1$ and $a_n = b_n + 2$. Hence, $q = b_n q_{n-1} + q_{n-2} = b_1$. From this we obtain that

$$\frac{1}{\frac{1}{a_n-b_n+r}+\frac{q_{n-1}}{q}} = \frac{1}{\frac{1}{2+r}+\frac{1}{b_1}}.$$

• Let $b_1 \geq 3$. Then, $q = b_1 \geq 3$ and $\frac{1}{1-\frac{1}{2q}} \leq \frac{6}{5}$. It yields

$$\frac{1}{\frac{1}{2+r}+\frac{1}{b_1}} \geq \frac{1}{\frac{1}{2}+\frac{1}{3}} = \frac{6}{5} = \frac{1}{1-\frac{1}{2q}}.$$

The equality occurs in the sixth exception.

• Suppose that $b_1 = 2$. Then, $q = b_1 = 2$ and $\frac{1}{1-\frac{1}{2q}} = \frac{4}{3}$. Hence,

$$\frac{1}{\frac{1}{2+r}+\frac{1}{b_1}} = \frac{1}{\frac{1}{2+r}+\frac{1}{2}}.$$

★ Assume that $a_2$ does not exist. Then, $r = 0$ and we obtain that

$$\frac{1}{\frac{1}{2+r}+\frac{1}{2}} = \frac{1}{\frac{1}{2}+\frac{1}{2}} = 1 < \frac{4}{3} = \frac{1}{1-\frac{1}{2q}}.$$

This is the seventh exception when $a_2$ does not exist.
★ Let $a_2 \geq 1$. It implies that

$$\frac{1}{\frac{1}{2+r}+\frac{1}{2}} < \frac{1}{\frac{1}{2+1}+\frac{1}{2}} = \frac{6}{5} < \frac{4}{3} = \frac{1}{1-\frac{1}{2q}}.$$

This is the seventh exception.

b. Suppose that $b_n > a_n$ then (4.10) has the form

$$\frac{c}{\frac{1}{c(b_n-a_n-r)+d} - \frac{q_{n-1}}{q}} > \frac{1}{1-\frac{1}{2q}}. \tag{4.13}$$

(i) Suppose that $c \geq 2$, $d \geq 1$ and $b_n \geq a_n + 1$. It yields

$$\frac{c}{\frac{1}{c(b_n-a_n-r)+d} - \frac{q_{n-1}}{q}} > \frac{c}{\frac{1}{d} - \frac{q_{n-1}}{q}} \geq \frac{2}{1-\frac{1}{q}} > \frac{1}{1-\frac{1}{2q}},$$

which is obviously true and inequality (4.13) follows.

(ii) Let $c = 1$, $d = 0$ and $b_n \geq a_n + 2$. From this we obtain that

$$\frac{1}{\frac{1}{b_n-a_n-r} - \frac{q_{n-1}}{q}} > \frac{1}{\frac{1}{2-1} - \frac{q_{n-1}}{q}} \geq \frac{1}{1-\frac{1}{q}} > \frac{1}{1-\frac{1}{2q}},$$

which is obviously true and inequality (4.13) follows.

The proof of Theorem 3.4 is complete. □

*Proof of Example* 3.1. We have

$$\left|\alpha - \frac{p}{q}\right| = \sum_{n=1}^{+\infty} \frac{1}{2^{2^n-1}A^{2^n}} - \sum_{n=1}^{N} \frac{1}{2^{2^n-1}A^{2^n}} = \sum_{n=N+1}^{+\infty} \frac{1}{2^{2^n-1}A^{2^n}}.$$

At the same time

$$\begin{aligned} \frac{1}{2q^2} &= \frac{1}{2\left(2^{2^N-1}A^{2^N}\right)^2} = \frac{1}{2.2^{2^{N+1}-2}A^{2^{N+1}}} \\ &= \frac{1}{2^{2^{N+1}-1}A^{2^{N+1}}} < \sum_{n=N+1}^{+\infty} \frac{1}{2^{2^n-1}A^{2^n}} = \left|\alpha - \frac{p}{q}\right|. \end{aligned}$$

Hence, we cannot use Legendre's theorem. On the other side we have

$$\begin{aligned} \frac{1}{q^2\left(2-\frac{1}{q}\right)} &= \frac{1}{\left(2^{2^N-1}A^{2^N}\right)^2\left(2-\frac{1}{2^{2^N-1}A^{2^N}}\right)} = \frac{1}{\left(2^{2^N-1}A^{2^N}\right)\left(2^{2^N}A^{2^N}-1\right)} \\ &= \frac{1}{2^{2^{N+1}-1}A^{2^{N+1}}} \cdot \sum_{n=0}^{+\infty} \frac{1}{2^{n2^N}A^{n2^N}} = \sum_{n=0}^{+\infty} \frac{1}{2^{(n+2)2^N-1}A^{(n+2)2^N}} \\ &> \sum_{n=N+1}^{+\infty} \frac{1}{2^{2^n-1}A^{2^n}} = \left|\alpha - \frac{p}{q}\right|. \end{aligned}$$

Thus, from Theorem 3.2 we obtain that $p/q$ is a convergent of $\alpha$. □

*Proof of Example* 3.2. Example 3.2 is an immediate consequence of Example 3.1 when we set $A = 1$. $\square$

*Proof of Example* 3.3. We have

$$\left|\alpha - \frac{p}{q}\right| = \sum_{n=1}^{+\infty} \frac{1}{2^{2^n} A^{2^n}} - \sum_{n=1}^{N} \frac{1}{2^{2^n} A^{2^n}} = \sum_{n=N+1}^{+\infty} \frac{1}{2^{2^n} A^{2^n}}.$$

At the same time

$$\frac{1}{q^2} = \frac{1}{2^{2.2^N} A^{2.2^N}} < \left|\alpha - \frac{p}{q}\right|.$$

Hence, we cannot use Barbolosi and Jager's theorem. On the other side we have

$$\frac{1}{\left(1 - \frac{1}{2q}\right) q^2} = \frac{1}{2^{2.2^N} A^{2.2^N}} \sum_{n=0}^{+\infty} \frac{1}{\left(2.2^{2^N} A^{2^N}\right)^n}$$

$$= \sum_{n=0}^{+\infty} \frac{1}{\left(2^{(n+2)2^N + n} A^{(n+2)2^N}\right)^n} > \left|\alpha - \frac{p}{q}\right|.$$

Therefore, from Theorem 3.4 we obtain that $p/q$ is a convergent or nearest mediant of $\alpha$.

$\square$

**Acknowledgements.** Tho Phuoc Nguyen is supported by grant SGS07/PřF/2025. The authors thank to unknown referee for valuable suggestions.

## References

[1] D. Barbolosi, *Fractions Continues à Quotients Partiels Impairs*, Thèse, Université de Provence, Marseille, 1988.

[2] D. Barbolosi and H. Jager, *On a theorem of Legendre in the theory of continued fractions*, Journal de Théories des Nombres de Bordeaux **6**(1) (1994), 81–94. https://doi.org/10.5802/jtnb.106

[3] S. Bahnerová and J. Hančl, *Sharpening of the theorem of Vahlen and related theorems*, J. Ramanujan Math. Soc. **36**(2) (2021), 109–121.

[4] P. Billingsley, *Ergodic Theory and Information*, John Wiley and Sons, New York, London, Sydney, 1965.

[5] É. Borel, *Contribution à l'analyse arithmétique du continu*, J. Math. Pures Appl. **9**(5) (1903), 329–375.

[6] J. Borwein, P. Borwein, Pi and the AGM, *A Study in Analytic Number Theory and Computational Complexity*, John Wiley & Sons, New York, 1987.

[7] L. G. P. Dirichlet, *Verallgemeinerung eines satzes aus der lehre von den kettenbruchen nebst einige anwendungen auf die theorie der zahlen*, S.-B. Preuss. Akad. Wiss. (1842), 93–95. https://doi.org/10.1017/CBO9781139237338.037

[8] N. I. Fel'dman and Y. V. Nesterenko, *Transcendental Numbers*, Springer-Verlag, New York, 1998.

[9] J. H. Grace, *The classification of rational approximations*, Proc. London Math. Soc. **17** (1918), 247–258. https://doi.org/10.1112/plms/s2-17.1.247

[10] J. Hančl, *Sharpening of theorems of Vahlen and Hurwitz and approximation properties of the golden ratio*, Arch. Math. (Basel) **105**(2) (2015), 129–137. https://doi.org/10.1007/s00013-015-0788-8

[11] J. Hančl, *Second basic theorem of Hurwitz*, Lith. Math. J. **56**(1) (2016), 72–76. https://doi.org/10.1007/s10986-016-9305-4
[12] J. Hančl, *On a theorem of A. A. Markoff*, Results Math. **76**(4) (2021), Article ID 192. https://doi.org/10.1007/s00025-021-01501-7
[13] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, Oxford, 2008.
[14] D. Hensley, *Continued Fractions*, Word Scientific Publishing, 2006.
[15] A. Hurwitz, *Über die angenäherte darstellung der irrationalzahlen durch rationale brüche*, Math. Ann. **2** (1891), 279–284. https://doi.org/10.1007/BF01206656
[16] S. Ito, *On Legendre's theorem related to Diophantine approximations*, Séminaire de Théorie des Nombres, Bordeaux **44** (1987–1988).
[17] S. Ito, *Algorithms with mediant convergents and their metrical theory*, Osaka J. Math. (1989), 557–578.
[18] S. Ito and H. Nakada, *On natural extensions of transformations related to Diophantine approximations*, Number Theory and Combinatorics (1985), 185–207.
[19] H. Jager and C. Kraaikamp, *On the approximation by continued fractions*, Indag. Math. **51** (1989), 289–307. https://doi.org/10.1016/S1385-7258(89)80004-6
[20] W. B. Jones and W. J. Thron, *Continued Fractions Analytic Theory and Applications*, Cambridge University Press, Cambridge, 1984.
[21] O. Karpenkov, *Geometry of Continued Fractions*, Springer, Heidelberg, 2013.
[22] A. Ya. Khinchin, *Continued Fractions*, The University of Chicago Press, Chicago, 1964.
[23] J. F. Koksma, *Bewijs van een stelling over kettingbreuken*, Mathematica A **6** (1937), 226–233.
[24] J. F. Koksma, *On continued fractions*, Bull. Belg. Math. Soc. Simon Stevin **29** (1951/52), 96–102.
[25] J. F. Koksma, *Diophantische Approximationen*, Julius Springer, Berlin, 1936.
[26] C. Kraaikamp, *A new class of continued fractions*, Acta Arith. **57** (1991), 1–39. https://doi.org/10.4064/aa-57-1-1-39
[27] A. M. Legendre, *Théorie des Nombres*, Paris, 1830.
[28] H. Nakada, *Metrical theory for a class of continued fractions transformations and their natural extensions*, Tokyo J. Math. **4** (1981), 399–426. https://doi.org/10.3836/tjm/1270215165
[29] K. H. Rosen, *Elementary Number Theory and its Applications*, Addison Wesley, 2005.
[30] W. Schmidt, *Diophantine Approximation*, Springer, Berlin, 1980.
[31] K. Th. Vahlen, *Über näherungswerte und kettenbrüche*, J. Reine Angew. Math. **115** (1895), 221–233.
[32] H. S. Wall, *Analytic Theory of Continued Fractions*, New York, Chelsea, 1948.

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