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TRIANGULAR SYSTEM OF HIGHER ORDER SINGULAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we introduce a high dimensional system of singular fractional differential equations. Using Schauder fixed point theorem, we prove an existence result. We also investigate the uniqueness of solution using the Banach contraction principle. Moreover, we study the Ulam-Hyers stability and the generalized-Ulam-Hyers stability of solutions. Some illustrative examples are also presented.

1. Introduction and Preliminaries

Recently, the fractional calculus has attracted the attention of researchers in various fields of applied sciences. For details, see [12,16,19,20] and the references therein. It is important to note that some research studies deal with the existence and uniqueness of solutions for some fractional differential equations are obtained in [1,6–9]. Other studies in [2,3,5,17,23] have been done for the singular fractional differential equations. On the other hand, the Ulam stability of fractional differential equations is quite significant in more realistic problems, numerical analysis, biology and economics. Considerable work has been done in this area, for instance, see [10,11,13–15,18,22,24].

Let us now present some important research papers that inspired our work: We begin by [4], where C. Bai and J. Fang established the existence of solutions for the following singular fractional coupled system:

$$\begin{cases} D^{\delta} u\left(t\right) = f\left(t, v\left(t\right)\right), \quad 0<t<1, \\ D^{\rho} v\left(t\right) = g\left(t, u\left(t\right)\right), \quad 0<t<1, \end{cases}$$

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where $0 < \delta,\ \rho < 1$, D^{δ}, D^{ρ} are two standard Riemann-Liouville fractional derivatives, $f,\ g : [0,1) \times [0,\infty) \to [0,\infty)$ are two given continuous functions, $\lim\limits_{t\to 0^+} f(t) = \infty$ and $\lim_{t\to 0^+} g(t) = \infty$.

In [25], A. Yang and W. Ge considered the following fractional coupled system

$$\begin{cases} D^{\alpha_1} u_1(t) + f_1(t, u_2(t), D^{\mu_1} u_2(t)) = 0, \\ \vdots \\ D^{\alpha_{n-1}} u_{n-1}(t) + f_{n-1}(t, u_n(t), D^{\mu_{n-1}} u_n(t)) = 0, \\ D^{\alpha_n} u_n(t) + f_n(t, u_1(t), D^{\mu_n} u_1(t)) = 0, \end{cases}$$

associated with the boundary conditions

$$\begin{cases} u_1(0) = u_2(0) = \cdots = u_n(0) = 0, \\ u_1(1) = u_2(1) = \cdots = u_n(1) = 0, \end{cases}$$

for $1 < \alpha_j < 2$, $\mu_j > 0$, $\alpha_j - \mu_{j-1} > 1$, $j = 1, 2, \ldots, n$, $\mu_0 = \mu_n$ and $f_j : [0,1] \times [0,\infty) \times \mathbb{R} \to [0,\infty)$ is continuous function. Some existence and multiplicity results of solutions are obtained.

In [21], A. Taïeb and Z. Dahmani established new existence and uniqueness results for the following problem:

$$\begin{cases} D^{\alpha} u(t) + \sum\limits_{i=1}^{m} f_i(t, u(t), v(t), D^{\gamma} u(t), D^{\rho} v(t)) = 0, \quad t \in J, \\ D^{\beta} v(t) + \sum\limits_{i=1}^{m} g_i(t, u(t), v(t), D^{\gamma} u(t), D^{\rho} v(t)) = 0, \quad t \in J, \\ u(0) = u_0^*,\ v(0) = v_0^*, \\ u'(0) = u''(0) = v'(0) = v''(0) = 0, \\ u'''(0) = J^r u(\tau),\ v'''(0) = J^{\varphi} v(\varsigma), \quad r > 0, \varphi > 0, \end{cases}$$

where $\alpha, \beta \in (3,4)$, $\gamma, \rho \in (0,3)$, $\tau, \varsigma \in (0,1)$, D^{α}, D^{ρ}, D^{β} and D^{γ} denote the Caputo fractional derivatives and J^r, J^{φ} denote the Riemann-Liouville fractional integrals, $J := [0,1]$, $u_0^*, v_0^* \in \mathbb{R}$. For each $i = 1, \ldots, m$, f_i and $g_i : J \times \mathbb{R}^4 \to \mathbb{R}$ are specific functions.

In this paper, we discuss the existence, uniqueness and Ulam stability of solutions for the following singular fractional coupled system:

$$\begin{cases} D^{\alpha_1} x_1(t) = f_1(t, x_1(t)), \\ D^{\alpha_2} x_2(t) = f_2(t, x_1(t), x_2(t)), \\ \vdots \\ D^{\alpha_n} x_n(t) = f_n(t, x_1(t), x_2(t) \ldots, x_n(t)), \\ 0 < t \le 1, \quad k-1 < \alpha_k < k, \quad k = 1, 2, \ldots, n, \\ x_1(0) = a_0^1, \quad k = 1, \\ x_k^{(j)}(0) = a_j^k, \quad j = 0, 1, \ldots, k-2, \quad k = 2, 3, \ldots, n, \\ D^{\delta_{k-1}} x_k(1) = 0, \quad k-2 < \delta_{k-1} < k-1, \quad k = 2, 3, \ldots, n, \end{cases} \tag{1.1}$$

where $n \in \mathbb{N} - \{0,1\}$. For all $k = 1,2,\ldots,n$, the functions $f_k : (0,1] \times \mathbb{R}^k \to \mathbb{R}$ are continuous, singular at $t = 0$, $\lim_{t\to 0^+} f_k(t) = \infty$ and there exist $\beta_k \in (0,1)$, $k = 1,2,\ldots,n$, such that $t^{\beta_k} f_k$, $k = 1,2,\ldots,n$, are continuous on $[0,1]$.

To the best of our knowledge, there are no papers that have considered this kind of singular fractional coupled system.

We present some basic definitions and lemmas that we need to prove our main results. It can be found in [16].

Definition 1.1. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ for a continuous function f on $[0,\infty)$ is defined as:

$$J^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, & \alpha > 0, \\ f(t), & \alpha = 0, \end{cases}$$

where $t \geq 0$ and $\Gamma(\alpha) := \int_0^{+\infty} e^{-u} u^{\alpha-1} du$.

Definition 1.2. The Caputo derivative of order α for a function $x : [0,+\infty) \to \mathbb{R}$, which is at least k-times differentiable can be defined as the following:

$$D^{\alpha} x(t) = \frac{1}{\Gamma(k-\alpha)} \int_0^t (t-s)^{k-\alpha-1} x^{(k)}(s)\, ds = J^{k-\alpha} x^{(k)}(t),$$

for $k-1 < \alpha < k$, $k \in \mathbb{N} - \{0\}$.

Lemma 1.1. *Let $\alpha, \beta > 0$, and $k-1 < \alpha < k$, $k \in \mathbb{N} - \{0\}$, and let j be a positive integer. Then*

$$D^{\alpha} t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-\alpha-1}, \quad \beta > k,$$

and

$$D^{\alpha} t^{j} = 0, \quad j = 0,1,\ldots,k-1.$$

Lemma 1.2. *Let $q > p > 0$ and $f \in L^1([a,b])$. Then for all $t \in [a,b]$, we have*

$$D^{p} J^{q} f(t) = J^{q-p} f(t), \quad t \in [a,b].$$

Lemma 1.3. *Let $k-1 < \alpha < k$, $k \in \mathbb{N} - \{0\}$, and let j be a positive integer. Then, the general solution of the fractional differential equation $D^{\alpha}x(t) = 0$, is given by:*

$$x(t) = \sum_{j=0}^{k-1} c_j\, t^{j}, \quad (c_j)_{j=0,1,\ldots,k-1} \in \mathbb{R}.$$

Lemma 1.4. *Let $k \in \mathbb{N} - \{0\}$, $k-1 < \alpha < k$, and let j be a positive integer. Then,*

$$J^{\alpha} D^{\alpha} x(t) = x(t) + \sum_{j=0}^{k-1} c_j\, t^{j}, \quad (c_j)_{j=0,1,\ldots,k-1} \in \mathbb{R}.$$

Lemma 1.5 (Shauder fixed point theorem)**.** *Let (E,d) be a complete metric space, let U be a closed convex subset of E, and let $T : E \to E$ be a mapping such that the set $V := \{Tx : x \in U\}$ is relatively compact in E. Then T has at least one fixed point.*

We also prove the following auxiliary result to give the integral representation of (1.1).

Lemma 1.6. *Assume that* $k-1<\alpha_k<k,\ k=1,2,\ldots,n,\ n\in\mathbb{N}-\{0,1\}$ *and* $F_k\in C([0,1],\mathbb{R})$. *Then, the following system*

$$\begin{cases} D^{\alpha_1}x_1(t)=F_1(t),\\ D^{\alpha_2}x_2(t)=F_2(t),\\ \vdots\\ D^{\alpha_n}x_n(t)=F_n(t),\end{cases}$$

associated with the conditions:

$$\begin{cases} x_1(0)=a_0^1,\\ x_k^{(j)}(0)=a_j^k,\quad k=2,3,\ldots,n, j=0,1,\ldots,k-2,\\ D^{\delta_{k-1}}x_k(1)=0,\quad k-2<\delta_{k-1}<k-1,\end{cases}\tag{1.2}$$

has a unique solution $(x_1,x_2,\ldots,x_n)$, *where*

$$x_k(t)=\begin{cases}\displaystyle\int_0^t\frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}F_1(s)\,ds+a_0^1,\quad k=1,\\[2ex] \displaystyle\int_0^t\frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)}F_k(s)\,ds+\sum_{j=0}^{k-2}\frac{a_j^k}{j!}t^j\\ \displaystyle-\frac{\Gamma(k-\delta_{k-1})}{(k-1)!}t^{k-1}\int_0^1\frac{(1-s)^{\alpha_k-\delta_{k-1}-1}}{\Gamma(\alpha_k-\delta_{k-1})}F_k(s)\,ds,\quad k=2,3,\ldots,n.\end{cases}\tag{1.3}$$

Proof. Using Lemma 1.4, we obtain the following integral equation:

$$x_k(t)=\int_0^t\frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)}F_k(s)\,ds-\sum_{j=0}^{k-1}c_j^kt^j,\quad k=1,2,\ldots,n,\tag{1.4}$$

where

$$\begin{pmatrix} c_0^1 & 0 & \ldots & \ldots & \ldots & 0\\ c_0^2 & c_1^2 & 0 & \ldots & \ldots & 0\\ c_0^3 & c_1^3 & c_2^3 & 0 & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & 0 & 0\\ c_0^{n-1} & c_1^{n-1} & c_2^{n-1} & \ldots & c_{n-2}^{n-1} & 0\\ c_0^n & c_1^n & c_2^n & \ldots & c_{n-2}^n & c_{n-1}^n\end{pmatrix}\in M_n(\mathbb{R}).$$

Applying the conditions given in (1.2), we observe that

$$x_1(0)=-c_0^1=a_0^1,$$

and for all $k=2,3,\ldots,n$, we get

$$\begin{cases} x_k^{(j)}(0)=-j!c_j^k=a_j^k,\quad j=0,1,\ldots,k-2,\\ D^{\delta_{k-1}}x_k(1)=\displaystyle\int_0^1\frac{(1-s)^{\alpha_k-\delta_{k-1}-1}}{\Gamma(\alpha_k-\delta_{k-1})}F_k(s)\,ds-\frac{\Gamma(k)}{\Gamma(k-\delta_{k-1})}c_{k-1}^k=0,\\ k-2<\delta_{k-1}<k-1,\end{cases}$$

which implies that

$$c_0^1 = -a_0^1, \tag{1.5}$$

and

$$c_j^k = \begin{cases} -\dfrac{a_j^k}{j!}, \quad j = 0, 1, \dots, k-2, \\ \dfrac{\Gamma(k-\delta_{k-1})}{\Gamma(k)} \displaystyle\int_0^1 \dfrac{(1-s)^{\alpha_k-\delta_{k-1}-1}}{\Gamma(\alpha_k-\delta_{k-1})} F_k(s)\, ds, \quad j = k-1, \end{cases} \tag{1.6}$$

where $k = 2, 3, \dots, n$.

Substituting (1.5) and (1.6) in (1.4), we find (1.3). The proof of Lemma 1.6 is thus achieved. □

Now, we introduce the Banach space

$$S := \{(x_1, x_2, \dots, x_n) : x_k \in C([0,1], \mathbb{R}), k = 1, 2, \dots, n\},$$

endowed with the norm:

$$\|(x_1, x_2, \dots, x_n)\|_S = \max_{1\le k\le n} \|x_k\|_\infty, \quad \|x_k\|_\infty = \max_{t\in[0,1]} |x_k(t)|.$$

2. Existence and Uniqueness

In this section, we try to establish sufficient conditions for the existence and uniqueness of solutions to the problem (1.1).

Define the nonlinear operator $A : S \to S$ by

$$A(x_1, x_2, \dots, x_n)(t) := (A_1(x_1)(t), A_2(x_1, x_2)(t), \dots, A_n(x_1, x_2, \dots, x_n)(t)),$$

such that, for all $t \in [0,1]$,

$$A_k(x_1, \dots, x_k)(t) := \begin{cases} \displaystyle\int_0^t \dfrac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} f_1(s)\, ds + a_0^1, \quad k = 1, \\ \displaystyle\int_0^t \dfrac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} f_k(s, \dots)\, ds + \sum_{j=0}^{k-2} \dfrac{a_j^k}{j!} t^j - \dfrac{\Gamma(k-\delta_{k-1})}{(k-1)!} t^{k-1} \\ \times \displaystyle\int_0^1 \dfrac{(1-s)^{\alpha_k-\delta_{k-1}-1}}{\Gamma(\alpha_k-\delta_{k-1})} f_k(s, \dots)\, ds, \quad k = 2, 3, \dots, n. \end{cases}$$

Lemma 2.1. *Let $k-1 < \alpha_k < k$, $k = 1, 2, \dots, n$, $n \in \mathbb{N} - \{0, 1\}$, $0 < \beta_k < 1$, $T_k : (0,1] \to \mathbb{R}$ be continuous function and $\lim_{t\to 0^+} T_k(t) = \infty$. Assume that $t^{\beta_k} T_k(t)$*

is continuous on $[0,1]$. *Then*

$$x_k(t)=\begin{cases}\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}T_1(s)\,ds+a_0^1, \quad k=1,\\ \int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)}T_k(s)\,ds+\sum_{j=0}^{k-2}\frac{a_j^k}{j!}t^j-\frac{\Gamma(k-\delta_{k-1})}{(k-1)!}t^{k-1}\\ \times\int_0^1 \frac{(1-s)^{\alpha_k-\delta_{k-1}-1}}{\Gamma(\alpha_k-\delta_{k-1})}T_k(s)\,ds, \quad k=2,3,\dots,n,\end{cases}$$

is continuous on $[0,1]$.

Proof. By the continuity of $t^{\beta_k}T_k$ and

$$x_k(t)=\begin{cases}\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}s^{-\beta_1}s^{\beta_1}T_1(s)\,ds+a_0^1, \quad k=1,\\ \int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)}s^{-\beta_k}s^{\beta_k}T_k(s)\,ds+\sum_{j=0}^{k-2}\frac{a_j^k}{j!}t^j-\frac{\Gamma(k-\delta_{k-1})}{(k-1)!}t^{k-1}\\ \times\int_0^1 \frac{(1-s)^{\alpha_k-\delta_{k-1}-1}}{\Gamma(\alpha_k-\delta_{k-1})}s^{-\beta_k}s^{\beta_k}T_k(s)\,ds, \quad k=2,3,\dots,n,\end{cases}$$

we get $x_k(0)=a_0^k$, $k=1,2,\dots,n$. Then, we will divide the proof into three cases.

Case 1. For $t_0=0$ and for all $t\in(0,1]$, by the continuity of $t^{\beta_k}T_k$, there exist $M_1,\dots,M_n>0$, such that for all $t\in[0,1]$, $\left|t^{\beta_k}T_k(t)\right|\le M_k$. Therefore, we get

$$\begin{aligned}&|x_k(t)-x_k(0)|\\ &=\begin{cases}\left|\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}s^{-\beta_1}s^{\beta_1}T_1(s)\,ds\right|, \quad k=1,\\ \left|\int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)}s^{-\beta_k}s^{\beta_k}T_k(s)\,ds+\sum_{j=1}^{k-2}\frac{a_j^k}{j!}t^j\right.\\ \left.-\frac{\Gamma(k-\delta_{k-1})}{(k-1)!}t^{k-1}\int_0^1 \frac{(1-s)^{\alpha_k-\delta_{k-1}-1}}{\Gamma(\alpha_k-\delta_{k-1})}s^{-\beta_k}s^{\beta_k}T_k(s)\,ds\right|, \quad k=2,3,\dots,n,\end{cases}\\ &\le\begin{cases}\frac{M_1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}s^{-\beta_1}ds, \quad k=1,\\ \frac{M_k}{\Gamma(\alpha_k)}\int_0^t (t-s)^{\alpha_k-1}s^{-\beta_k}ds+\sum_{j=1}^{k-2}\frac{\left|a_j^k\right|}{j!}t^j+\frac{\Gamma(k-\delta_{k-1})M_k}{(k-1)!\Gamma(\alpha_k-\delta_{k-1})}t^{k-1}\\ \times\int_0^1 (1-s)^{\alpha_k-\delta_{k-1}-1}s^{-\beta_k}ds, \quad k=2,3,\dots,n.\end{cases}\end{aligned}$$

Using Beta Euler function denoted by B, we obtain

$$
\begin{aligned}
&\left|x_k(t)-x_k(0)\right| \\
&\leq \begin{cases}
\dfrac{M_1 t^{\alpha_1-\beta_1}}{\Gamma(\alpha_1)}\displaystyle\int_0^1 (1-u)^{\alpha_1-1} u^{-\beta_1} du, \quad k=1, \\
\dfrac{M_k t^{\alpha_k-\beta_k}}{\Gamma(\alpha_k)}\displaystyle\int_0^1 (1-u)^{\alpha_k-1} u^{-\beta_k} du + \sum_{j=1}^{k-2} \dfrac{\left|a_j^k\right|}{j!} t^j \\
+\dfrac{\Gamma(k-\delta_{k-1}) M_k B(\alpha_k-\delta_{k-1}, 1-\beta_k)}{(k-1)!\Gamma(\alpha_k-\delta_{k-1})} t^{k-1}, \quad k=2,3,\ldots,n,
\end{cases} \\
&\leq \begin{cases}
\dfrac{M_1 B(\alpha_1, 1-\beta_1) t^{\alpha_1-\beta_1}}{\Gamma(\alpha_1)}, \quad k=1, \\
\dfrac{M_k B(\alpha_k, 1-\beta_k) t^{\alpha_k-\beta_k}}{\Gamma(\alpha_k)} + \sum_{j=1}^{k-2} \dfrac{\left|a_j^k\right|}{j!} t^j \\
+\dfrac{\Gamma(k-\delta_{k-1}) M_k B(\alpha_k-\delta_{k-1}, 1-\beta_k)}{(k-1)!\Gamma(\alpha_k-\delta_{k-1})} t^{k-1}, \quad k=2,3,\ldots,n,
\end{cases} \\
&\to 0 \ \text{ as } t \to 0, \quad k=1,2,\ldots,n.
\end{aligned}
$$

Case 2. For $t_0 \in (0,1)$ and for all $t \in (t_0, 1]$, we have

$$
\begin{aligned}
&\left|x_k(t)-x_k(t_0)\right| \\
&\leq \begin{cases}
\left|\displaystyle\int_0^t \dfrac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} s^{-\beta_1} s^{\beta_1} T_1(s) ds - \int_0^{t_0} \dfrac{(t_0-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} s^{-\beta_1} s^{\beta_1} T_1(s) ds\right|, \quad k=1, \\
\left|\displaystyle\int_0^t \dfrac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} s^{-\beta_k} s^{\beta_k} T_k(s) ds - \int_0^{t_0} \dfrac{(t_0-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} s^{-\beta_k} s^{\beta_k} T_k(s) ds\right| \\
+\displaystyle\sum_{j=0}^{k-2} \dfrac{\left|a_j^k\right|}{j!}\left(t^j-t_0^j\right) + \dfrac{\Gamma(k-\delta_{k-1})}{(k-1)!}\left(t^{k-1}-t_0^{k-1}\right) \\
\times\left|\displaystyle\int_0^1 \dfrac{(1-s)^{\alpha_k-\delta_{k-1}-1}}{\Gamma(\alpha_k-\delta_{k-1})} s^{-\beta_k} s^{\beta_k} T_k(s) ds\right|, \quad k=2,3,\ldots,n,
\end{cases} \\
&\leq \begin{cases}
\dfrac{M_1}{\Gamma(\alpha_1)}\left(\displaystyle\int_0^t (t-s)^{\alpha_1-1} s^{-\beta_1} ds - \int_0^{t_0} (t_0-s)^{\alpha_1-1} s^{-\beta_1} ds\right), \quad k=1, \\
\dfrac{M_k}{\Gamma(\alpha_k)}\left(\displaystyle\int_0^t (t-s)^{\alpha_k-1} s^{-\beta_k} ds - \int_0^{t_0} (t_0-s)^{\alpha_k-1} s^{-\beta_k} ds\right) \\
+\displaystyle\sum_{j=1}^{k-2} \dfrac{\left|a_j^k\right|}{j!}\left(t^j-t_0^j\right) + \dfrac{\Gamma(k-\delta_{k-1}) M_k}{(k-1)!\Gamma(\alpha_k-\delta_{k-1})}\left(t^{k-1}-t_0^{k-1}\right) \\
\times\displaystyle\int_0^1 (1-s)^{\alpha_k-\delta_{k-1}-1} s^{-\beta_k} ds, \quad k=2,3,\ldots,n.
\end{cases}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left|x_k(t)-x_k(t_0)\right| \\
&\leq \begin{cases}
\dfrac{M_1\left(t^{\alpha_1-\beta_1}-t_0^{\alpha_1-\beta_1}\right)B\left(\alpha_1,1-\beta_1\right)}{\Gamma\left(\alpha_1\right)}, \quad k=1,\\
\dfrac{M_k\left(t^{\alpha_k-\beta_k}-t_0^{\alpha_k-\beta_k}\right)B\left(\alpha_k,1-\beta_k\right)}{\Gamma\left(\alpha_k\right)}+\sum_{j=1}^{k-2}\dfrac{\left|a_j^k\right|}{j!}\left(t^j-t_0^j\right)\\
+\dfrac{\Gamma\left(k-\delta_{k-1}\right)M_kB\left(\alpha_k-\delta_{k-1},1-\beta_k\right)}{(k-1)!\Gamma\left(\alpha_k-\delta_{k-1}\right)}\left(t^{k-1}-t_0^{k-1}\right), \quad k=2,3,\ldots,n,
\end{cases}\\
&\to 0, \text{ as } t\to t_0, \quad k=1,2,\ldots,n.
\end{aligned}
$$

Case 3. For t_0 $(0,1]$ and for all $t\in[0,t_0)$. Similarly, as in Case 2, it can be shown that

$$
\begin{aligned}
&\left|x_k(t)-x_k(t_0)\right| \\
&\leq \begin{cases}
\dfrac{M_1\left(t_0^{\alpha_1-\beta_1}-t^{\alpha_1-\beta_1}\right)B\left(\alpha_1,1-\beta_1\right)}{\Gamma\left(\alpha_1\right)}, \quad k=1,\\
\dfrac{M_k\left(t_0^{\alpha_k-\beta_k}-t^{\alpha_k-\beta_k}\right)B\left(\alpha_k,1-\beta_k\right)}{\Gamma\left(\alpha_k\right)}+\sum_{j=1}^{k-2}\dfrac{\left|a_j^k\right|}{j!}\left(t_0^j-t^j\right)\\
+\dfrac{\Gamma\left(k-\delta_{k-1}\right)M_kB\left(\alpha_k-\delta_{k-1},1-\beta_k\right)}{(k-1)!\Gamma\left(\alpha_k-\delta_{k-1}\right)}\left(t_0^{k-1}-t^{k-1}\right), \quad k=2,3,\ldots,n,
\end{cases}\\
&\to 0, \text{ as } t\to t_0, \quad k=1,2,\ldots,n.
\end{aligned}
$$

This ends the proof. □

Lemma 2.2. *Let $k-1<\alpha_k<k$, $k=1,2,\ldots,n$, $n\in\mathbb{N}-\{0,1\}$, $0<\beta_k<1$, $f_k:(0,1]\times\mathbb{R}^k\to\mathbb{R}$ be continuous, and $\lim_{t\to0^+}f_k(t,\ldots)=\infty$. Assume that $t^{\beta_k}f_k(t,\ldots)$ is continuous on $[0,1]\times\mathbb{R}^k$. Then, the operator $A:S\to S$ is completely continuous.*

Proof. For all $(x_1,\ldots,x_n)\in S$, let

$$A(x_1,x_2\ldots,x_n)(t)=\left(A_1(x_1),A_2(x_1,x_2),\ldots,A_n(x_1,\ldots,x_n)\right)(t),$$

where

$$
\begin{aligned}
&A_k(x_1,\ldots,x_k)(t)\\
&:=\begin{cases}
\displaystyle\int_0^t\frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}f_1(s,x_1(s))\,ds+a_0^1, \quad k=1,\\
\displaystyle\int_0^t\frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)}f_k(s,x_1(s),\ldots,x_k(s))\,ds+\sum_{j=0}^{k-2}\frac{a_j^k}{j!}t^j-\frac{\Gamma(k-\delta_{k-1})}{(k-1)!}t^{k-1}\\
\displaystyle\times\int_0^1\frac{(1-s)^{\alpha_k-\delta_{k-1}-1}}{\Gamma(\alpha_k-\delta_{k-1})}f_k(s,x_1(s),\ldots,x_k(s))\,ds, \quad k=2,3,\ldots,n.
\end{cases}
\end{aligned}
$$

By Lemma 2.1, we have $A : S \to S$. Let

$$\left(x_1^0, x_2^0, \dots, x_n^0\right) \in S : \left\|\left(x_1^0, x_2^0, \dots, x_n^0\right)\right\|_S = \lambda_0$$

and

$$(x_1, x_2, \dots, x_n) \in S : \ \left\|(x_1, x_2, \dots, x_n) - \left(x_1^0, x_2^0, \dots, x_n^0\right)\right\|_S < 1,$$

then

$$\|(x_1, x_2, \dots, x_n)\|_S < 1 + \lambda_0 = \lambda.$$

By the continuity of $t^{\beta_k} f_k(t, x_1, \dots, x_k)$, we know that $t^{\beta_k} f_k(t, x_1, \dots, x_k)$ is uniformly continuous on $[0,1] \times [-\lambda, \lambda]^k$.

Hence, for all $t \in [0,1]$ and for each $\epsilon > 0$, there exists $\rho > 0\,(\rho < 1)$, with

$$\left|t^{\beta_k} f_k(t, x_1(t), \dots, x_k(t)) - t^{\beta_k} f_k\left(t, x_1^0(t), \dots, x_k^0(t)\right)\right| < \epsilon, \tag{2.1}$$

where $(x_1, x_2, \dots, x_n) \in S$, and $\|(x_1, x_2, \dots, x_n) - (x_1^0, x_2^0, \dots, x_n^0)\|_S < \rho$. Then

$$\begin{aligned} &\left\|A(x_1, x_2 \dots, x_n) - A\left(x_1^0, x_2^0, \dots, x_n^0\right)\right\|_S \\ &= \max_{1 \le k \le n} \left\|A_k(x_1, \dots, x_k)(t) - A_k\left(x_1^0, \dots, x_k^0\right)(t)\right\|_\infty . \end{aligned} \tag{2.2}$$

We have

$$\begin{aligned} &\left\|A_k(x_1, \dots, x_k)(t) - A_k\left(x_1^0, \dots, x_k^0\right)(t)\right\|_\infty \\ &\le \begin{cases} \max\limits_{t\in[0,1]} \displaystyle\int_0^t \frac{(t-s)^{\alpha_1 - 1} s^{-\beta_1}}{\Gamma(\alpha_1)} \left|s^{\beta_1} f_1(s, x_1(s)) - s^{\beta_1} f_1\left(s, x_1^0(s)\right)\right| ds, \quad k = 1, \\ \max\limits_{t\in[0,1]} \displaystyle\int_0^t \frac{(t-s)^{\alpha_k - 1} s^{-\beta_k}}{\Gamma(\alpha_k)} \left|s^{\beta_k} f_k(s, x_1(s), \dots, x_k(s))\right. \\ \left. -s^{\beta_k} f_k\left(s, x_1^0(s), \dots, x_k^0(s)\right)\right| ds + \max\limits_{t\in[0,1]} \dfrac{\Gamma(k - \delta_{k-1})}{(k-1)!} t^{k-1} \\ \times \displaystyle\int_0^1 \frac{(1-s)^{\alpha_k - \delta_{k-1} - 1} s^{-\beta_k}}{\Gamma(\alpha_k - \delta_{k-1})} \left|s^{\beta_k} f_k(s, x_1(s), \dots, x_k(s))\right. \\ \left. -s^{\beta_k} f_k\left(s, x_1^0(s), \dots, x_k^0(s)\right)\right| ds, \quad k = 2, 3, \dots, n. \end{cases} \end{aligned}$$

Using (2.1), we obtain

$$\begin{aligned} &\left\|A_k(x_1, \dots, x_k)(t) - A_k\left(x_1^0, \dots, x_k^0\right)(t)\right\|_\infty \\ &\le \begin{cases} \dfrac{\epsilon}{\Gamma(\alpha_1)} \max\limits_{t\in[0,1]} \displaystyle\int_0^t (t-s)^{\alpha_1 - 1} s^{-\beta_1} ds, \ k = 1, \\ \\ \dfrac{\epsilon}{\Gamma(\alpha_k)} \max\limits_{t\in[0,1]} \displaystyle\int_0^t (t-s)^{\alpha_k - 1} s^{-\beta_k} ds \\ + \dfrac{\epsilon \Gamma(k - \delta_{k-1})}{(k-1)! \Gamma(\alpha_k - \delta_{k-1})} \max\limits_{t\in[0,1]} t^{k-1} \displaystyle\int_0^1 (1-s)^{\alpha_k - \delta_{k-1} - 1} s^{-\beta_k} ds, \\ k = 2, 3, \dots, n, \end{cases} \end{aligned} \tag{2.3}$$

$$\leq \begin{cases} \epsilon \dfrac{B(\alpha_1, 1-\beta_1)}{\Gamma(\alpha_1)} \max\limits_{t\in[0,1]} t^{\alpha_1-\beta_1}, \quad k=1, \\ \epsilon \left(\dfrac{B(\alpha_k, 1-\beta_k)}{\Gamma(\alpha_k)} \max\limits_{t\in[0,1]} t^{\alpha_k-\beta_k} + \dfrac{\Gamma(k-\delta_{k-1}) B(\alpha_k-\delta_{k-1}, 1-\beta_k)}{(k-1)!\Gamma(\alpha_k-\delta_{k-1})} \right), \\ k=2,3,\ldots,n, \end{cases}$$

$$= \begin{cases} \epsilon \dfrac{\Gamma(1-\beta_1)}{\Gamma(\alpha_1+1-\beta_1)}, \quad k=1, \\ \\ \epsilon \left(\dfrac{\Gamma(1-\beta_k)}{\Gamma(\alpha_k+1-\beta_k)} + \dfrac{\Gamma(k-\delta_{k-1})\Gamma(1-\beta_k)}{(k-1)!\Gamma(\alpha_k-\delta_{k-1}+1-\beta_k)} \right), \quad k=2,3,\ldots,n. \end{cases}$$

We pose:

$$\Lambda_1 := \frac{\Gamma(1-\beta_1)}{\Gamma(\alpha_1+1-\beta_1)}, \tag{2.4}$$
$$\Lambda_k := \frac{\Gamma(1-\beta_k)}{\Gamma(\alpha_k+1-\beta_k)} + \frac{\Gamma(k-\delta_{k-1})\Gamma(1-\beta_k)}{(k-1)!\Gamma(\alpha_k-\delta_{k-1}+1-\beta_k)}.$$

By (2.3) and (2.4), we have

$$\left\| A_k(x_1,\ldots,x_k)(t) - A_k\left(x_1^0,\ldots,x_k^0\right)(t) \right\|_\infty \leq \begin{cases} \epsilon\Lambda_1, & k=1, \\ \epsilon\Lambda_k, & k=2,3,\ldots,n. \end{cases} \tag{2.5}$$

Thanks to (2.2) and (2.5), we get

$$\left\| A(x_1,x_2\ldots,x_n) - A\left(x_1^0,x_2^0,\ldots,x_n^0\right) \right\|_S \leq \epsilon \max_{1\leq k\leq n} \Lambda_k.$$

Therefore,

$$\left\| A(x_1,x_2\ldots,x_n) - A\left(x_1^0,x_2^0,\ldots,x_n^0\right) \right\|_S \to 0,$$

as

$$\left\| (x_1,x_2\ldots,x_n) - \left(x_1^0,x_2^0,\ldots,x_n^0\right) \right\|_S \to 0.$$

Hence, $A: S \to S$ is continuous.

Let $\theta \subset S$ be bounded. Then, there exists a positive constant ς such that $\|(x_1,x_2\ldots,x_n)\|_S \leq \varsigma$, for all $(x_1,x_2\ldots,x_n) \in \theta$. Since $t^{\beta_k} f_k(t,x_1,\ldots,x_k)$, $k=1,2,\ldots,n$, are continuous on $[0,1]\times[-\varsigma,\varsigma]^k$, there exist positive constants L_k, $k=1,2,\ldots,n$, such that

$$\left| t^{\beta_k} f_k(t,x_1(t),\ldots,x_k(t)) \right| \leq L_k, \quad \text{for all } t\in[0,1], \text{ for all } (x_1,x_2\ldots,x_n)\in\theta. \tag{2.6}$$

Then

$$\|A(x_1,x_2\ldots,x_n)\|_S = \max_{1\leq k\leq n} \|A_k(x_1,\ldots,x_k)\|_\infty. \tag{2.7}$$

We have

$$\|A_k(x_1,\ldots,x_k)\|_\infty$$
$$\leq \begin{cases} \max\limits_{t\in[0,1]}\int_0^t \frac{(t-s)^{\alpha_1-1}s^{-\beta_1}}{\Gamma(\alpha_1)}\left|s^{\beta_1}f_1(s,x_1)\right|ds+\left|a_0^1\right|, \quad k=1,\\ \max\limits_{t\in[0,1]}\int_0^t \frac{(t-s)^{\alpha_k-1}s^{-\beta_k}}{\Gamma(\alpha_k)}\left|s^{\beta_k}f_k(s,x_1,\ldots,x_k)\right|ds \\ +\sum\limits_{j=0}^{k-2}\frac{\left|a_j^k\right|}{j!}\max\limits_{t\in[0,1]}t^j+\frac{\Gamma(k-\delta_{k-1})}{(k-1)!}\max\limits_{t\in[0,1]}t^{k-1} \\ \times\int_0^1\frac{(1-s)^{\alpha_k-\delta_{k-1}-1}s^{-\beta_k}}{\Gamma(\alpha_k-\delta_{k-1})}\left|s^{\beta_k}f_k(s,x_1,\ldots,x_k)\right|ds, \\ k=2,3,\ldots,n. \end{cases}$$

Using (2.6), we get

$$\|A_k(x_1,\ldots,x_k)\|_\infty \tag{2.8}$$
$$\leq \begin{cases} \frac{L_1}{\Gamma(\alpha_1)}\max\limits_{t\in[0,1]}\int_0^t(t-s)^{\alpha_1-1}s^{-\beta_1}ds+\left|a_0^1\right|, \quad k=1,\\ \frac{L_k}{\Gamma(\alpha_k)}\max\limits_{t\in[0,1]}\int_0^t(t-s)^{\alpha_k-1}s^{-\beta_k}ds+\sum\limits_{j=0}^{k-2}\frac{\left|a_j^k\right|}{j!} \\ +\frac{\Gamma(k-\delta_{k-1})L_k}{(k-1)!\Gamma(\alpha_k-\delta_{k-1})}\int_0^1(1-s)^{\alpha_k-\delta_{k-1}-1}s^{-\beta_k}ds, \quad k=2,3,\ldots,n, \end{cases}$$
$$\leq \begin{cases} \frac{L_1\Gamma(1-\beta_1)}{\Gamma(\alpha_1+1-\beta_1)}\max\limits_{t\in[0,1]}t^{\alpha_1-\beta_1}+\left|a_0^1\right|, \quad k=1,\\ L_k\left(\frac{\Gamma(1-\beta_k)}{\Gamma(\alpha_k+1-\beta_k)}\max\limits_{t\in[0,1]}t^{\alpha_k-\beta_k}+\frac{\Gamma(k-\delta_{k-1})\Gamma(1-\beta_k)}{(k-1)!\Gamma(\alpha_k-\delta_{k-1}+1-\beta_k)}\right) \\ +\sum\limits_{j=0}^{k-2}\frac{\left|a_j^k\right|}{j!}, \quad k=2,3,\ldots,n, \end{cases}$$
$$\leq \begin{cases} L_1\Lambda_1+\left|a_0^1\right|, \quad k=1,\\ L_k\Lambda_k+\sum\limits_{j=0}^{k-2}\frac{\left|a_j^k\right|}{j!}, \quad k=2,3,\ldots,n. \end{cases}$$

Then by (2.7) and (2.8), we get

$$\|A(x_1,x_2\ldots,x_n)\|_S\leq\max_{2\leq k\leq n}\left\{L_1\Lambda_1+\left|a_0^1\right|,L_k\Lambda_k+\sum_{j=0}^{k-2}\frac{\left|a_j^k\right|}{j!}\right\}.$$

Thus, $A(\theta)$ is bounded.

For all $(x_1,x_2\ldots,x_n)\in\theta$, and for all $t_1,t_2\in[0,1]$, $t_1<t_2$, we have:

$$\|A(x_1,x_2\ldots,x_n)(t_2)-A(x_1,x_2\ldots,x_n)(t_1)\|_S$$

$$= \max_{1\leq k\leq n} \left\| A_k\left(x_1,\ldots,x_k\right)\left(t_2\right) - A_k\left(x_1,\ldots,x_k\right)\left(t_1\right)\right\|_\infty . \tag{2.9}$$

Then

$$\left\| A_k\left(x_1,\ldots,x_k\right)\left(t_2\right) - A_k\left(x_1,\ldots,x_k\right)\left(t_1\right)\right\|_\infty$$
$$\leq \begin{cases} \max\limits_{t\in[0,1]} \left| \int_0^{t_2} \frac{\left(t_2-s\right)^{\alpha_1-1} s^{-\beta_1}}{\Gamma\left(\alpha_1\right)} s^{\beta_1} f_1\left(s,x_1\right) ds \right. \\ \left. - \int_0^{t_1} \frac{\left(t_1-s\right)^{\alpha_1-1} s^{-\beta_1}}{\Gamma\left(\alpha_1\right)} s^{\beta_1} f_1\left(s,x_1\right) ds \right|, \quad k=1, \\ \max\limits_{t\in[0,1]} \left| \int_0^{t_2} \frac{\left(t_2-s\right)^{\alpha_k-1} s^{-\beta_k}}{\Gamma\left(\alpha_k\right)} s^{\beta_k} f_k\left(s,x_1,\ldots,x_k\right) ds \right. \\ \left. - \int_0^{t_1} \frac{\left(t_1-s\right)^{\alpha_k-1} s^{-\beta_k}}{\Gamma\left(\alpha_k\right)} s^{\beta_k} f_k\left(s,x_1,\ldots,x_k\right) ds \right| \\ + \sum_{j=0}^{k-2} \frac{\left|a_j^k\right|}{j!} \left(t_2^j - t_1^j\right) + \frac{\Gamma\left(k-\delta_{k-1}\right)}{(k-1)!} \left(t_2^{k-1} - t_1^{k-1}\right) \\ \times \int_0^1 \frac{(1-s)^{\alpha_k-\delta_{k-1}-1} s^{-\beta_k}}{\Gamma\left(\alpha_k-\delta_{k-1}\right)} \left| s^{\beta_k} f_k\left(s,x_1,\ldots,x_k\right)\right| ds, \quad k=2,3,\ldots,n. \end{cases}$$

Hence,

$$\left\| A_k\left(x_1,\ldots,x_k\right)\left(t_2\right) - A_k\left(x_1,\ldots,x_k\right)\left(t_1\right)\right\|_\infty \tag{2.10}$$
$$\leq \begin{cases} \frac{L_1\Gamma\left(1-\beta_1\right)}{\Gamma\left(\alpha_1+1-\beta_1\right)} \left(t_2^{\alpha_1-\beta_1} - t_1^{\alpha_1-\beta_1}\right), \quad k=1, \\ \frac{L_k\Gamma\left(1-\beta_k\right)\left(t_2^{\alpha_k-\beta_k} - t_1^{\alpha_k-\beta_k}\right)}{\Gamma\left(\alpha_k+1-\beta_k\right)} + \sum_{j=0}^{k-2} \frac{\left|a_j^k\right|\left(t_2^j - t_1^j\right)}{j!} \\ + \frac{\Gamma\left(k-\delta_{k-1}\right) L_k\Gamma\left(1-\beta_k\right)\left(t_2^{k-1} - t_1^{k-1}\right)}{(k-1)!\Gamma\left(\alpha_k-\delta_{k-1}+1-\beta_k\right)}, \quad k=2,3,\ldots,n. \end{cases}$$

Then, by (2.9) and (2.10), we obtain

$$\left\| A\left(x_1,x_2\ldots,x_n\right)\left(t_2\right) - A\left(x_1,x_2\ldots,x_n\right)\left(t_1\right)\right\|_S \tag{2.11}$$
$$\leq \max\left\{ \frac{L_1\Gamma\left(1-\beta_1\right)}{\Gamma\left(\alpha_1+1-\beta_1\right)} \left(t_2^{\alpha_1-\beta_1} - t_1^{\alpha_1-\beta_1}\right), \frac{L_k\Gamma\left(1-\beta_k\right)\left(t_2^{\alpha_k-\beta_k} - t_1^{\alpha_k-\beta_k}\right)}{\Gamma\left(\alpha_k+1-\beta_k\right)} \right.$$
$$\left. + \sum_{j=0}^{k-2} \frac{\left|a_j^k\right|\left(t_2^j - t_1^j\right)}{j!} + \frac{\Gamma\left(k-\delta_{k-1}\right) L_k\Gamma\left(1-\beta_k\right)\left(t_2^{k-1} - t_1^{k-1}\right)}{(k-1)!\Gamma\left(\alpha_k-\delta_{k-1}+1-\beta_k\right)} \right\}.$$

The right-hand side of (2.11) is independent of $(x_1, x_2 \ldots, x_n)$ and tends to zero as $t_1 \to t_2$. Thus $A(\theta)$ is equicontinuous. By Arzela-Ascoli theorem, A is completely continuous. □

Theorem 2.1. *Assume that there exist nonnegative constants* $\left(\omega_j^k\right)_{j=1,\dots,k}^{k=1,\dots,n}$, *satisfying*

$$t^{\beta_k}\left|f_k\left(t,x_1,\dots,x_k\right)-f_k\left(t,y_1,\dots,y_k\right)\right|\leq\sum_{j=1}^{k}\omega_j^k\left|x_j-y_j\right|, \tag{2.12}$$

for all $t\in[0,1]$ *and all* $\left(x_1,\dots,x_k\right),\left(y_1,\dots,y_k\right)\in\mathbb{R}^k$.
If

$$\Sigma:=\max_{2\leq k\leq n}\left(\omega_1^1\Lambda_1,\sum_{j=1}^{k}\omega_j^k\Lambda_k\right)<1, \tag{2.13}$$

then the system (1.1) *has a unique solution on* $[0,1]$.

Proof. We will prove that A is a contractive operator on S.
Let $\left(x_1,x_2\dots,x_n\right),\left(y_1,y_2\dots,y_n\right)\in S$ and $t\in[0,1]$, we have

$$\begin{aligned}&\left\|A\left(x_1,x_2\dots,x_n\right)-A\left(y_1,y_2\dots,y_n\right)\right\|_S\\&=\max_{1\leq k\leq n}\left\|A_k\left(x_1,\dots,x_k\right)(t)-A_k\left(y_1,\dots,y_k\right)(t)\right\|_\infty.\end{aligned} \tag{2.14}$$

Then

$$\begin{aligned}&\left\|A_k\left(x_1,\dots,x_k\right)(t)-A_k\left(y_1,\dots,y_k\right)(t)\right\|_\infty\\&\leq\begin{cases}\displaystyle\max_{t\in[0,1]}\int_0^t\frac{(t-s)^{\alpha_1-1}s^{-\beta_1}}{\Gamma\left(\alpha_1\right)}s^{\beta_1}\left|f_1\left(s,x_1(s)\right)-f_1\left(s,y_1(s)\right)\right|ds, \quad k=1,\\ \displaystyle\max_{t\in[0,1]}\int_0^t\frac{(t-s)^{\alpha_k-1}s^{-\beta_k}}{\Gamma\left(\alpha_k\right)}s^{\beta_k}\left|f_k\left(s,x_1(s),\dots,x_k(s)\right)\right.\\ \displaystyle\left.-f_k\left(s,y_1(s),\dots,y_k(s)\right)\right|ds+\max_{t\in[0,1]}\frac{\Gamma\left(k-\delta_{k-1}\right)}{(k-1)!}t^{k-1}\\ \displaystyle\times\int_0^1\frac{(1-s)^{\alpha_k-\delta_{k-1}-1}s^{-\beta_k}}{\Gamma\left(\alpha_k-\delta_{k-1}\right)}s^{\beta_k}\left|f_k\left(s,x_1(s),\dots,x_k(s)\right)\right.\\ \left.-f_k\left(s,y_1(s),\dots,y_k(s)\right)\right|ds, \quad k=2,3,\dots,n.\end{cases}\end{aligned}$$

Thanks to (2.12), we can write

$$\begin{aligned}&\left\|A_k\left(x_1,\dots,x_k\right)(t)-A_k\left(y_1,\dots,y_k\right)(t)\right\|_\infty\\&\leq\begin{cases}\displaystyle\frac{\omega_1^1}{\Gamma\left(\alpha_1\right)}\left\|x_1-y_1\right\|_\infty\max_{t\in[0,1]}\int_0^t(t-s)^{\alpha_1-1}s^{-\beta_1}ds, \quad k=1,\\ \left(\omega_1^k\left\|x_1-y_1\right\|_\infty+\cdots+\omega_k^k\left\|x_k-y_k\right\|_\infty\right)\\ \displaystyle\times\left(\max_{t\in[0,1]}\int_0^t\frac{(t-s)^{\alpha_k-1}s^{-\beta_k}}{\Gamma\left(\alpha_k\right)}ds\right.\\ \displaystyle\left.+\frac{\Gamma\left(k-\delta_{k-1}\right)}{(k-1)!\Gamma\left(\alpha_k-\delta_{k-1}\right)}\int_0^1(1-s)^{\alpha_k-\delta_{k-1}-1}s^{-\beta_k}ds\right), \quad k=2,3,\dots,n,\end{cases}\end{aligned} \tag{2.15}$$

$$\leq \begin{cases} \dfrac{\omega_1^1 B(\alpha_1, 1-\beta_1) \|x_1 - y_1\|_\infty}{\Gamma(\alpha_1)} \max\limits_{t\in[0,1]} t^{\alpha_1-\beta_1}, \quad k=1, \\ \sum\limits_{j=1}^{k} \omega_j^k \max\limits_{1\leq k\leq n} \|x_k - y_k\|_\infty \left(\dfrac{B(\alpha_k, 1-\beta_k)}{\Gamma(\alpha_k)} \max\limits_{t\in[0,1]} t^{\alpha_k-\beta_k} \right. \\ \left. + \dfrac{\Gamma(k-\delta_{k-1}) B(\alpha_k - \delta_{k-1}, 1-\beta_k)}{(k-1)!\Gamma(\alpha_k - \delta_{k-1})} \right), \quad k = 2,3,\ldots,n, \end{cases}$$

$$\leq \begin{cases} \dfrac{\omega_1^1 \Gamma(1-\beta_1)}{\Gamma(\alpha_1+1-\beta_1)} \|x_1 - y_1\|_\infty, \quad k=1, \\ \sum\limits_{j=1}^{k} \omega_j^k \left(\dfrac{\Gamma(1-\beta_k)}{\Gamma(\alpha_k+1-\beta_k)} + \dfrac{\Gamma(k-\delta_{k-1})\Gamma(1-\beta_k)}{(k-1)!\Gamma(\alpha_k - \delta_{k-1}+1-\beta_k)} \right) \\ \times \|(x_1 - y_1, \ldots, x_k - y_k)\|_S, \quad k = 2,3,\ldots,n. \end{cases}$$

By (2.14) and (2.15), we obtain

$$\begin{aligned} &\|A(x_1, x_2, \ldots, x_n) - A(y_1, y_2, \ldots, y_n)\|_S \\ &\leq \max_{2\leq k\leq n} \left(\omega_1^1 \Lambda_1, \sum_{j=1}^{k} \omega_j^k \Lambda_k \right) \|(x_1 - y_1, \ldots, x_k - y_k)\|_S. \end{aligned}$$

By (2.13), we have $\Sigma := \max_{2\leq k\leq n} \left(\omega_1^1 \Lambda_1, \sum_{j=1}^{k} \omega_j^k \Lambda_k \right) < 1$. Hence, A is a contractive operator. Consequently, by Banach fixed point theorem, A has a fixed point which is the unique solution of system (1.1). This completes the proof. □

Example 2.1. Consider the following singular fractional system:

$$(2.16) \quad \begin{cases} D^{\frac{3}{4}} x_1(t) = \dfrac{\sin x_1(t)}{12\pi\sqrt{t}}, \quad 0 < t \leq 1, \\ D^{\frac{3}{2}} x_2(t) = \dfrac{\cos x_1(t) - \cos x_2(t)}{16\pi^3 t^{\frac{5}{7}}}, \quad 0 < t \leq 1, \\ D^{\frac{7}{3}} x_3(t) = \dfrac{(\sin x_1(t) + \sin x_2(t) + \cos x_3(t))}{24\pi t^{\frac{3}{8}}}, \quad 0 < t \leq 1, \\ D^{\frac{7}{2}} x_4(t) = \dfrac{|x_1(t) + x_2(t) + x_3(t) + x_4(t)|}{32\pi t^{\frac{1}{3}} (1 + |x_1(t) + x_2(t) + x_3(t) + x_4(t)|)}, \quad 0 < t \leq 1, \\ x_1(0) = 1, \\ x_2(0) = \sqrt{2},\ D^{\frac{1}{2}} x_2(1) = 0, \\ x_3(0) = \dfrac{3}{5},\ x_3'(0) = 2\sqrt{3},\ D^{\frac{4}{3}} x_3(1) = 0, \\ x_4(0) = \dfrac{1}{2},\ x_4'(0) = \sqrt{5},\ x_4''(0) = 1,\ D^{\frac{5}{2}} x_4(1) = 0. \end{cases}$$

We have:

$$\begin{aligned} &n = 4, \quad \alpha_1 = \frac{3}{4}, \quad \alpha_2 = \frac{3}{2}, \quad \alpha_3 = \frac{7}{3}, \quad \alpha_4 = \frac{7}{2}, \quad \delta_1 = \frac{1}{2}, \quad \delta_2 = \frac{4}{3}, \quad \delta_3 = \frac{5}{2}, \\ &a_0^1 = 1, \quad a_0^2 = \sqrt{2}, \quad a_0^3 = \frac{3}{5}, \quad a_1^3 = 2\sqrt{3}, \quad a_0^4 = \frac{1}{2}, \quad a_1^4 = \sqrt{5}, \quad a_2^4 = 1. \end{aligned}$$

Then, for each $t \in [0,1]$ and $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$, we have:

$$t^{\frac{2}{3}} \left| f_1(t, x_1) - f_1(t, y_1) \right| \le \frac{t^{\frac{1}{6}}}{12\pi} \left| x_1 - y_1 \right|,$$

$$t^{\frac{6}{7}} \left| f_2(t, x_1, x_2) - f_2(t, y_1, y_2) \right| \le \frac{t^{\frac{1}{7}}}{16\pi^3} \left| x_1 - y_1 \right| + \frac{t^{\frac{1}{7}}}{16\pi^3} \left| x_2 - y_2 \right|,$$

$$t^{\frac{7}{8}} \left| f_3(t, x_1, x_2, x_3) - f_3(t, y_1, y_2, y_3) \right|$$
$$\le \left(\frac{\sqrt{t}}{24\pi} \left| x_1 - y_1 \right| + \frac{\sqrt{t}}{24\pi} \left| x_2 - y_2 \right| + \frac{\sqrt{t}}{24\pi} \left| x_3 - y_3 \right| \right),$$

$$t^{\frac{1}{2}} \left| f_4(t, x_1, x_2, x_3, x_4) - f_4(t, y_1, y_2, y_3, y_4) \right|$$
$$\le \left(\frac{t^{\frac{1}{6}}}{32\pi} \left| x_1 - y_1 \right| + \frac{t^{\frac{1}{6}}}{32\pi} \left| x_2 - y_2 \right| + \frac{t^{\frac{1}{6}}}{32\pi} \left| x_3 - y_3 \right| + \frac{t^{\frac{1}{6}}}{32\pi} \left| x_4 - y_4 \right| \right),$$

where $\beta_1 = \frac{2}{3}$, $\beta_2 = \frac{6}{7}$, $\beta_3 = \frac{7}{8}$, $\beta_4 = \frac{1}{2}$.

Moreover, we can take:

$$\omega_1^1 = \frac{1}{12\pi},$$

$$\omega_1^2 = \omega_2^2 = \frac{1}{16\pi^3}, \quad \sum_{j=1}^{2} \omega_j^2 = \frac{1}{8\pi^3},$$

$$\omega_1^3 = \omega_2^3 = \omega_3^3 = \frac{1}{24\pi}, \quad \sum_{j=1}^{3} \omega_j^3 = \frac{1}{8\pi},$$

$$\omega_1^4 = \omega_2^4 = \omega_3^4 = \omega_4^4 = \frac{1}{32\pi}, \quad \sum_{j=1}^{4} \omega_j^4 = \frac{1}{8\pi}.$$

On the other hand, we get

$$\Lambda_1 = 2.7958, \quad \Lambda_2 = 13.4869, \quad \Lambda_3 = 9.4443, \quad \Lambda_4 = 0.5908.$$

Thus,

$$\omega_1^1 \Lambda_1 = 0.0742, \quad \sum_{j=1}^{2} \omega_j^2 \Lambda_2 = 0.0544, \quad \sum_{j=1}^{3} \omega_j^3 \Lambda_3 = 0.3759, \quad \sum_{j=1}^{4} \omega_j^4 \Lambda_4 = 0.0235.$$

Then the singular fractional system (2.16) has a unique solution on $[0,1]$.

Theorem 2.2. *Let $k-1 < \alpha_k < k$, $k = 1, 2, \ldots, n$, $n \in \mathbb{N} - \{0,1\}$, $0 < \beta_k < 1$. Assume that $f_k : (0,1] \times \mathbb{R}^k \to \mathbb{R}$ is continuous with $\lim_{t \to 0^+} f_k(t, \ldots) = \infty$ and $t^{\beta_k} f_k(t, \ldots)$ is continuous on $[0,1] \times \mathbb{R}^k$. Then, the system (1.1) has at least one solution on $[0,1]$.*

Proof. Let $P_k = \max_{t \in [0,1]} t^{\beta_k} \left| f_k(t, x_1(t), \ldots, x_k(t)) \right|$, and define the set $\Delta \subset S$ by

$$\Delta := \left\{ (x_1, x_2, \ldots, x_n) \in S : \left\| (x_1, x_2, \ldots, x_n) \right\|_S \le r \right\},$$

where

$$r=\max_{2\leq k\leq n}\left(P_1\Lambda_1+\left|a_0^1\right|,P_k\Lambda_k+\sum_{j=0}^{k-2}\frac{\left|a_j^k\right|}{j!}\right).$$

We will prove that $A:\Delta\to\Delta$. For $(x_1,x_2,\ldots,x_n)\in\Delta$ and $t\in[0,1]$, we have

$$\|A(x_1,x_2,\ldots,x_n)\|_S=\max_{1\leq k\leq n}\|A_k(x_1,\ldots,x_k)(t)\|_\infty. \tag{2.17}$$

Then

$$\|A_k(x_1,\ldots,x_k)(t)\|_\infty$$
$$\leq\begin{cases}\max_{t\in[0,1]}\int_0^t\frac{(t-s)^{\alpha_1-1}s^{-\beta_1}}{\Gamma(\alpha_1)}s^{\beta_1}|f_1(s,x_1)|ds+\left|a_0^1\right|, & k=1,\\ \max_{t\in[0,1]}\int_0^t\frac{(t-s)^{\alpha_k-1}s^{-\beta_k}}{\Gamma(\alpha_k)}s^{\beta_k}|f_k(s,x_1,\ldots,x_k)|ds+\sum_{j=0}^{k-2}\frac{\left|a_j^k\right|}{j!}\max_{t\in[0,1]}t^j\\ +\frac{\Gamma(k-\delta_{k-1})}{(k-1)!}\max_{t\in[0,1]}t^{k-1}\int_0^1\frac{(1-s)^{\alpha_k-\delta_{k-1}-1}s^{-\beta_k}}{\Gamma(\alpha_k-\delta_{k-1})}s^{\beta_k}|f_k(s,x_1,\ldots,x_k)|ds,\\ k=2,3,\ldots,n,\end{cases}$$
$$\leq\begin{cases}\frac{P_1}{\Gamma(\alpha_1)}\max_{t\in[0,1]}\int_0^t(t-s)^{\alpha_1-1}s^{-\beta_1}ds+\left|a_0^1\right|, & k=1,\\ \frac{P_k}{\Gamma(\alpha_k)}\max_{t\in[0,1]}\int_0^t(t-s)^{\alpha_k-1}s^{-\beta_k}ds+\sum_{j=0}^{k-2}\frac{\left|a_j^k\right|}{j!}\\ +\frac{\Gamma(k-\delta_{k-1})P_k}{(k-1)!\Gamma(\alpha_k-\delta_{k-1})}\int_0^1(1-s)^{\alpha_k-\delta_{k-1}-1}s^{-\beta_k}ds, & k=2,3,\ldots,n.\end{cases}$$

Thus,

$$\|A_k(x_1,\ldots,x_k)(t)\|_\infty \tag{2.18}$$
$$\leq\begin{cases}\frac{P_1\Gamma(1-\beta_1)}{\Gamma(\alpha_1+1-\beta_1)}\max_{t\in[0,1]}t^{\alpha_1-\beta_1}+|a_0^1|, & k=1,\\ P_k\left(\frac{\Gamma(1-\beta_k)}{\Gamma(\alpha_k+1-\beta_k)}\max_{t\in[0,1]}t^{\alpha_k-\beta_k}+\frac{\Gamma(k-\delta_{k-1})\Gamma(1-\beta_k)}{(k-1)!\Gamma(\alpha_k-\delta_{k-1}+1-\beta_k)}\right),\\ +\sum_{j=0}^{k-2}\frac{\left|a_j^k\right|}{j!}, & k=2,3,\ldots,n,\end{cases}$$
$$\leq\begin{cases}P_1\Lambda_1+|a_0^1|, & k=1,\\ P_k\Lambda_k+\sum_{j=0}^{k-2}\frac{\left|a_j^k\right|}{j!}, & k=2,3,\ldots,n.\end{cases}$$

Using (2.17) and (2.18), we can write

$$\|A(x_1,x_2,\ldots,x_n)\|_S\leq\max_{2\leq k\leq n}\left(P_1\Lambda_1+\left|a_0^1\right|,P_k\Lambda_k+\sum_{j=0}^{k-2}\frac{\left|a_j^k\right|}{j!}\right). \tag{2.19}$$

Hence, $\|A(x_1,x_2,\ldots,x_n)\|_S \le r$. By Lemma 2.1, we have $A(x_1,x_2,\ldots,x_n)(t) \in C([0,1])$. Moreover, for $(x_1,x_2,\ldots,x_n) \in \Delta$, we have $A(x_1,x_2,\ldots,x_n) \in \Delta$. So, $A(\Delta) \subset \Delta$, and $A:\Delta \to \Delta$. Then from Lemma 2.2, we get A is completely continuous.

By Lemma 1.5, the system (1.1) has at least one solution on $[0,1]$. Theorem 2.2 is thus proved. □

Example 2.2. Consider the following system:

$$\left\{\begin{array}{l}
D^{\frac{1}{2}}x_1(t) = t^{-\frac{1}{3}}e^{-t}\sin x_1(t), \quad 0<t\le 1,\\
D^{\frac{4}{3}}x_2(t) = \dfrac{\cos x_1(t)}{\sqrt{t}\,(\pi+\sin x_2(t))}, \quad 0<t\le 1,\\
D^{\frac{9}{4}}x_3(t) = \dfrac{e^t\cos x_3}{t^{\frac{1}{5}}\left(4\pi+\sin(x_1+x_2)\right)}, \quad 0<t\le 1,\\
D^{\frac{7}{2}}x_4(t) = t^{-\frac{2}{9}}\dfrac{e^{-2t}\sin(x_1+x_2)}{16+\cos(x_3+x_4)}, \quad 0<t\le 1,\\
D^{\frac{14}{3}}x_5(t) = \dfrac{\cos(u_1+u_2+u_3+u_4)}{t^{\frac{1}{4}}e^t}, \quad 0<t\le 1,\\
x_1(0)=\sqrt{3},\\
x_2(0)=\dfrac{2}{3}, \quad D^{\frac{1}{4}}x_2(1)=0,\\
x_3(0)=-1, \quad x_3'(0)=\dfrac{1}{2}, \quad D^{\frac{3}{2}}x_3(1)=0,\\
x_4(0)=\dfrac{\sqrt{7}}{2}, \quad x_4'(0)=\dfrac{1}{4}, \quad x_4''(0)=\dfrac{\sqrt{5}}{3}, \quad D^{\frac{11}{5}}x_4(1)=0,\\
x_5(0)=1, \quad x_5'(0)=\dfrac{4}{3}, \quad x_5''(0)=\dfrac{3}{7}, \quad x_5'''(0)=\dfrac{2\sqrt{3}}{5}, \quad D^{\frac{10}{3}}x_5(1)=0.
\end{array}\right. \tag{2.20}$$

We have:

$$n=5, \quad \alpha_1=\frac{1}{2}, \quad \alpha_2=\frac{4}{3}, \quad \alpha_3=\frac{9}{4}, \quad \alpha_4=\frac{7}{2}, \quad \alpha_5=\frac{14}{3}, \quad \delta_1=\frac{1}{4}, \quad \delta_2=\frac{3}{2},$$
$$\delta_3=\frac{11}{5}, \quad \delta_4=\frac{10}{3}, \quad a_0^1=\sqrt{3}, \quad a_0^2=\frac{2}{3}, \quad a_0^3=-1, \quad a_1^3=\frac{1}{2}, \quad a_0^4=\frac{\sqrt{7}}{2},$$
$$a_1^4=\frac{1}{4}, \quad a_2^4=\frac{\sqrt{5}}{3}, \quad a_0^5=1, \quad a_1^5=\frac{4}{3}, \quad a_2^5=\frac{3}{7}, \quad a_3^5=\frac{2\sqrt{3}}{5}.$$

For $\beta_1=\frac{2}{3}$, $\beta_2=\frac{3}{4}$, $\beta_3=\frac{2}{5}$, $\beta_4=\frac{4}{9}$, $\beta_5=\frac{1}{2}$, the system (2.20) has at least one solution on $[0,1]$.

3. Ulam Stability

In this section, we study the Ulam-Hyers stability and the generalized Ulam-Hyers stability of solutions for system (1.1).

Definition 3.1. The singular fractional system (1.1) is Ulam-Hyers stable if there exists a real number $\mu>0$, such that for all $(\epsilon_1,\epsilon_2,\ldots,\epsilon_n)>0$, and for all solution

$(x_1, x_2, \ldots, x_n) \in S$ of

$$\left\{ \begin{array}{l} \left| D^{\alpha_1} x_1(t) - f_1(t, x_1(t)) \right| \leq \epsilon_1, \\ \left| D^{\alpha_2} x_2(t) - f_2(t, x_1(t), x_2(t)) \right| \leq \epsilon_2, \\ \vdots \\ \left| D^{\alpha_n} x_n(t) - f_n(t, x_1(t), x_2(t), \ldots, x_n(t)) \right| \leq \epsilon_n, \quad 0 < t \leq 1, \end{array} \right. \tag{3.1}$$

there exists a solution $(y_1, y_2, \ldots, y_n) \in S$ satisfying

$$\left\{ \begin{array}{l} D^{\alpha_1} y_1(t) = f_1(t, y_1(t)), \\ D^{\alpha_2} x_2(t) = f_2(t, y_1(t), y_2(t)), \\ \vdots \\ D^{\alpha_n} x_n(t) = f_n(t, y_1(t), y_2(t), \ldots, y_n(t)), \quad 0 < t \leq 1, k = 1, \\ y_1(0) = a_0^1, \\ y_k^{(j)}(0) = a_j^k, \quad k = 2, 3, \ldots, n, j = 0, 1, \ldots, k-2, \\ D^{\delta_{k-1}} y_k(1) = 0, \quad k = 2, 3, \ldots, n, k-2 < \delta_{k-1} < k-1, \\ k-1 < \alpha_k < k, \quad k = 1, 2, \ldots, n, \end{array} \right. \tag{3.2}$$

with

$$\left\| (x_1 - y_1, \ldots, x_n - y_n) \right\|_S \leq \mu\epsilon, \quad \epsilon > 0.$$

Definition 3.2. The singular fractional system (1.1) is generalized Ulam-Hyers stable if there exists $\phi \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\phi(0) = 0$, such that for all $\epsilon > 0$, and for each solution $(x_1, x_2, \ldots, x_n) \in S$ of (3.1), there exists a solution $(y_1, y_2, \ldots, y_n) \in S$ of (3.2) with

$$\left\| (x_1 - y_1, \ldots, x_n - y_n) \right\|_S \leq \phi(\epsilon), \quad \epsilon > 0.$$

Theorem 3.1. *Let $k-1 < \alpha_k < k$, $k = 1, 2, \ldots, n$, $n \in \mathbb{N} - \{0, 1\}$ and $0 < \beta_k < 1$.*

Assume that:

(H_1) $f_k : (0,1] \times \mathbb{R}^k \to \mathbb{R}$ is continuous with $\lim_{t \to 0^+} f_k(t, \ldots) = \infty$ and $t^{\beta_k} f_k(t, \ldots)$ is continuous on $[0,1] \times \mathbb{R}^k$;

(H_2) $\left\| t^{\beta_k} D^{\alpha_k} x_k \right\|_\infty \geq \left\{ \begin{array}{ll} P_1\Lambda_1 + \left| a_0^1 \right|, & k = 1, \\ P_k\Lambda_k + \sum_{j=0}^{k-2} \frac{\left| a_j^k \right|}{j!}, & k = 2, 3, \ldots, n, \end{array} \right.$

(H_3) all the assumptions of Theorem 2.1 are satisfied;

$(H4)$ $\sum_{j=1}^{k} \omega_j^k < 1$, $k = 1, 2, \ldots, n$.

Then, the singular fractional system (1.1) is generalized Ulam-Hyers stable.

Proof. Using (H_1) we receive (2.19). Thus, for all solution $(x_1, x_2, \ldots, x_n) \in S$ of (3.1), we can write

$$\left\| (x_k) \right\|_\infty \leq \left\{ \begin{array}{ll} P_1\Lambda_1 + \left| a_0^1 \right|, & k = 1, \\ P_k\Lambda_k + \sum_{j=0}^{k-2} \frac{\left| a_j^k \right|}{j!}, & k = 2, 3, \ldots, n. \end{array} \right. \tag{3.3}$$

Then, by combining (H_2) with (3.3), we get

$$\|x_k\|_\infty \leq \left\|t^{\beta_k} D^{\alpha_k} x_k\right\|_\infty . \tag{3.4}$$

On the other hand, using (H_3), there exists a solution $(y_1, y_2, \ldots, y_n) \in S$ satisfying (3.2). Therefore, by (3.4) we can write:

$$\begin{aligned}
\|x_k - y_k\|_\infty \leq & \left\|t^{\beta_k} D^{\alpha_k} (x_k - y_k)\right\|_\infty \\
\leq & \left\|t^{\beta_k} \left(D^{\alpha_k} x_k - f_k (t, x_1, \ldots, x_k)\right) - t^{\beta_k} \left(D^{\alpha_k} y_k - f_k (t, y_1, \ldots, y_k)\right)\right. \\
& \left. + t^{\beta_k} \left(f_k (t, x_1, \ldots, x_k) - f_k (t, y_1, \ldots, y_k)\right)\right\|_\infty \\
\leq & \left\|t^{\beta_k} \left(D^{\alpha_k} x_k - f_k (t, x_1, \ldots, x_k)\right)\right\|_\infty + \left\|t^{\beta_k} \left(D^{\alpha_k} y_k - f_k (t, y_1, \ldots, y_k)\right)\right\|_\infty \\
& + \left\|t^{\beta_k} \left(f_k (t, x_1, \ldots, x_k) - f_k (t, y_1, \ldots, y_k)\right)\right\|_\infty \\
\leq & \left\|t^{\beta_k}\right\|_\infty \left\|(D^{\alpha_k} x_k - f_k (t, x_1, \ldots, x_k))\right\|_\infty \\
& + \left\|t^{\beta_k}\right\|_\infty \left\|(D^{\alpha_k} y_k - f_k (t, y_1, \ldots, y_k))\right\|_\infty \\
& + \left\|t^{\beta_k} \left(f_k (t, x_1, \ldots, x_k) - f_k (t, y_1, \ldots, y_k)\right)\right\|_\infty
\end{aligned}$$

From (2.12), (3.1) and (3.2), we obtain

$$\|(x_k - y_k)\|_\infty \leq \epsilon_k + \sum_{j=1}^{k} \omega_j^k \max_{1 \leq k \leq n} \|(x_k - y_k)\|_\infty .$$

Then

$$\max_{1 \leq k \leq n} \|(x_k - y_k)\|_\infty \leq \frac{\epsilon}{1 - \sum_{j=1}^{k} \omega_j^k} := \mu\epsilon, \quad \epsilon = \max_{1 \leq k \leq n} \epsilon_k, \quad \mu = \frac{1}{1 - \sum_{j=1}^{k} \omega_j^k}.$$

Hence,

$$\|(x_1 - y_1, \ldots, x_n - y_n)\|_S \leq \mu\epsilon.$$

Using (H_4), we get $\mu > 0$. Thus, system (1.1) is Ulam-Hyers stable. Taking $\phi(\epsilon) = \mu\epsilon$, we get system (1.1) is generalized Ulam-Hyers stable. This ends the proof. $\square$

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