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GENERAL CLASSES OF SHRINKAGE ESTIMATORS FOR THE MULTIVARIATE NORMAL MEAN WITH UNKNOWN VARIANCE: MINIMAXITY AND LIMIT OF RISKS RATIOS

ABDELKADER BENKHALED[1] AND ABDENOUR HAMDAOUI[2]

Abstract. In this paper, we consider two forms of shrinkage estimators of the mean θ of a multivariate normal distribution $X \sim N_p\left(\theta, \sigma^2 I_p\right)$ in $\mathbb{R}^p$ where σ^2 is unknown and estimated by the statistic S^2 ($S^2 \sim \sigma^2 \chi_n^2$). Estimators that shrink the components of the usual estimator X to zero and estimators of Lindley-type, that shrink the components of the usual estimator to the random variable $\overline{X}$. Our aim is to improve under appropriate condition the results related to risks ratios of shrinkage estimators, when n and p tend to infinity and to ameliorate the results of minimaxity obtained previously of estimators cited above, when the dimension p is finite. Some numerical results are also provided.

1. Introduction

Shrinkage estimates are alternative estimates that use information from all studies to provide potentially better estimates for each study. While these estimates is biased, they have a considerably smaller variance, and thus tend to be better in terms of total mean squared error. For example, Xie et al. [21] introduced a class of semiparametric/parametric shrinkage estimators and established their asymptotic optimality properties, Hansen [9] compared the mean-squared error of ordinary least squares (OLS), James-Stein, and least absolute shrinkage and selection operator (Lasso) shrinkage estimators and shows that neither James-Stein nor Lasso uniformly dominates the other, Selahattin et al. [15] provided several alternative methods for derivation of the restricted ridge regression estimator (RRRE).

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Mean vector parameter estimation is an important problem in the context of shrinkage estimation and has been widely applied in many scientific and engineering problems. This fact is certainly reflected by the abundant literature on the subject, let us cite for instance. Stein [16] showed the inadmissibility of the usual estimator X of the mean θ of a multivariate normal distribution $X \sim N_p\left(\theta, \sigma^2 I_p\right)$ when the dimension of the space of the observations $p \geqslant 3$. James and Stein [10], introduced the class of shrinkage estimators $\delta_a = (1 - aS^2/\left\|X\right\|^2)X$, that improving the usual estimator X under the quadratic loss function. Many developments in this field has realized by Lindley [12], Baranchik [1], Stein [17] and Selahattin and Issam [13]. Tsukuma and Kubokawa [20] addresses the problem of estimating the mean vector of a singular multivariate normal distribution with an unknown singular covariance matrix. Selahattin and Issam [14], introduced and derived the optimal extended balanced loss function (EBLF) estimators and pridictors and discuss their performances.

When the dimension p is infinite, Casella and Hwang [4], studied the case where σ^2 is known ($\sigma^2 = 1$) and showed that if the limit of the ratio $\left\|\theta\right\|^2/p$ is a constant $c > 0$, then the risks ratios of the James-Stein estimator δ^{JS} and the positive-part of the James-Stein estimator δ^{JS+}, to the maximum likelihood estimator X, tend to a constant value $c/(1+c)$. Benmansour and Hamdaoui [2], have taken the same model given by Casella and Hwang [4], where the parameter σ^2 is unknown and they established the same results. Hamdaoui and Benmansour [6], considered the model $X \sim N_p\left(\theta, \sigma^2 I_p\right)$ where σ^2 is unknown and estimated by S^2 ($S^2 \sim \sigma^2\chi^2_n$). They studied the following class of shrinkage estimators $\delta_\phi = \delta^{JS} + l(S^2\phi(S^2, \|X\|^2)/\|X\|^2)X$, where l is a real parameter. The authors showed that, when the sample size n and the dimension of space parameters p tend to infinity, the estimators δ_ϕ have a lower bound $B_m = c/(1+c)$ and if the shrinkage function ϕ satisfies some conditions, the risks ratio $R(\delta_\phi, \theta)/R(X, \theta)$ attains this lower bound B_m, in particulary the risks ratios $R(\delta^{JS}, \theta)/R(X, \theta)$ and $R(\delta^{JS+}, \theta)/R(X, \theta)$. In Hamdaoui et al. [8], the authors studied the limit of risks ratios of two forms of shrinkage estimators. The first one has been introduced by Benmansour and Mourid [3], $\delta_\psi = \delta^{JS} + l(S^2\psi(S^2, \|X\|^2)/\|X\|^2)X$, where l is a real parameter and $\psi\left(\cdot, u\right)$ is a function with support $[0, b]$ and satisfies some conditions different from the one given in Hamdaoui and Benmensour [6]. The second is the polynomial form of shrinkage estimator introduced by Li and Kio [11]. Hamdaoui and Mezouar [7], studied the general class of shrinkage estimators $\delta_\phi = \left(1 - S^2\phi\left(S^2, \|X\|^2\right)/\left\|X\right\|^2\right)X$. They showed the same results given in Hamdaoui and Benmansour [6], with different conditions on the shrinkage function ϕ.

In this work, we consider the model $X \sim N_p\left(\theta, \sigma^2 I_p\right)$ and independently of the observations X, we observe $S^2 \sim \sigma^2\chi^2_n$ an estimator of σ^2. It's well known that the quadratic risk of the usual estimator X is $p\sigma^2$. Consequently, any estimator of θ which has a quadratic risk less than $p\sigma^2$ dominate X, then it is minimax. We consider two different forms of shrinkage estimators of θ: estimators of the form $\delta^\psi = (1 - \psi(S^2, \left\|X\right\|^2)S^2/\left\|X\right\|^2)X$, and estimators of Lindley-type given by $\delta^\varphi =$

$(1-\varphi(S^2,T^2)S^2/T^2)(X-\overline{X})+\overline{X}$, that shrink the components of the maximum likelihood estimator X to the random variable $\overline{X}$. Our aim in this work is based on two points. First, when n and p tend to infinity, we give results of the limit of risks ratios of estimators defined above to the maximum likelihood estimator X, different from the one obtained in our published papers. The second point is to generalize and to improve the results of minimaxity obtained by Strawderman [18], Sun [19] and Hamdaoui and Benmansour [6].

The paper is outlined as follows: In Section 2, we consider the form of shrinkage estimators defined in (2.2) and we study the minimaxity and the limit of risks ratio to these estimators to the usual estimator X. In Section 3, we consider the second form of shrinkage estimators defined in (3.1) of Lindley-type. In this case, we follow the same steps as we treated the first form (2.2). In Section 4, we graphically illustrate some results given in this paper. In the end, we give an Appendix which contains technical lemmas used in the proofs of our results.

2. Shrinkage to Zero

Let $X \sim N_p(\theta, \sigma^2 I_p)$ where σ^2 is unknown and estimated by S^2 ($S^2 \sim \sigma^2\chi^2_n$). The aim is to estimate θ by an estimator δ relatively at the quadratic loss function

$$L(\delta,\theta) = \|\delta-\theta\|_p^2,$$

with $\|\cdot\|_p$ is the usual norm in $\mathbb{R}^p$. We associate its risk function

$$R(\delta,\theta) = E_\theta(L(\delta,\theta)).$$

We denote the general form of a shrinkage estimator as follows

$$\delta_j^\phi\left(X,S^2\right) = \left(1-\phi\left(S^2,\|X\|^2\right)\right)X_j, \quad j=1,\dots,p. \tag{2.1}$$

We recall that $\frac{\|X\|^2}{\sigma^2} \sim \chi^2_p(\lambda)$, where $\chi^2_p(\lambda)$ denotes the non-central chi-square distribution with p degrees of freedom and non-centrality parameter $\lambda = \frac{\|\theta\|^2}{2\sigma^2}$. We also recall the following Lemma given by Fourdrinier et al. [5], that we will use often in the next.

Lemma 2.1. *Let $X \sim N_p(\theta,\sigma^2 I_p)$ with $\theta\in\mathbb{R}^p$. Then*

(a) *for $p\geq 3$ we have $E\left(\frac{1}{\|X\|^2}\right) = \frac{1}{\sigma^2}E\left(\frac{1}{p-2+2K}\right)$;*

(b) *for $p\geq 5$ we have $E\left(\frac{1}{\|X\|^4}\right) = \frac{1}{\sigma^4}E\left(\frac{1}{(p-2+2K)(p-4+2K)}\right)$,*

where $K\sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$ being the Poisson's distribution of parameter $\frac{\|\theta\|^2}{2\sigma^2}$.

For the next, we need the following results obtained by Hamdaoui and Benmansour [6].

Proposition 2.1 (Hamdaoui and Benmansour [6])**.** *The risk of the estimator given in* (2.1) *is*

$$R\left(\delta^\phi(X,S^2),\theta\right) = \sigma^2 E\left\{\phi_K^2\ \chi^2_{p+2K} - 2\phi_K\left(\chi^2_{p+2K}-2K\right)+p\right\},$$

where $\phi_K = \phi\left(\sigma^2\chi^2_n, \sigma^2\chi^2_{p+2K}\right)$ *and* $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$ *being the Poisson's distribution of parameter* $\frac{\|\theta\|^2}{2\sigma^2}$ *and* χ^2_n *is the central chi-square distribution with* n *degrees of freedom. Furthermore,* $R\left(\delta^\phi\left(X, S^2\right), \theta\right) \geq B_p(\theta)$ *with*

$$B_p(\theta) = \sigma^2\left\{p-2-E\left\{\frac{(p-2)^2}{p-2+2K}\right\}\right\}.$$

We set by $b_p(\theta) = \frac{B_p(\theta)}{R(X,\theta)}$, it is clear that if $\lim\limits_{p\to+\infty} \frac{\|\theta\|^2}{p\sigma^2} = c\ (>0)$, then

$$\lim_{p\to\infty} b_p(\theta) = \frac{c}{1+c}.$$

In the particular case where $\phi(S^2, \|X\|^2) = d\frac{S^2}{\|X\|^2}$ we have

$$\delta^d\left(X, S^2\right) = \left(1 - d\frac{S^2}{\|X\|^2}\right)X,$$

hence

$$R\left(\delta^d(X, S^2), \theta\right) = \sigma^2\left\{p + n\left[d^2(n+2) - 2d(p-2)\right]E\left(\frac{1}{p-2+2K}\right)\right\}.$$

For $d = \frac{p-2}{n+2}$ we obtain the James-Stein estimator which minimizes the risk of $\delta^d\left(X, S^2\right)$ whose quadratic risk is

$$R\left(\delta^{JS}(X, S^2), \theta\right) = \sigma^2\left\{p - \frac{n}{n+2}(p-2)^2E\left(\frac{1}{p-2+2K}\right)\right\}.$$

Proposition 2.2 (Hamdaoui and Benmansour [6])**.** *If* $\lim\limits_{p\to+\infty} \frac{\|\theta\|^2}{p\sigma^2} = c$, *then*

$$\lim_{n,p\to\infty} \frac{R\left(\delta^\phi(X, S^2), \theta\right)}{R(X,\theta)} \geq \frac{c}{1+c}$$

and

$$\lim_{n,p\to\infty} \frac{R\left(\delta^{JS}(X, S^2), \theta\right)}{R(X,\theta)} = \frac{c}{1+c}.$$

We note that from the Proposition 2.2, the risks ratio of any shrinkage estimator $\delta^\phi\left(X, S^2\right)$ of the form (2.1) dominating the James-Stein estimator $\delta^{JS}\left(X, S^2\right)$, to the maximum likelihood estimator attains the limiting lower bound $B_m = \frac{c}{1+c}$ (<1), when n and p tend simultaneously to infinity.

Now we rewrite the estimator in (2.1) by letting $\phi\left(S^2, \|X\|^2\right) = \psi\left(S^2, \|X\|^2\right)\frac{S^2}{\|X\|^2}$, as given by

$$\delta^\psi_j\left(X, S^2\right) = \left(1 - \psi\left(S^2, \|X\|^2\right)\frac{S^2}{\|X\|^2}\right)X_j, \quad j = 1, \ldots, p. \tag{2.2}$$

Using the Proposition 2.1, the risk function of estimator $\delta^{\psi}\left(X,S^2\right)$ given in (2.2), is

$$\begin{aligned} R\left(\delta^{\psi}(X,S^2),\theta\right) &= \sigma^2 E\left\{\frac{\psi_K^2}{\sigma^2}\frac{\left(\sigma^2\chi_n^2\right)^2}{\left(\sigma^2\chi_{p+2K}^2\right)} - 2\psi_K\frac{\left(\sigma^2\chi_n^2\right)}{\left(\sigma^2\chi_{p+2K}^2\right)}\left(\chi_{p+2K}^2 - 2K\right) + p\right\} \\ &= p\sigma^2 + \sigma^2 E\left\{\chi_n^2\ \psi_K\left[\frac{\psi_K\ \chi_n^2}{\chi_{p+2K}^2} - 2\left(1 - \frac{2K}{\chi_{p+2K}^2}\right)\right]\right\}, \end{aligned}$$

where $\psi_K = \psi\left(\sigma^2\chi_n^2\ , \sigma^2\chi_{p+2K}^2\right).$

We write $\Delta_\psi = R\left(\delta^{\psi}\left(X,S^2\right),\theta\right) - R\left(X,\theta\right)$. As $R\left(X,\theta\right) = p\sigma^2$, then

$$\Delta_\psi = \sigma^2 E\left\{\chi_n^2\ \psi_K\left[\frac{\chi_n^2\ \psi_K}{\chi_{p+2K}^2} - 2\left(1 - \frac{2K}{\chi_{p+2K}^2}\right)\right]\right\}. \tag{2.3}$$

2.1. **Limit of risks ratios.** In this part, we are interested in studying of the limit of risks ratios of estimators defined in (2.2), to the usual estimator X. So, we give results different from the one given in our published papers.

Theorem 2.1. *Assume that $\delta^{\psi}\left(X,S^2\right)$ is given in (2.2), such that $p \geq 3$ and ψ satisfies:*

(H) $\left|\frac{p-2}{n+2} - \psi\left(S^2, \|X\|^2\right)\right| \leq g\left(S^2\right)$ *a.s., where* $E\left\{g^2\left(\sigma^2\chi_{n+4}^2\right)\right\} = O\left(\frac{1}{n^2}\right)$, *when n is in the neighborhood of $+\infty$.*

If $\lim\limits_{p\to+\infty}\frac{\|\theta\|^2}{p\sigma^2} = c$, *then*

$$\lim_{n,p\to+\infty}\frac{R\left(\delta^{\psi}\left(X,S^2\right),\theta\right)}{R\left(X,\theta\right)} = \frac{c}{1+c}.$$

Proof. We note $\alpha = \frac{p-2}{n+2}$ and $\psi\left(S^2, \|X\|^2\right) = \psi$. As

$$R\left(\delta^{\psi}(X,S^2),\theta\right) = E\left\{\sum_{i=1}^{p}\left[\left(1 - \psi\frac{S^2}{\|X\|^2}\right)X_i - \theta_i\right]^2\right\}$$

and

$$R\left(\delta^{JS}(X,S^2),\theta\right) = E\left\{\sum_{i=1}^{p}\left[\left(1 - \alpha\frac{S^2}{\|X\|^2}\right)X_i - \theta_i\right]^2\right\},$$

then

$$\begin{aligned} \Delta_{JS} =& R\left(\delta^{\psi}(X,S^2),\theta\right) - R\left(\delta^{JS}(X,S^2),\theta\right) \\ =& E\left\{\sum_{i=1}^{p}\left\{\left(\left[\left(1 - \psi\frac{S^2}{\|X\|^2}\right)X_i - \theta_i\right] - \left[\left(1 - \alpha\frac{S^2}{\|X\|^2}\right)X_i - \theta_i\right]\right)\right.\right. \\ &\times\left.\left.\left(\left[\left(1 - \psi\frac{S^2}{\|X\|^2}\right)X_i - \theta_i\right] + \left[\left(1 - \alpha\frac{S^2}{\|X\|^2}\right)X_i - \theta_i\right]\right)\right\}\right\} \end{aligned}$$

$$
\begin{aligned}
=&2E\left\{\sum_{i=1}^{p}\left(\left[(\alpha-\psi)\frac{S^2}{\|X\|^2}X_i\right]\left[\left(1-\frac{(\alpha+\psi)}{2}\frac{S^2}{\|X\|^2}\right)X_i-\theta_i\right]\right)\right\}\\
=&2E\left\{\sum_{i=1}^{p}\left[(\alpha-\psi)\left(1-\frac{(\alpha+\psi)}{2}\frac{S^2}{\|X\|^2}\right)\frac{S^2}{\|X\|^2}X_i^2\right]\right.\\
&\left.-\sum_{i=1}^{p}\left[(\alpha-\psi)\frac{S^2}{\|X\|^2}X_i\ \theta_i\right]\right\}\\
=&2E\left\{\sum_{i=1}^{p}\left[(\alpha-\psi)\left(1+\frac{(-\alpha+\alpha-\psi-\alpha)}{2}\frac{S^2}{\|X\|^2}\right)\frac{S^2}{\|X\|^2}X_i^2\right]\right.\\
&\left.-\sum_{i=1}^{p}\left[(\alpha-\psi)\frac{S^2}{\|X\|^2}X_i\ \theta_i\right]\right\}\\
=&2E\left\{\left[(\alpha-\psi)\frac{S^2}{\|X\|^2}\sum_{i=1}^{p}X_i^2\right]+\frac{1}{2}\left[(\alpha-\psi)^2\frac{S^4}{\|X\|^4}\sum_{i=1}^{p}X_i^2\right]\right.\\
&\left.-\alpha\left[(\alpha-\psi)\frac{S^4}{\|X\|^4}\sum_{i=1}^{p}X_i^2\right]-\sum_{i=1}^{p}\left[(\alpha-\psi)\frac{S^2}{\|X\|^2}X_i\ \theta_i\right]\right\}\\
=&2E\left\{(\alpha-\psi)S^2+\frac{1}{2}(\alpha-\psi)^2\frac{S^4}{\|X\|^2}-\alpha(\alpha-\psi)\frac{S^4}{\|X\|^2}\right.\\
&\left.-\sum_{i=1}^{p}\left[(\alpha-\psi)\frac{S^2}{\|X\|^2}X_i\ \theta_i\right]\right\}.
\end{aligned}
$$

Using the conditional expectation and the formula (2.7) given in Benmansour and Mourid [3], we have

$$
\begin{aligned}
E\left[(\alpha-\psi)\frac{S^2}{\|X\|^2}\langle X,\theta\rangle\right]&=E\left\{\sum_{i=1}^{p}\left[(\alpha-\psi)\frac{S^2}{\|X\|^2}X_i\ \theta_i\right]\right\}\\
&=\lambda E\left[\left(\alpha-\psi\left(\sigma^2\chi_n^2,\sigma^2\chi_{p+2}^2(\lambda)\right)\right)\frac{\chi_n^2}{\chi_{p+2}^2(\lambda)}\right],
\end{aligned}
$$

where $\lambda=\frac{\|\theta\|^2}{\sigma^2}$. Then

$$
\begin{aligned}
\Delta_{JS}\leq&2E\left\{\left[(|\alpha-\psi|)S^2\right]+\frac{1}{2}(\alpha-\psi)^2\frac{S^4}{\|X\|^2}+\alpha(|\alpha-\psi|)\frac{S^4}{\|X\|^2}\right.\\
&\left.+\lambda E\left[\left(\left|\alpha-\psi\left(\sigma^2\chi_n^2,\sigma^2\chi_{p+2}^2(\lambda)\right)\right|\right)\frac{\chi_n^2}{\chi_{p+2}^2(\lambda)}\right]\right\}.
\end{aligned}
$$

From the hypothesis (H) and the independence of two variables S^2 and $\|X\|^2$, we have

$$
\Delta_{JS}\leq 2E\left[S^2g\left(S^2\right)\right]+E\left[S^4g^2\left(S^2\right)\right]E\left(\frac{1}{\|X\|^2}\right)
$$

$$
\begin{aligned}
&+ 2\alpha E\left[S^4 g\left(S^2\right)\right] E\left(\frac{1}{\|X\|^2}\right) + 2\lambda E\left[S^2 g\left(S^2\right)\right] E\left(\frac{1}{\chi^2_{p+2}(\lambda)}\right)\\
=&2E\left[S^4 \frac{g\left(S^2\right)}{S^2}\right] + E\left[S^4 g^2\left(S^2\right)\right] E\left(\frac{1}{\|X\|^2}\right)\\
&+ 2\alpha E\left[S^4 g\left(S^2\right)\right] E\left(\frac{1}{\|X\|^2}\right) + 2\lambda E\left[S^4 \frac{g\left(S^2\right)}{S^2}\right] E\left(\frac{1}{\chi^2_{p+2}(\lambda)}\right).
\end{aligned}
$$

Using the Lemma 5.1 of the Appendix and the fact that $E\left(\frac{1}{\chi^2_p(\lambda)}\right) \leq \frac{1}{p-2}$, we obtain

$$
\begin{aligned}
&\Delta_{JS}\\
\leq&2n(n+2)\sigma^2 E\left[\frac{g\left(\sigma^2\chi^2_{n+4}\right)}{\chi^2_{n+4}}\right] + n(n+2)\sigma^2 E\left[g^2\left(\sigma^2\chi^2_{n+4}\right)\right] E\left(\frac{1}{\chi^2_p(\lambda)}\right)\\
&+ 2n(n+2)\sigma^2\left[\alpha E\left[g\left(\sigma^2\chi^2_{n+4}\right)\right] E\left(\frac{1}{\chi^2_p(\lambda)}\right) + \lambda E\left[\frac{g\left(\sigma^2\chi^2_{n+4}\right)}{\chi^2_{n+4}}\right] E\left(\frac{1}{\chi^2_{p+2}(\lambda)}\right)\right]\\
\leq&2n(n+2)\sigma^2 E\left[\frac{g\left(\sigma^2\chi^2_{n+4}\right)}{\chi^2_{n+4}}\right] + \frac{n(n+2)}{p-2}\sigma^2 E\left[g^2\left(\sigma^2\chi^2_{n+4}\right)\right]\\
&+ 2n\sigma^2 E\left[g\left(\sigma^2\chi^2_{n+4}\right)\right] + 2\lambda\frac{n(n+2)}{p}\sigma^2 E\left[\frac{g\left(\sigma^2\chi^2_{n+4}\right)}{\chi^2_{n+4}}\right].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\frac{\Delta_{JS}}{p\sigma^2} \leq&\frac{2n(n+2)}{p} E\left[\frac{g\left(\sigma^2\chi^2_{n+4}\right)}{\chi^2_{n+4}}\right] + \frac{n(n+2)}{p(p-2)} E\left[g^2\left(\sigma^2\chi^2_{n+4}\right)\right]\\
&+ \frac{2n}{p} E\left[g\left(\sigma^2\chi^2_{n+4}\right)\right] + \frac{2\lambda}{p}\frac{n(n+2)}{p} E\left[\frac{g\left(\sigma^2\chi^2_{n+4}\right)}{\chi^2_{n+4}}\right].
\end{aligned}
$$

From condition $E\left[g^2\left(\sigma^2\chi^2_{n+4}\right)\right] = O\left(\frac{1}{n^2}\right)$ and using the Schwarz inequality, when n is in the neighborhood of $+\infty$, we obtain

$$
\begin{aligned}
E\left[\frac{g\left(\sigma^2\chi^2_{n+4}\right)}{\chi^2_{n+4}}\right] &\leq E^{1/2}\left[g^2\left(\sigma^2\chi^2_{n+4}\right)\right] \times E^{1/2}\left[\frac{1}{\left(\chi^2_{n+4}\right)^2}\right]\\
&\leq \sqrt{M}\frac{1}{n} \times \sqrt{\frac{1}{n(n+2)}} \leq \sqrt{M}\frac{1}{n^2}
\end{aligned}
$$

and

$$
E\left[g\left(\sigma^2\chi^2_{n+4}\right)\right] \leq E^{1/2}\left[g^2\left(\sigma^2\chi^2_{n+4}\right)\right] \leq \sqrt{M}\frac{1}{n},
$$

where M is a real strictly positive. Then, when n is in the neighborhood of $+\infty$, we have

$$\frac{\Delta_{JS}}{p\sigma^2} \leq \frac{2(n+2)}{np}\sqrt{M} + \frac{n+2}{np(p-2)}M + \frac{2}{p}\sqrt{M} + \frac{2\lambda}{p\sigma^2} \cdot \frac{(n+2)}{np}M.$$

As $\lim\limits_{p\to+\infty} \frac{\lambda}{p} = \lim\limits_{p\to+\infty} \frac{\|\theta\|^2}{p\sigma^2} = c$, then

$$\lim_{n,p\to+\infty} \frac{\Delta_{JS}}{p\sigma^2} \leq 0.$$

Using the Proposition 2.2, we have

$$\lim_{n,p\to+\infty} \frac{R\left(\delta^{\psi}(X,S^2),\theta\right)}{R(X,\theta)} = \frac{c}{1+c}. \qquad \square$$

Example 2.1. Let $\psi_1 = \frac{p-2}{n+2} - \frac{S^2}{(1+S^2)^2}$, therefore

$$\delta^{\psi_1}\left(X,S^2\right) = \left(1 - \left(\frac{p-2}{n+2} - \frac{S^2}{(1+S^2)^2}\right)\frac{S^2}{\|X\|^2}\right)X.$$

It is sufficient to take $g(S^2) = \frac{S^2}{(1+S^2)^2}$, then from the Lemma 5.1 of the Appendix, we have

$$\begin{aligned}
E\left[g^2\left(\sigma^2\chi^2_{n+4}\right)\right] &= E\left[\frac{\left(\sigma^2\chi^2_{n+4}\right)^2}{\left(1+\sigma^2\chi^2_{n+4}\right)^4}\right] \\
&= (n+4)(n+6)\sigma^4 E\left[\frac{1}{\left(1+\sigma^2\chi^2_{n+8}\right)^4}\right] \\
&\leq \frac{(n+4)(n+6)}{\sigma^4} E\left[\frac{1}{\left(\chi^2_{n+8}\right)^4}\right] \\
&= \frac{1}{\sigma^4} \cdot \frac{(n+4)(n+6)}{n(n+2)(n+4)(n+6)} \overset{+\infty}{\sim} \frac{1}{\sigma^4} \cdot \frac{1}{n^2}.
\end{aligned}$$

Thus,

$$E\left[g^2\left(\sigma^2\chi^2_{n+4}\right)\right] = O\left(\frac{1}{n^2}\right).$$

2.2. **Minimaxity.** In this part we study the minimaxity of estimators defined in (2.2). We give another results that improve the one given in Strawderman [18], Sun [19] and Hamdaoui and Benmansour [6].

Theorem 2.2. *Assume that $\delta^{\psi}(X,S^2)$ is given in (2.2), such that $p \geq 3$ and ψ satisfies:*

(a) $\psi\left(S^2, \|X\|^2\right)$ *is monotone non-decreasing in* $\|X\|^2$*;*

(b) $0 \leq \psi\left(S^2, \|X\|^2\right) \leq \frac{2(p-2)}{n+2}$.

A sufficient condition so that the estimator $\delta^{\psi}(X, S^2)$ *is minimax is, for any* k, $k = 0, 1, 2, \ldots,$

$$E\left\{\psi\left(\sigma^2\chi^2_{n+4}\ ,\sigma^2\chi^2_{p+2k}\right)\right\} \le E\left\{\psi\left(\sigma^2\chi^2_{n+2}\ ,\sigma^2\chi^2_{p+2k}\right)\right\}.$$

Proof. From the formula (2.3) and the condition (b), we have

$$\Delta_\psi \le \sigma^2 E\left\{\chi^2_n\ \psi_K\left[\frac{\frac{2(p-2)}{n+2}\chi^2_n}{\chi^2_{p+2K}} - 2\left(1-\frac{2K}{\chi^2_{p+2K}}\right)\right]\right\}.$$

We will prove that the expectation on the right hand side being non-positive for any $K = k$, $k = 0, 1, 2, \ldots$.

By using the conditional expectation, we obtain

$$\Delta_\psi \le \sigma^2 E\left[E\left\{\psi_k\ \chi^2_n\left[\frac{\frac{2(p-2)}{n+2}\chi^2_n}{\chi^2_{p+2k}} - 2\left(1-\frac{2k}{\chi^2_{p+2k}}\right)\right]\Bigg|\ \chi^2_n\right\}\right]$$
$$\le \sigma^2 E\left\{\chi^2_n\ E\left(\psi_k \mid \chi^2_n\right)\ E\left[\left(\frac{\frac{2(p-2)}{n+2}\ \chi^2_n}{\chi^2_{p+2k}} - 2\left(1-\frac{2k}{\chi^2_{p+2k}}\right)\right)\Bigg|\ \chi^2_n\right]\right\},$$

the last inequality according to the condition (a) and the fact that the covariance of two functions one increasing and the other decreasing is non-positive.

Using the Lemma 2.1, we obtain

$$E\left[\left(\frac{\frac{2(p-2)}{n+2}\ \chi^2_n}{\chi^2_{p+2k}} - 2\left(1-\frac{2k}{\chi^2_{p+2k}}\right)\right)\Bigg|\ \chi^2_n\right]$$
$$=E\left[\left(\frac{\frac{2(p-2)}{n+2}\ \chi^2_n}{p-2+2k} - 2 + \frac{4k}{p-2+2k}\right)\Bigg|\ \chi^2_n\right] = \frac{2(p-2)\left(\frac{\chi^2_n}{n+2}-1\right)}{p-2+2k}.$$

Then

$$\Delta_\psi \le \sigma^2 E\left\{\chi^2_n\ \frac{2(p-2)\left(\frac{\chi^2_n}{n+2}-1\right)}{p-2+2k}\ E\left(\psi_k \mid \chi^2_n\right)\right\}$$
$$= \frac{2(p-2)\sigma^2}{p-2+2k} E\left\{\chi^2_n\ \left(\frac{\chi^2_n}{n+2}-1\right)\psi_k\right\}. \tag{2.4}$$

From the Lemma 5.1 of the Appendix, we have

$$E\left\{\chi_n^2\left(\frac{\chi_n^2}{n+2}-1\right)\psi_k\right\}=nE\left\{\psi\left(\sigma^2\chi_{n+4}^2\ ,\sigma^2\chi_{p+2k}^2\right)-\psi\left(\sigma^2\chi_{n+2}^2\ ,\sigma^2\chi_{p+2k}^2\right)\right\}.$$

Using the sufficient condition

$$E\left[\psi\left(\sigma^2\chi_{n+4}^2\ ,\sigma^2\chi_{p+2k}^2\right)\right]\leq E\left[\psi\left(\sigma^2\chi_{n+2}^2\ ,\sigma^2\chi_{p+2k}^2\right)\right],$$

we obtain

$$E\left\{\chi_n^2\left(\frac{\chi_n^2}{n+2}-1\right)\psi_k\right\}\leq 0.$$

Thus,

$$\Delta_\psi\leq 0.$$

□

Example 2.2. Let $\psi_2=\frac{2(p-2)}{n+2}\ln\left(1+S^2\right)\exp\left(-S^2\right)$, therefore

$$\delta^{\psi_2}\left(X,S^2\right)=\left(1-\frac{2\,(p-2)}{n+2}\frac{S^2\ln\left(1+S^2\right)\exp\left(-S^2\right)}{\|X\|^2}\right)X.$$

Remark 2.1. (i) Using the Lemma 5.2 of the Appendix, it is clear that if $\psi\left(S^2,\|X\|^2\right)$ is monotone non-increasing in S^2, then the sufficient condition:

$$E\left\{\psi\left(\sigma^2\chi_{n+4}^2\ ,\sigma^2\chi_{p+2k}^2\right)\right\}\leq E\left\{\psi\left(\sigma^2\chi_{n+2}^2\ ,\sigma^2\chi_{p+2k}^2\right)\right\}$$

is satisfied. Thus, the theorem 2.2 gives an improvement of the results of minimaxity given in the first Theorem of Strawderman [18], Theorem 4.1 of Sun [19] and Theorem 4.1 of Hamdaoui and Benmansour [6].

(ii) Note that the James-Stein estimator satisfies the conditions of Theorem 2.2, thus Theorem 2.2 gives another proof of the minimaxity of the James-Stein estimator.

3. Estimator of Lindley-Type

Let the model be $X/\theta,\sigma^2\sim N_p\left(\theta,\sigma^2I_p\right)$, where the parameters θ and σ^2 are unknown and σ^2 is estimated by S^2 ($S^2\sim\sigma^2\chi_n^2$). The aim is to estimate the mean $\theta=\left(\theta_1,\theta_2,\dots,\theta_p\right)^t$ by shrinkage estimators of the form

$$\delta_j^{\phi}\left(X,S^2,T^2\right)=\left(1-\phi\left(S^2,T^2\right)\right)\left(X_j-\overline{X}\right)+\overline{X},\quad j=1,2,\dots,p,\tag{3.1}$$

where

$$\overline{X}=\frac{1}{p}\sum_{i=1}^{p}X_i\quad\text{and}\quad T^2=\sum_{i=1}^{p}\left(X_i-\overline{X}\right)^2,$$

with the two random variables S^2 and T^2 are independent. In the next, we follow the same steps that we treated in Section 2, then we give a similar results to those given in Section 2 with some changes in the proofs.

Lemma 3.1. *For any functions f and g of the two variables S^2 and T^2, such that all expectations of* (a) *and* (b) *exist, we have*

(a) $E\left\{f\left(S^2,T^2\right)\right\}=E\left\{f\left(\sigma^2\chi_n^2,\sigma^2\chi_{p-1+2K}^2\right)\right\}$;

(b) $E\left\{g\left(S^2,T^2\right)\sum\limits_{i=1}^{p}\left(\theta_i-\overline{\theta}\right)\left(X_i-\overline{X}\right)\right\}=2\sigma^2E\left\{Kg\left(\sigma^2\chi_n^2,\sigma^2\chi_{p-1+2K}^2\right)\right\}$,

where $K\sim P\left(\sum\limits_{i=1}^{p}\left(\theta_i-\overline{\theta}\right)^2/2\sigma^2\right)$ *being the Poisson's distribution of parameter* $\sum\limits_{i=1}^{p}\left(\theta_i-\overline{\theta}\right)^2/2\sigma^2$ *and* $\overline{\theta}=\frac{1}{p}\sum\limits_{i=1}^{p}\theta_i$.

Proof. Analogous to the proof of the Lemma 2.1 given by Sun [19]. □

The following proposition, gives the explicit formula of the risk of the estimator $\delta^{\phi}\left(X,S^2,T^2\right)$ given in (3.1). For the proof see Appendix.

Proposition 3.1. *Let* $\delta^{\phi}\left(X,S^2,T^2\right)$ *is given in* (3.1)*, then for any* $p\geq 4$ *we have*

(i) $R\left(\delta^{\phi}(X,S^2,T^2),\theta\right)=\sigma^2\left\{\phi_K^2\chi_{p-1+2K}^2-2\phi_K\left(\chi_{p-1+2K}^2-2K\right)+p\right\}$;

(ii) $R\left(\delta^{\phi}(X,S^2,T^2),\theta\right)\geq B_p\left(\theta\right)$, *where*

$$\phi_K=\phi\left(\sigma^2\chi_n^2,\sigma^2\chi_{p-1+2K}^2\right)\quad \textit{and}\quad B_p\left(\theta\right)=\sigma^2E\left\{p-\frac{\left(\chi_{p-1+2K}^2-2K\right)^2}{\chi_{p-1+2K}^2}\right\};$$

(iii) *if* $c=\lim\limits_{p\to+\infty}\sum\limits_{i=1}^{p}\left(\theta_i-\overline{\theta}\right)^2/p\sigma^2$ *exists, then*

$$\lim_{p\to+\infty}\frac{B_p\left(\theta\right)}{R\left(X,\theta\right)}=\lim_{p\to+\infty}\frac{B_p\left(\theta\right)}{p\sigma^2}=\lim_{p\to+\infty}b_p\left(\theta\right)=\frac{c}{1+c}.$$

Now, we consider the special case when $\phi\left(S^2,T^2\right)=d\frac{S^2}{T^2}$, where d is a constant, then the estimator given in (3.1) is written as

$$\delta_j^d\left(X,S^2,T^2\right)=\left(1-d\frac{S^2}{T^2}\right)\left(X_j-\overline{X}\right)+\overline{X},\quad j=1,2,\ldots,p. \tag{3.2}$$

From the Proposition 3.1, we have

$$R\left(\delta^d(X,S^2,T^2),\theta\right)=\sigma^2\left\{p-\left[2dn\left(p-3\right)-d^2n\left(n+2\right)\right]E\left(\frac{1}{p-3+2K}\right)\right\}.$$

We note that when $d=0$, the estimator $\delta^0\left(X,S^2,T^2\right)$ given in (3.2) becomes the maximum likelihood estimator X, its risk equal $p\sigma^2$. In this case, the James-Stein estimator is obtained by minimizing the risk $R\left(\delta^d(X,S^2,T^2),\theta\right)$, the James-Stein estimator is given by

$$\delta_j^{JS}\left(X,S^2,T^2\right)=\left(1-\frac{p-3}{n+2}\frac{S^2}{T^2}\right)\left(X_j-\overline{X}\right)+\overline{X},\quad j=1,2,\ldots,p. \tag{3.3}$$

Its risk is

$$R\left(\delta^{JS}(X,S^2,T^2),\theta\right)=\sigma^2\left\{p-\frac{n}{n+2}\left(p-3\right)^2E\left(\frac{1}{p-3+2K}\right)\right\}, \tag{3.4}$$

where $K\sim P\left(\sum\limits_{i=1}^{p}\left(\theta_i-\overline{\theta}\right)^2/2\sigma^2\right)$.

Proposition 3.2. (a) *If $p \geq 4$, the James-Stein estimator $\delta^{JS}\left(X, S^2, T^2\right)$ given in* (3.3) *is minimax.*

(b) *If* $\lim\limits_{p \to +\infty} \sum\limits_{i=1}^{p} \left(\theta_i - \overline{\theta}\right)^2 / p\sigma^2 = c(>0)$, *then*

$$\lim_{n,p \to +\infty} \frac{R\left(\delta^{JS}\left(X, S^2, T^2\right), \theta\right)}{R\left(X, \theta\right)} = \frac{c}{1+c}.$$

Proof. (a) It is obviously from the formula (3.4).

(b) For $p \geq 6$ and from the Lemma 3.1 given by Sun [19], we have

$$\frac{1}{p-3+\frac{\sum_{i=1}^{p}\left(\theta_i-\overline{\theta}\right)^2}{\sigma^2}} \leq E\left(\frac{1}{p-3+2K}\right) \leq \frac{1}{p-5+\frac{\sum_{i=1}^{p}\left(\theta_i-\overline{\theta}\right)^2}{\sigma^2}},$$

then

$$\frac{R\left(\delta^{JS}\left(X, S^2, T^2\right), \theta\right)}{R\left(X, \theta\right)} \geq 1 - \frac{n}{n+2} \cdot \frac{(p-3)^2}{p^2} \cdot \frac{1}{\frac{p-5}{p} + \frac{\sum_{i=1}^{p}\left(\theta_i-\overline{\theta}\right)^2}{p\sigma^2}}$$

and

$$\frac{R\left(\delta^{JS}\left(X, S^2, T^2\right), \theta\right)}{R\left(X, \theta\right)} \leq 1 - \frac{n}{n+2} \cdot \frac{(p-3)^2}{p^2} \cdot \frac{1}{\frac{p-3}{p} + \frac{\sum_{i=1}^{p}\left(\theta_i-\overline{\theta}\right)^2}{p\sigma^2}}.$$

Thus,

$$\frac{c}{1+c} = 1 - \frac{1}{1+c} \leq \lim_{n,p \to +\infty} \frac{R\left(\delta^{JS}\left(X, S^2, T^2\right), \theta\right)}{R\left(X, \theta\right)} \leq 1 - \frac{1}{1+c} = \frac{c}{1+c}. \qquad \square$$

Remark 3.1. From Propositions 3.1 and 3.2, we note that the risks ratio of any shrinkage estimator $\delta^{\phi}\left(X, S^2, T^2\right)$ of the form (3.1) dominating the James-Stein estimator $\delta^{JS}\left(X, S^2, T^2\right)$, to the maximum likelihood estimator attains the limiting lower bound $B_m = \frac{c}{1+c}$, when n and p tend simultaneously to infinity.

Next, we consider the general form of shrinkage estimators of Lindley-type, defined by

$$\delta_j^{\varphi}\left(X, S^2, T^2\right) = \left(1 - \varphi\left(S^2, T^2\right)\frac{S^2}{T^2}\right)\left(X_j - \overline{X}\right) + \overline{X}, \quad j = 1, 2, \ldots, p. \tag{3.5}$$

We write $\Delta_{\varphi} = R\left(\delta^{\varphi}\left(X, S^2, T^2\right), \theta\right) - R\left(X, \theta\right)$. Then

$$\Delta_{\varphi} = \sigma^2 E\left\{\chi_n^2\ \varphi_K\left[\frac{\chi_n^2 \varphi_K}{\chi_{p-1+2K}^2} - 2\left(1 - \frac{2K}{\chi_{p-1+2K}^2}\right)\right]\right\},$$

where $\varphi_K = \varphi\left(\sigma^2\chi_n^2, \sigma^2\chi_{p-1+2K}^2\right)$.

3.1. **Limit of risks ratios.**

Proposition 3.3. *Assume that* $\delta^{\varphi}\left(X,S^2,T^2\right)$ *is given in* (3.5), *such that* $p\geq 3$ *and* φ *satisfies*

(H) $\left|\frac{p-3}{n+2}-\varphi\left(S^2,T^2\right)\right|\leq g\left(S^2\right)$ *a.s., where* $E\left\{g^2\left(\sigma^2\chi^2_{n+4}\right)\right\}=O\left(\frac{1}{n^2}\right).$

If $\lim\limits_{p\to+\infty}\sum\limits_{i=1}^{p}\left(\theta_i-\overline{\theta}\right)^2/p\sigma^2=c,$ *then*

$$\lim_{n,p\to+\infty}\frac{R\left(\delta^{\varphi}\left(X,S^2,T^2\right),\theta\right)}{R\left(X,\theta\right)}=\frac{c}{1+c}.$$

Proof. We follow the same steps of the proof of Theorem 2.1, endeed we write $\alpha=\frac{p-3}{n+2}$ and $\varphi\left(S^2,T^2\right)=\varphi$. As

$$R\left(\delta^{\varphi}(X,S^2,T^2),\theta\right)=E\left\{\sum_{i=1}^{p}\left[\left(1-\varphi\frac{S^2}{T^2}\right)\left(X_i-\overline{X}\right)-\theta_i\right]^2\right\}$$

and

$$R\left(\delta^{JS}(X,S^2,T^2),\theta\right)=E\left\{\sum_{i=1}^{p}\left[\left(1-\alpha\frac{S^2}{T^2}\right)\left(X_i-\overline{X}\right)-\theta_i\right]^2\right\},$$

we have

$$\begin{aligned}\Delta_{JS}=&R\left(\delta^{\varphi}(X,S^2,T^2),\theta\right)-R\left(\delta^{JS}(X,S^2,T^2),\theta\right)\\=&2E\left\{\sum_{i=1}^{p}\left[\left(\alpha-\varphi\right)\frac{S^2}{T^2}\left(X_i-\overline{X}\right)\right]\left[\left(1-\frac{\left(\alpha+\varphi\right)}{2}\frac{S^2}{T^2}\right)\left(X_i-\overline{X}\right)-\theta_i\right]\right\}\\=&2E\left\{\sum_{i=1}^{p}\left[\left(\alpha-\varphi\right)\left(1-\frac{\left(\alpha+\varphi\right)}{2}\frac{S^2}{T^2}\right)\frac{S^2}{T^2}\left(X_i-\overline{X}\right)^2\right]\right.\\&\left.-\sum_{i=1}^{p}\left[\left(\alpha-\varphi\right)\frac{S^2}{T^2}\left(X_i-\overline{X}\right)\ \theta_i\right]\right\}\\=&2E\left\{\left[\left(\alpha-\varphi\right)\frac{S^2}{T^2}\sum_{i=1}^{p}\left(X_i-\overline{X}\right)^2\right]+\frac{1}{2}\left[\left(\alpha-\varphi\right)^2\frac{S^4}{T^4}\sum_{i=1}^{p}\left(X_i-\overline{X}\right)^2\right]\right.\\&\left.-\alpha\left[\left(\alpha-\varphi\right)\frac{S^4}{T^4}\sum_{i=1}^{p}\left(X_i-\overline{X}\right)^2\right]-\sum_{i=1}^{p}\left[\left(\alpha-\varphi\right)\frac{S^2}{T^2}\left(X_i-\overline{X}\right)\ \left(\theta_i-\overline{X}\right)\right]\right\}\\=&2E\left\{\left[\left(\alpha-\varphi\right)S^2\right]+\frac{1}{2}\left[\left(\alpha-\varphi\right)^2\frac{S^4}{T^2}\right]-\alpha\left[\left(\alpha-\varphi\right)\frac{S^4}{T^2}\right]\right.\\&\left.-\sum_{i=1}^{p}\left[\left(\alpha-\varphi\right)\frac{S^2}{T^2}\left(X_i-\overline{X}\right)\ \left(\theta_i-\overline{X}\right)\right]\right\}.\end{aligned}$$

As $\sum\limits_{i=1}^{p}\left(X_i-\overline{X}\right)=0$, then

$$E\left\{(\alpha-\varphi)\frac{S^2}{T^2}\sum_{i=1}^{p}(X_i-\overline{X})(\theta_i-\overline{X})\right\}$$

$$
\begin{aligned}
=&E\left\{(\alpha-\varphi)\frac{S^2}{T^2}\sum_{i=1}^{p}(X_i-\overline{X})(\theta_i-\overline{\theta})+(\overline{\theta}-\overline{X})(\alpha-\varphi)\frac{S^2}{T^2}\sum_{i=1}^{p}(X_i-\overline{X})\right\}\\
=&E\left\{(\alpha-\varphi)\frac{S^2}{T^2}\sum_{i=1}^{p}(X_i-\overline{X})(\theta_i-\overline{\theta})\right\}\\
=&2\sigma^2E\left\{K\left(\alpha-\varphi(\sigma^2\chi_n^2,\sigma^2\chi_{p-1+2K}^2)\right)\frac{\chi_n^2}{\chi_{p-1+2K}^2}\right\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\Delta_{JS}\leq&2E\left\{\left(|\alpha-\varphi|\right)S^2+\frac{1}{2}\left(\alpha-\varphi\right)^2\frac{S^4}{T^2}+\alpha\left(|\alpha-\varphi|\right)\frac{S^4}{T^2}\right.\\
&\left.+2\sigma^2K\left(\left|\alpha-\varphi\left(\sigma^2\chi_n^2,\sigma^2\chi_{p-1+2K}^2\right)\right|\right)\frac{\chi_n^2}{\chi_{p-1+2K}^2}\right\}.
\end{aligned}
$$

From hypothesis (H) and the independence to two variables S^2 and T^2, we have

$$
\begin{aligned}
\Delta_{JS}\leq&2\left\{E\left[S^2g\left(S^2\right)\right]+\frac{1}{2}E\left[S^4g^2\left(S^2\right)\right]\ E\left(\frac{1}{T^2}\right)\right.\\
&\left.+\alpha E\left[S^4g\left(S^2\right)\right]\ E\left(\frac{1}{T^2}\right)\right\}+2\sigma^2E\left[S^2g\left(S^2\right)\frac{K}{\chi_{p-1+2K}^2}\right]\\
=&2E\left[S^4\frac{g\left(S^2\right)}{S^2}\right]+E\left[S^4g^2\left(S^2\right)\right]E\left(\frac{1}{T^2}\right)\\
&+2\alpha E\left[S^4g\left(S^2\right)\right]E\left(\frac{1}{T^2}\right)+2\sigma^2E\left[S^4\frac{g\left(S^2\right)}{S^2}\cdot\frac{K}{\chi_{p-1+2K}^2}\right].
\end{aligned}
$$

Using the conditional expectation, we have

$$
\begin{aligned}
E\left[S^4\frac{g\left(S^2\right)}{S^2}\frac{K}{\chi_{p-1+2K}^2}\right]=&E\left\{E\left(\left[S^4\frac{g\left(S^2\right)}{S^2}\cdot\frac{K}{\chi_{p-1+2K}^2}\right]\middle|S^2\right)\right\}\\
=&\frac{1}{2}E\left\{E\left(\left[S^4\frac{g\left(S^2\right)}{S^2}\cdot\frac{2K}{p-3+2K}\right]\middle|S^2\right)\right\}\\
\leq&\frac{1}{2}E\left[S^4\frac{g\left(S^2\right)}{S^2}\right].
\end{aligned}
$$

From the Lemma 5.1 of the Appendix, the independence of two variables χ_{n+4}^2 and χ_{p-1+2K}^2 and the fact that $E\left(\frac{1}{\chi_{p-1+2K}^2}\right)=E\left(\frac{1}{p-3+2K}\right)\leq\frac{1}{p-3}$, we obtain

$$
\begin{aligned}
\Delta_{JS}\leq&2n\left(n+2\right)\sigma^2\left\{E\left[\frac{g\left(\sigma^2\chi_{n+4}^2\right)}{\chi_{n+4}^2}\right]+\frac{1}{2}E\left[g^2\left(\sigma^2\chi_{n+4}^2\right)\right]\ E\left(\frac{1}{\chi_{p-1+2K}^2}\right)\right\}\\
&+2n\left(n+2\right)\sigma^2\left(\alpha E\left[g\left(\sigma^2\chi_{n+4}^2\right)\right]+E\left[\frac{g\left(\sigma^2\chi_{n+4}^2\right)}{\chi_{n+4}^2}\right]\right)\ E\left(\frac{1}{\chi_{p-1+2K}^2}\right)
\end{aligned}
$$

$$\leq 2n\left(n+2\right)\sigma^2 E\left[\frac{g\left(\sigma^2\chi^2_{n+4}\right)}{\chi^2_{n+4}}\right]+\frac{n\left(n+2\right)}{p-3}\sigma^2 E\left[g^2\left(\sigma^2\chi^2_{n+4}\right)\right]$$
$$+2n\sigma^2 E\left[g\left(\sigma^2\chi^2_{n+4}\right)\right]+2\sigma^2\frac{n\left(n+2\right)}{p-3}E\left[\frac{g\left(\sigma^2\chi^2_{n+4}\right)}{\chi^2_{n+4}}\right].$$

Thus,

$$\frac{\Delta_{JS}}{p\sigma^2}\leq\frac{2n\left(n+2\right)}{p}E\left[\frac{g\left(\sigma^2\chi^2_{n+4}\right)}{\chi^2_{n+4}}\right]+\frac{n\left(n+2\right)}{p\left(p-3\right)}E\left[g^2\left(\sigma^2\chi^2_{n+4}\right)\right]$$
$$+\frac{2n}{p}E\left[g\left(\sigma^2\chi^2_{n+4}\right)\right]+\frac{2\lambda}{p\sigma^2}\frac{n\left(n+2\right)}{p-3}E\left[\frac{g\left(\sigma^2\chi^2_{n+4}\right)}{\chi^2_{n+4}}\right],$$

where $\lambda=\sum\limits_{i=1}^{p}\left(\theta_i-\overline{\theta}\right)^2/\sigma^2$.

From the condition $E\left[g^2\left(\sigma^2\chi^2_{n+4}\right)\right]=O\left(\frac{1}{n^2}\right)$, when n is in the neighborhood of $+\infty$, we have

$$\frac{\Delta_{JS}}{p\sigma^2}\leq\frac{2\left(n+2\right)}{np}\sqrt{M}+\frac{n+2}{np\left(p-3\right)}M+\frac{2}{p}\sqrt{M}+\frac{2\lambda}{p\sigma^2}\cdot\frac{\left(n+2\right)}{np}M,$$

where M is real strictly positive.

As $\lim\limits_{p\to+\infty}\frac{\lambda}{p}=\lim\limits_{p\to+\infty}\sum\limits_{i=1}^{p}\left(\theta_i-\overline{\theta}\right)^2/p\sigma^2=c$, hence

$$\lim_{n,p\to+\infty}\frac{\Delta_{JS}}{p\sigma^2}\leq 0.$$

Thus, from Propositions 3.1 and 3.2, we have

$$\lim_{n,p\to+\infty}\frac{R\left(\delta^{\varphi}\left(X,S^2,T^2\right),\theta\right)}{R\left(X,\theta\right)}=\frac{c}{1+c}.$$ □

3.2. **Minimaxity.**

Proposition 3.4. *Assume that $\delta^{\varphi}\left(X,S^2,T^2\right)$ is given in (3.5), such that $p\geq 4$. If*

(a) *$\varphi\left(S^2,T^2\right)$ is monotone non-decreasing in T^2;*

(b) $0\leq\varphi\left(S^2,T^2\right)\leq\dfrac{2\left(p-3\right)}{n+2}$.

A sufficient condition so that the estimator $\delta^{\varphi}\left(X,S^2,T^2\right)$ is minimax is, for any k, $k=0,1,2,\ldots$, and for each fixed T^2

$$E\left\{\varphi\left(\sigma^2\chi^2_{n+4}\,,\sigma^2\chi^2_{p-1+2k}\right)\right\}\leq E\left\{\varphi\left(\sigma^2\chi^2_{n+2}\,,\sigma^2\chi^2_{p-1+2k}\right)\right\}.$$

Proof. The proof is similar to proof of Theorem 2.2. Endeed, from condition (b), we obtain

$$\Delta_{\varphi}=\sigma^2 E\left\{\chi^2_n\;\varphi_K\left[\frac{\chi^2_n\;\varphi_K}{\chi^2_{p-1+2K}}-2\left(1-\frac{2K}{\chi^2_{p-1+2K}}\right)\right]\right\}$$

$$\leq \sigma^2 E\left\{\chi_n^2\ \varphi_K \left[\frac{\frac{2(p-3)}{n+2}\chi_n^2}{\chi_{p-1+2K}^2} - 2\left(1-\frac{2K}{\chi_{p-1+2K}^2}\right)\right]\right\}.$$

We will prove that the expectation on the right hand side being non-positive for any $K = k,\ k = 0, 1, 2, \ldots$

By using the conditional expectation, we have

$$\begin{aligned}\Delta_\varphi &\leq \sigma^2 E\left[E\left\{\chi_n^2\ \varphi_k \left[\frac{\frac{2(p-3)}{n+2}\chi_n^2}{\chi_{p-1+2k}^2} - 2\left(1-\frac{2k}{\chi_{p-1+2k}^2}\right)\right]\middle|\ \chi_n^2\right\}\right] \\ &\leq \sigma^2 E\left\{\chi_n^2\ E\left(\varphi_k\middle|\ \chi_n^2\right)\ E\left[\left(\frac{\frac{2(p-3)}{n+2}\ \chi_n^2}{\chi_{p-1+2k}^2} - 2\left(1-\frac{2k}{\chi_{p-1+2k}^2}\right)\right)\middle|\ \chi_n^2\right]\right\},\end{aligned}$$

the last inequality according to the condition (a) and the fact that the covariance of two functions one increasing and the other decreasing is non-positive.

As

$$\begin{aligned}E\left[\left(\frac{\frac{2(p-3)}{n+2}\ \chi_n^2}{\chi_{p-1+2k}^2} - 2\left(1-\frac{2k}{\chi_{p-1+2k}^2}\right)\right) \mid \chi_n^2\right] &= E\left[\frac{2\,(p-3)\left(\frac{\chi_n^2}{n+2}-1\right)}{p-3+2k}\middle|\ \chi_n^2\right] \\ &= \frac{2\,(p-3)\left(\frac{\chi_n^2}{n+2}-1\right)}{p-3+2k},\end{aligned}$$

then

$$\begin{aligned}\Delta_\varphi &\leq \sigma^2 E\left\{\chi_n^2\ \frac{2\,(p-3)\left(\dfrac{\chi_n^2}{n+2}-1\right)}{p-3+2k}\ E\left(\varphi_k \mid \chi_n^2\right)\right\} \\ &= \frac{2\,(p-3)\,\sigma^2}{p-3+2k}\ E\left\{\chi_n^2\left(\frac{\chi_n^2}{n+2}-1\right)\varphi_k\right\}.\end{aligned}$$

Using the sufficient condition

$$E\left\{\varphi\left(\sigma^2\chi_{n+4}^2\ ,\sigma^2\chi_{p-1+2k}^2\right)\right\} \leq E\left\{\varphi\left(\sigma^2\chi_{n+2}^2\,\sigma^2\chi_{p-1+2k}^2\right)\right\},$$

we have

$$E\left\{\chi_n^2\left(\frac{\chi_n^2}{n+2}-1\right)\varphi_k\right\} \leq 0,$$

hence $\Delta_\varphi \leq 0$. □

Remark 3.2. Note that the James-Stein estimator given in (3.3) satisfies the conditions of the Proposition 3.4, thus the James-Stein estimator is minimax.

4. SIMULATION

We illustrate the graph of the upper bound given by the formula (2.4) for the risk difference Δ_ψ of the estimator $\delta^{\psi_2}(X, S^2)$ given in the Example 2.2 and the maximum likelihood estimator, divided by the risk of the maximum likelihood estimator $R(X, \theta) = p\sigma^2$, as a function of $d = \|\theta\|^2$ and $s = \sigma^2$, for various values of n and p.

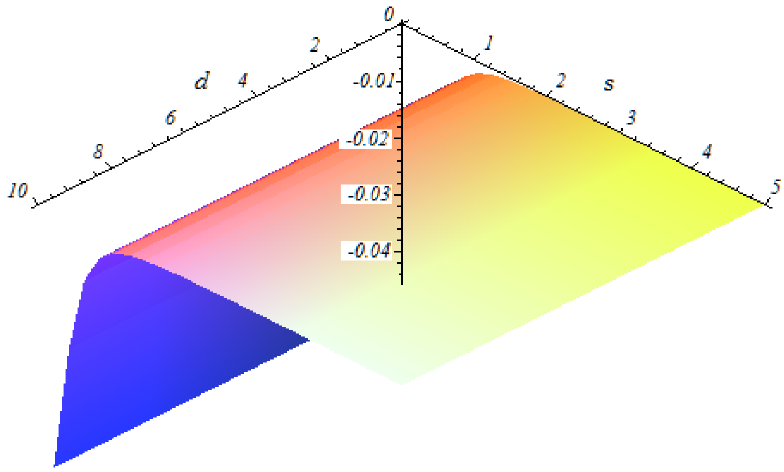

FIGURE 1. $n = 10$ and $p = 4$

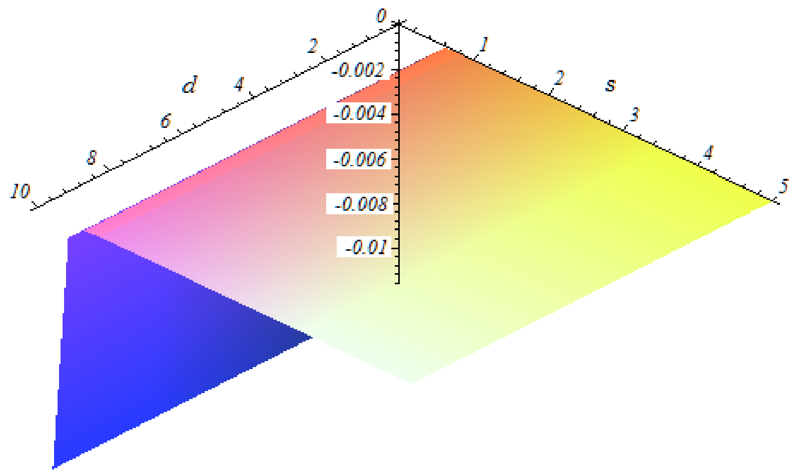

FIGURE 2. $n = 25$ and $p = 10$

In Figure 1 and Figure 2, we note that an upper bound of risks difference of the estimator $\delta^{\psi_2}(X, S^2)$ given in the Example 2.2 and the maximum likelihood estimator X, divided by the risk of the maximum likelihood estimator is negative, thus the estimator $\delta^{\psi_2}(X, S^2)$ is minimax for $n = 10$ and $p = 4$ and for $n = 25$ and $p = 10$.

5. Appendix

Lemma 5.1 (Casella and Hwang [4]). *For any real function h such that $E\left(h\left(\chi_q^2(\lambda)\right)\chi_q^2(\lambda)\right)$ exists, we have*

$$E\left\{h\left(\chi_q^2(\lambda)\right)\chi_q^2(\lambda)\right\} = qE\left\{h\left(\chi_{q+2}^2(\lambda)\right)\right\} + 2\lambda E\left\{h\left(\chi_{q+4}^2(\lambda)\right)\right\}.$$

Lemma 5.2 (Benmansour and Hamdaoui [2]). *Let f be a real function. If for $p \geq 3$, $E_{\chi_p^2(\lambda)}[(f(U)]$ exists, then*

(a) *if f is monotone non-increasing, we have*

$$E_{\chi_{p+2}^2(\lambda)}[(f(U)] \leq E_{\chi_p^2(\lambda)}[(f(U)];$$

(b) *if f is monotone non-decreasing, we have*

$$E_{\chi_{p+2}^2(\lambda)}[(f(U)] \geq E_{\chi_p^2(\lambda)}[(f(U)].$$

Proof. (Proposition 3.1) (i)

$$\begin{aligned} R\left(\delta^\phi\left(X,S^2,T^2\right),\theta\right) =& E\left[\sum_{i=1}^p\left[\left(1-\phi\left(S^2,T^2\right)\right)\left(X_i-\overline{X}\right)+\overline{X}\ -\theta_i\right]^2\right] \\ =& E\left[\left[1-\phi\left(S^2,T^2\right)\right]^2\sum_{i=1}^p\left(X_i-\overline{X}\right)^2\right]+E\left[\sum_{i=1}^p\left(\overline{X}\ -\theta_i\right)^2\right] \\ &+2E\left[\left[1-\phi\left(S^2,T^2\right)\right]\sum_{i=1}^p\left(X_i-\overline{X}\right)\left(\overline{X}-\theta_i\right)\right]. \end{aligned}$$

As

$$\begin{aligned} E\left[\left[1-\phi\left(S^2,T^2\right)\right]^2\sum_{i=1}^p\left(X_i-\overline{X}\right)^2\right] =& E\left[\left(1-\phi_K\right)^2T^2\right] \\ =& \sigma^2E\left[\left(1-\phi_K\right)^2\ \chi_{p-1+2K}^2\right] \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} E\left[\sum_{i=1}^p\left(\overline{X}-\theta_i\right)^2\right] =& E\left[\sum_{i=1}^p\left(\overline{X}-\overline{\theta}+\overline{\theta}-\theta_i\right)^2\right] \\ =& E\left[\sum_{i=1}^p\left(\overline{X}-\overline{\theta}\right)^2\right]+\sum_{i=1}^p\left(\overline{\theta}-\theta_i\right)^2+2\left(\sum_{i=1}^p\left(\overline{\theta}-\theta_i\right)\right)E\left(\overline{X}-\overline{\theta}\right) \\ =& \sigma^2+\sum_{i=1}^p\left(\overline{\theta}-\theta_i\right)^2. \end{aligned} \tag{5.2}$$

The last equality comes from the distribution of $\overline{X}$, $\overline{X} \sim N_p\left(\overline{\theta},\frac{\sigma^2}{p}\right)$ and the fact that $\sum_{i=1}^p\left(\overline{\theta}-\theta_i\right)=0$.

Furthermore, we have

$$\begin{aligned}
&2E\left[\left[1-\phi\left(S^2,T^2\right)\right]\sum_{i=1}^{p}\left(X_i-\overline{X}\right)\left(\overline{X}-\theta_i\right)\right] \\
=&-2E\left[\left[1-\phi\left(S^2,T^2\right)\right]\sum_{i=1}^{p}\left(X_i-\overline{X}\right)\left(\theta_i-\overline{\theta}+\overline{\theta}-\overline{X}\right)\right] \\
=&-2E\left[\left[1-\phi\left(S^2,T^2\right)\right]\sum_{i=1}^{p}\left(X_i-\overline{X}\right)\left(\theta_i-\overline{\theta}\right)\right] \\
&-2E\left[\left[1-\phi\left(S^2,T^2\right)\right]\left(\overline{\theta}-\overline{X}\ \right)\sum_{i=1}^{p}\left(X_i-\overline{X}\right)\right] \\
=&-2E\left[\left[1-\phi\left(S^2,T^2\right)\right]\sum_{i=1}^{p}\left(X_i-\overline{X}\right)\left(\theta_i-\overline{\theta}\right)\right].
\end{aligned}$$

The last equality follows from the fact that $\sum_{i=1}^{p}\left(X_i-\overline{X}\right)=0$.

Using (b) of Lemma 3.1, we have

$$-2E\left[\left[1-\phi\left(S^2,T^2\right)\right]\sum_{i=1}^{p}\left(X_i-\overline{X}\right)\left(\theta_i-\overline{\theta}\right)\right]=-4\sigma^2E\left[K\left(1-\phi_K\right)\right]. \tag{5.3}$$

From formulas (5.1), (5.2) and (5.3) and the fact that $E\left(K\right)=\frac{\sum_{i=1}^{p}\left(\theta_i-\overline{\theta}\right)^2}{2\sigma^2}$, we have

$$\begin{aligned}
R\left(\delta^{\phi}\left(X,S^2,T^2\right),\theta\right)&=E\left\{\sigma^2\left(1-\phi_K\right)^2\ \chi^2_{p-1+2K}+\sigma^2+2\sigma^2K-4\sigma^2K\left(1-\phi_K\right)\right\} \\
&=\sigma^2E\left\{\phi_K^2\ \chi^2_{p-1+2K}-2\phi_K\left(\chi^2_{p-1+2K}-2K\right)+p\right\}.
\end{aligned}$$

(ii) We note that $R\left(\delta^{\phi}\left(X,S^2,T^2\right),\theta\right)$ can be written as

$$\begin{aligned}
R\left(\delta^{\phi}\left(X,S^2,T^2\right),\theta\right)=&\sigma^2E\left\{p-\frac{\left(\chi^2_{p-1+2K}-2K\right)^2}{\chi^2_{p-1+2K}}\right\} \\
&+\sigma^2E\left\{\chi^2_{p-1+2K}\left(\phi_K-1+\frac{2K}{\chi^2_{p-1+2K}}\right)^2\right\} \\
\geq&\sigma^2E\left\{p-\frac{\left(\chi^2_{p-1+2K}-2K\right)^2}{\chi^2_{p-1+2K}}\right\}=B_p\left(\theta\right).
\end{aligned}$$

(iii)

$$B_p\left(\theta\right)=\sigma^2E\left\{p-\frac{\left(\chi^2_{p-1+2K}-2K\right)^2}{\chi^2_{p-1+2K}}\right\}$$

$$= \sigma^2 \left\{ p - E \left\{ E \left[\left(\chi^2_{p-1+2K} + \frac{4K^2}{\chi^2_{p-1+2K}} - 4K \right) \mid K \right] \right\} \right\}$$
$$= \sigma^2 \left\{ p - E \left(p - 1 + 2K + \frac{4K^2}{p-3+2K} - 4K \right) \right\}$$
$$= \sigma^2 \left\{ p - 2 - E \left[\frac{(p-3)^2}{p-3+2K} \right] \right\}.$$

Thus, from Lemma 3.1 given in Sun [19], we obtain

$$\lim_{p \to +\infty} b_p(\theta) = \lim_{p \to +\infty} \frac{B_p(\theta)}{p\sigma^2} = \frac{c}{1+c}. \qquad \square$$

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[1]Department of Biology,
University Mustapha Stambouli of Mascara,
29000 Mascara, Algeria
Email address: benkhaled08@yahoo.fr

[2]Department of Mathematics,
University of Sciences and Technologies Mohamed Boudiaf, Oran -USTO-,
El Mnaouar BP 1505, Bir El Djir 31000, Oran, Algeria
Email address: abdenour.hamdaoui@yahoo.fr, abdenour.hamdaoui@univ-usto.dz